

HARDER INDUCTION – WORKSHEET

COURSE/LEVEL

NSW Secondary High School Year 11 Preliminary Mathematics Extension. Syllabus reference: 7.4.

1. Prove the following by induction, where n is any positive integer:

$$(a) \quad 1^2 + 3^2 + 5^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{6}.$$

$$(b) \quad \frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + \dots + \frac{(2n-1)}{n(n+1)(n+2)} = \frac{3}{4} + \frac{1}{2(n+1)} - \frac{5}{2(n+2)}.$$

$$(c) \quad 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1.$$

$$(d) \quad \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

$$(e) \quad \sum_{i=1}^n i(n-i+1) = \frac{n(n+1)(n+2)}{6}.$$

$$(f) \quad 1 + 2x + 3x^2 + \dots + nx^{n-1} = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}.$$

$$(g) \quad \sum_{i=1}^n (i^3 + 3i^5) = 4 \left(\sum_{i=1}^n i \right)^3.$$

$$(h) \quad 8 + 88 + 888 + \dots + \underbrace{888\dots 8}_{n \text{ digits}} = \frac{8}{81} (10^{n+1} - 9n - 10).$$

$$(i) \quad \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{\sin \frac{1}{2}(2n+1)\theta}{2 \sin \frac{1}{2}\theta} - \frac{1}{2}.$$

$$(j) \quad \sin \theta + \sin 2\theta + \dots + \sin n\theta = \frac{\sin \frac{1}{2}n\theta \sin \frac{1}{2}(n+1)\theta}{\sin \frac{1}{2}\theta}.$$

Hint: You may use the fact that $2 \cos x \sin y = \sin(x+y) - \sin(x-y)$.

(k) Prove the Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Hint: You may use the fact that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

(j) If $y = \sin(ax)$, then $\frac{d^n y}{dx^n} = a^n \sin\left(ax + \frac{n\pi}{2}\right)$.

(k) If $y = \ln(1+x)$, then $\frac{d^n y}{dx^n} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$.

(l) If $y = \frac{1}{1-x}$, then $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$.

(m) If $y = \frac{1}{1-x^2}$, then $\frac{d^n y}{dx^n} = \frac{n!}{2} \left(\frac{1}{(1-x)^{n+1}} + \frac{(-1)^n}{(1+x)^{n+1}} \right)$.

2. Prove the following results for all **odd integers** $n \geq 1$.

(a) $3^n + 7^n$ is divisible by 10.

(b) $7^n + 11^n$ is divisible by 9.

3. Prove that each of the following expressions are divisible by 9 if n is any positive integer.

(a) $4^n + 6n - 1$.

(b) $5^{2n} + 3n - 1$.

(c) $(3n+1)7^n - 1$.

(d) $n^3 + (n+1)^3 + (n+2)^3$.

4. Prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ where n is any positive integer.

5. Prove the following for all integer values of n not less than 1.

(a) $\sum_{k=1}^{2^n} \frac{1}{k} \geq \frac{n+2}{2}$.

(b) $\sum_{k=1}^n \frac{1}{k^2} \leq 2 - \frac{1}{n}$. (HSC 1987)

(c) $\sum_{k=1}^{2^n-1} \frac{1}{k^2} \leq 2 - \frac{1}{2^{n-1}}$

(d) $\sum_{k=1}^{2^n-1} \frac{1}{k^r} \leq \frac{1 - \left(\frac{1}{2}\right)^{(r-1)n}}{1 - \left(\frac{1}{2}\right)^{r-1}}$

(Note that (c) is a special case of the inequality in (d) with $r = 2$.)

6. (HSC 1985)

(a) Show that for $k \geq 0$, $2k+3 > 2\sqrt{(k+1)(k+2)}$.

(b) Hence prove that for $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

7. Prove Bernoulli's inequality: $1 + nx \leq (1+x)^n$ where $x > -1$ (and n is any positive integer.)