NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 1/67

by James Coroneos*

1. Solve the following quadratic equations, expressing roots in the form $a + ib$, where a, b are real, and $i^2 = -1$.

(i) $x^2 - 4x + 5 = 0$ **(ii)** $x^2 - 4x - 3 = 0$
(iii) $x^2 - 4x + 4 = 0$ **(iv)** $x^2 + 4 = 0$
(v) $2x^2 - 3x + 4 = 0$ **(vi)** $5x^2 - 12x + 17 = 0$
(vii) $x^2 - 2x\cos\theta + 1 = 0$ **(viii)** $x^2 + 2ix + 1 = 0$
(ix) $ix^2 - x + 4i = 0$ **(x)** $2x^2 - 6ix - 3 = 0$
(xi) $x^2 - 2i\sec\theta x - 1 = 0$ **(xii)** $x^2\cos^2\theta + x\sin 2\theta + 1 = 0$

Verify that those quadratic equations with real coefficients have roots which are complex conjugates of each other, but that this is not so if the coefficients are themselves complex.

2. **(i)** Show that $x^2 + x + 2 = (x^2 + x + \frac{1}{4}) + 1\frac{3}{4} = (x + \frac{1}{2})^2 - \frac{7}{4}i^2 = [x + \frac{1}{2} - \frac{\sqrt{7}}{2}i][x + \frac{1}{2} + \frac{\sqrt{7}}{2}i]$

(ii) Similarly find the complex factors of
(a) $x^2 - 2x + 10$ **(b)** $x^2 + 4x + 5$ **(c)** $x^2 - 6x + 14$ **(d)** $x^2 + 2ax + a^2 + b^2$
(e) $x^2 + 1$ **(f)** $x^4 - 1$ **(g)** $x^2 + x + 1$

3. **(i)** Given that $b^2 - c = -k^2$, where $k > 0$; show that the roots of the equation $x^2 - 2bx + c = 0$ are $x = b \pm ki$.

(ii) If a, b, c are real and $b^2 < 4ac$, show that the roots of $ax^2 + bx + c = 0$ are complex conjugates.

*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{AMS}$-TEX.

4. Noting that $x^3 - 1 = (x-1)(x^2+x+1)$, prove that the solutions of $x^3 = 1$ are $x = 1, \frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$. These solutions are called the *3 cube roots of unity*, since they ate the roots of $x^3 = 1$. The solutions $\frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$ are the *complex cube roots of unity*, whilst $x = 1$ is the *real* cube root of unity. If these complex roots are denoted by $\alpha = \frac{-1+\sqrt{3}i}{2}, \beta = \frac{-1-\sqrt{i}}{2}$, verify that $\alpha^2 = \beta, \alpha = \beta^2, \alpha^3 = \beta^3 = 1, 1+\alpha+\beta = 0$. [*Note*, in actual practice, the roots of $x^3 = 1$ are usually written as $1, \omega, \omega^2$ and that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$]