Exercices 30(b)

Q.1) a) $y = x\log_e x$ The function is define for $x > 0$

$\frac{dy}{dx} = \log_e x + 1$ (gradient function)

Let $\frac{dy}{dx} = 0$ To get the stationary turning points.

$\therefore \log_e x = -1$ $\therefore x = 1/e$

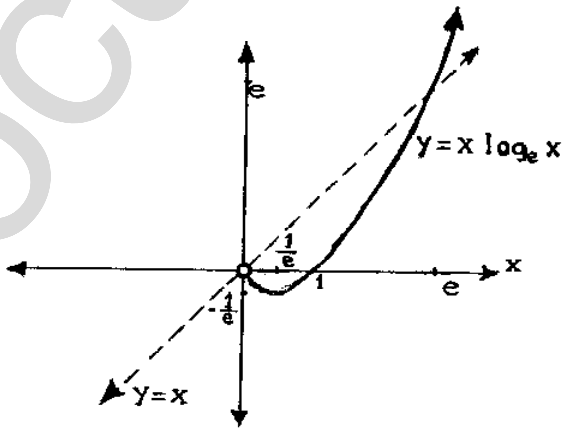
Note: $\lim_{x \to 0^+} x\log_e x = 0$

x		1/e	
$\frac{dy}{dx}$	−	0	+
y	↘	min −1/e	↗

When $x < 1/e$, $\frac{dy}{dx} < 0$

$x > 1/e$, $\frac{dy}{dx} > 0$

$\therefore$ *At* $(1/e, -1/e)$ *there are a minimum turning point.*



b) $y = \frac{\log_e x}{x}$ The function is define for $x > 0$

$\frac{dy}{dx} = \frac{1 - \log_e x}{x^2}$ (gradient function)

Let $\frac{dy}{dx} = 0$ To get the stationary turning points.

$\therefore \log_e x = 1$ $\therefore x = e$

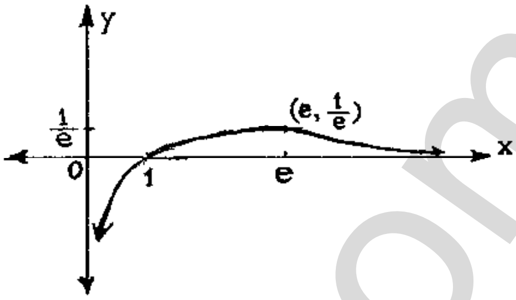
Note: $\lim_{x \to 0^+} \frac{\log_e x}{x} \to -\infty$

x		e	
$\frac{dy}{dx}$	+	0	−
y	↗	max 1/e	↘

When $x < e$, $\frac{dy}{dx} > 0$

$x > e$, $\frac{dy}{dx} < 0$

$\therefore$ *At* $(e, 1/e)$ *there are a maximum turning point.*



c) $y = \dfrac{x}{\log_e x}$ The function is define for $x > 0$ except $x = 1$

Asymptotic behaviour: *When* $x \to 0^+$, $y \to 0^+$

When $x \to 1$, $y \to \infty$ $\therefore$ *$x = 1$ is a vertical asymptote.*

$\dfrac{dy}{dx} = \dfrac{\log_e x - 1}{\log^2_e x}$ (gradient function)

Let $\dfrac{dy}{dx} = 0$ To get the stationary turning points.

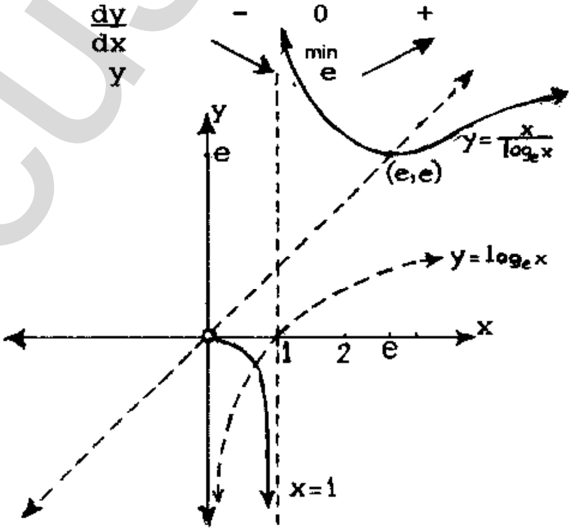
$\therefore \log_e x = 1$ $\therefore x = e$ $\therefore y = e$

When $x < e$, $\dfrac{dy}{dx} < 0$

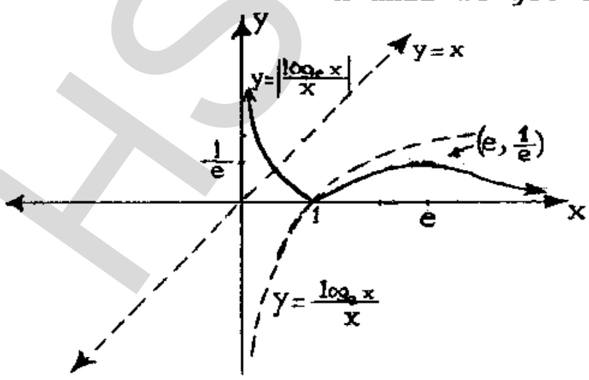
$x > e$, $\dfrac{dy}{dx} > 0$

$\therefore$ *At (e , e) there is a minimum turning point.*

x		e	
$\dfrac{dy}{dx}$	$-$	0	$+$
y		min e	



d) $y = \left|\dfrac{\log_e x}{x}\right|$ Using b) and by reflecting the portion of the graph which is in the 4th quadrant in the x-axis we get the graph desired.



Q.2) $y = x^2e^{-x}$ The function is define for all x (real numbers)

Asymptotic behaviour: *When* $x \to -\infty$, $y \to +\infty$

When $x \to +\infty$, $y \to 0$ $\therefore y = 0$ *is a horizontal asymptote.*

$\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2)$ (gradient function)

Let $\frac{dy}{dx} = 0$ To get the stationary turning points.

$\therefore x = 0$ or 2 Note: $e^{-x} > 0$

x		0		2	
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$
y		min 0		max $4/e^2$	

When $x < 0$, $\frac{dy}{dx} < 0$

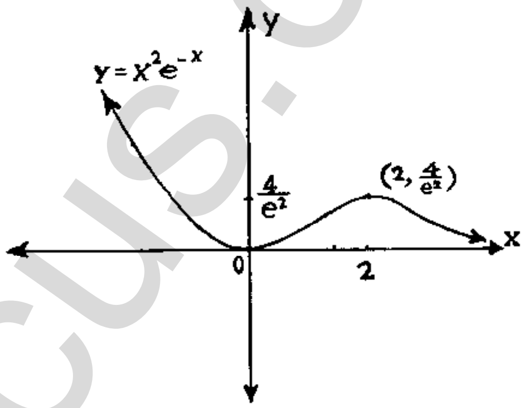
$x > 0$, $\frac{dy}{dx} > 0$

$\therefore$ *At* $(0, 0)$ *there is a minimum turning point.*

When $x < 2$, $\frac{dy}{dx} > 0$

$x > 2$, $\frac{dy}{dx} < 0$

$\therefore$ *At* $(2, 4/e^2)$ *there is a maximum turning point.*



b) $y = x\log_e(x^2 - 1)$

Domain: The function is define for all $x < -1$ or $x > 1$

Asymptotic behaviour:

When $x \to -\infty$, $y \to -\infty$

When $x \to +\infty$, $y \to +\infty$

Note: $f(-x) = -x\log_e[(-x)^2 - 1]$

$= -x\log_e(x^2 - 1)$

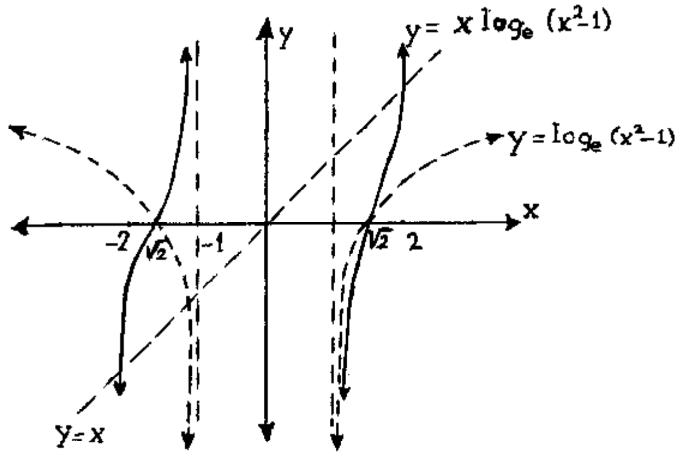
$= -f(x)$

$\therefore$ *The function is an odd function.* $\therefore$ *It is symmetric about the origin.*

Intersection with $y = x$

$\therefore \log_e(x^2 - 1) = 1$

$\therefore x = \pm\sqrt{e+1}$

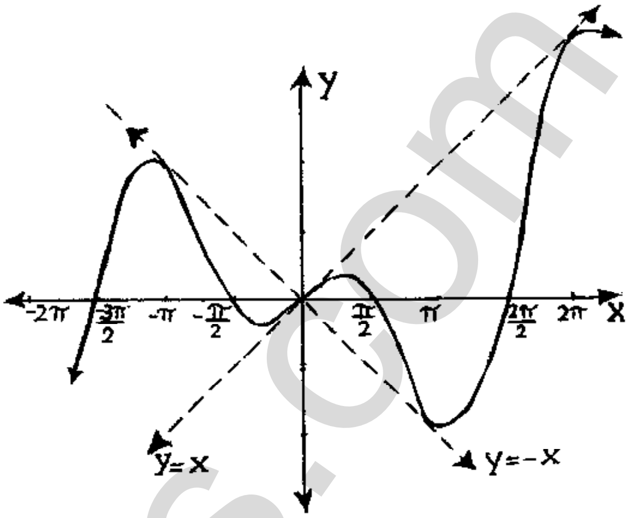


c) $y = x\cos x$ The function is define for all x (real numbers)

Note1: $f(-x) = -x\cos(-x)$
$= -x\cos x = -f(x)$
$\therefore$ *The function is an odd function.* $\therefore$ *It is symmetric about the origin.*
Note2: Since $-1 \le \cos x \le 1$
$\therefore$ *When* $y = x$, $\cos x = 1$
and $x = 0, 2\pi, \ldots$
when $y = -x$, $\cos x = -1$
$\therefore x = \pi$, $3\pi, \ldots$
And $y = 0$, $\cos x = 0$
$\therefore x = \pi/2$, $3\pi/2$, ...

$\frac{dy}{dx} = \cos x - x\sin x$

Let $dy/dx = 0$ To get the stationary turning points.
$\therefore \cot x = x$ $\therefore$ The stationary turning points do not lie on the lines $y = \pm x$

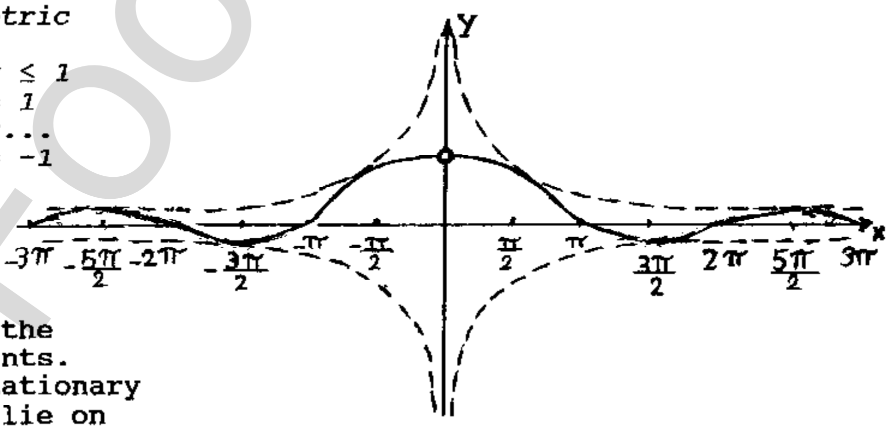


d) $y = \frac{\sin x}{x}$ The function is define for all x except $x = 0$.

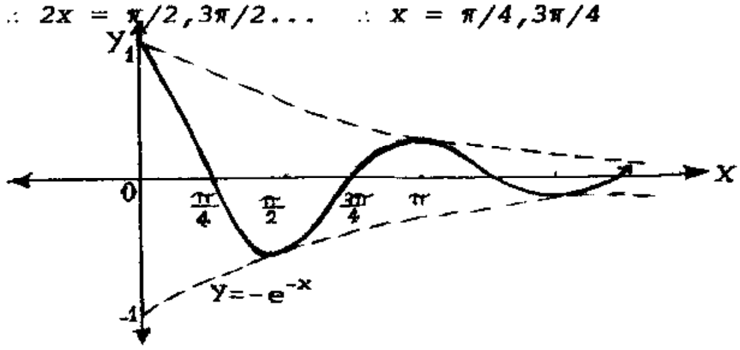
Note1: $f(-x) = \sin(-x)/(-x)$
$= \sin x/x = f(x)$
$\therefore$ *The function is an even function.* $\therefore$ *It is symmetric about the y - axis.*
Note2: Since $-1 \le \sin x \le 1$
$\therefore$ *When* $y = 1/x$, $\sin x = 1$
and $x = \pi/2, 5\pi/2, -3\pi/2\ldots$
when $y = -1/x$, $\sin x = -1$
$\therefore x = 3\pi/2, -\pi/2, \ldots$
And $y = 0$, $\sin x = 0$
$\therefore x = 0$, π , $-\pi\ldots$

$\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$

Let $dy/dx = 0$ To get the stationary turning points.
$\therefore \cot x = 1/x$ $\therefore$ The stationary turning points do not lie on the lines $y = \pm x$



Q.3) $y = e^{-x}\cos 2x$ The function is define for all $x \ge 0$.
Note: Since $-1 \le \cos 2x \le 1$ $\therefore$ *When* $y = e^{-x}$, $\cos 2x = 1$ $\therefore x = n\pi$
when $y = -e^{-x}$, $\cos 2x = -1$ $\therefore 2x = \pm\pi + 2n\pi$ $\therefore x = (n + 1/2)\pi$
And $y = 0$, $\cos 2x = 0$ $\therefore 2x = \pi/2, 3\pi/2\ldots$ $\therefore x = \pi/4, 3\pi/4$

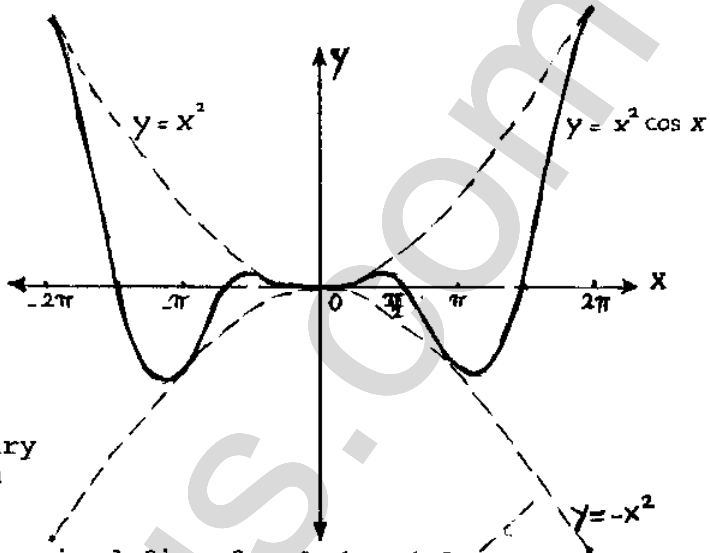


Q.4) a) $y = x^2\cos x$ The function is define for $-2\pi \le x \le 2\pi$

Note1: $f(-x) = (-x)^2\cos(-x)$
$= x^2\cos x = f(x)$
$\therefore$ *The function is an even function.* $\therefore$ *It is symmetric about the y - axis.*
Note2: Since $-1 \le \cos x \le 1$
$\therefore$ *When* $y = x^2$, $\cos x = 1$
and $x = 0, 2\pi, ...$
when $y = -x^2$, $\cos x = -1$
$\therefore x = \pi , 3\pi, ...$
And $y = 0$, $\cos x = 0$
$\therefore x = \pi/2 , 3\pi/2 , ...$

$\dfrac{dy}{dx} = 2x\cos x - x^2\sin x$

Let $dy/dx = 0$ To get the stationary turning points.
$\therefore \cot x = x/2$ $\therefore$ The stationary turning points do not lie on the lines $y = \pm x^2$

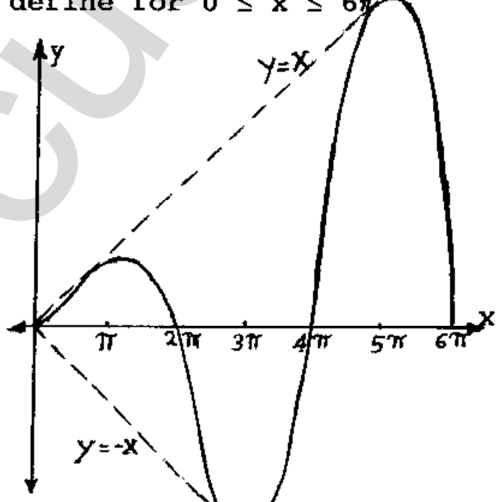


b) $y = x\sin(x/2)$ The function is define for $0 \le x \le 6\pi$

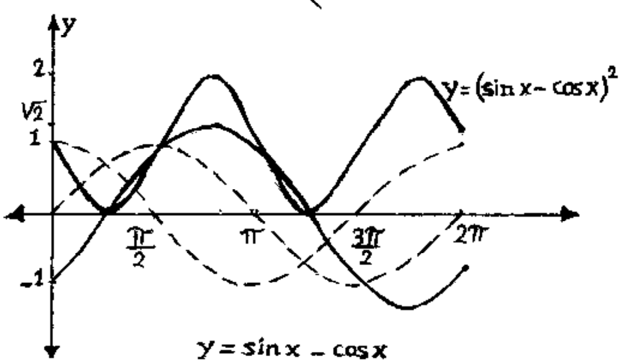
Note $-1 \le \sin(x/2) \le 1$
$\therefore$ *When* $y = x$, $\sin(x/2) = 1$
and $x = \pi$, $5\pi/2$
when $y = -x$, $\sin(x/2) = -1$
$\therefore x = 3\pi$
And $y = 0$, $\sin(x/2) = 0$
$\therefore x = 0 , 2\pi , 4\pi , 6\pi$

$\dfrac{dy}{dx} = \sin(x/2) + (x/2)\cos(x/2)$

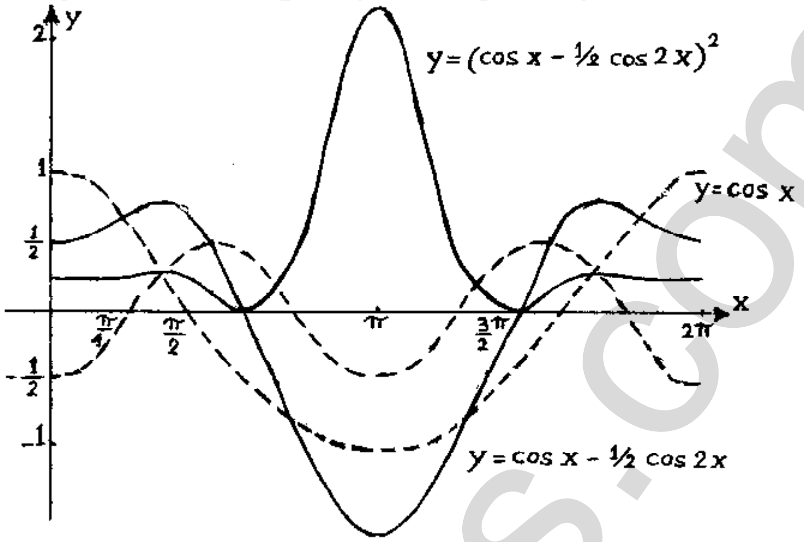
Let $dy/dx = 0$ To get the stationary turning points.
$\therefore \cot(x/2) = -x/2$
$\therefore$ The stationary turning points do not lie on the lines $y = \pm x$



Q.5) $y = \sin x - \cos x$ and $y = (\sin x - \cos x)^2$, $0 \le x \le 2\pi$



Q.6) $y = \cos x - \frac{1}{2}\cos 2x$ and $y = (\cos x - \frac{1}{2}\cos 2x)^2$; $0 \le x \le 2\pi$



Q.7) $y = \dfrac{x - 1}{x + 1}$ the function is defined for all x (real numbers) except $x = -1$

Asymptotic behaviour:

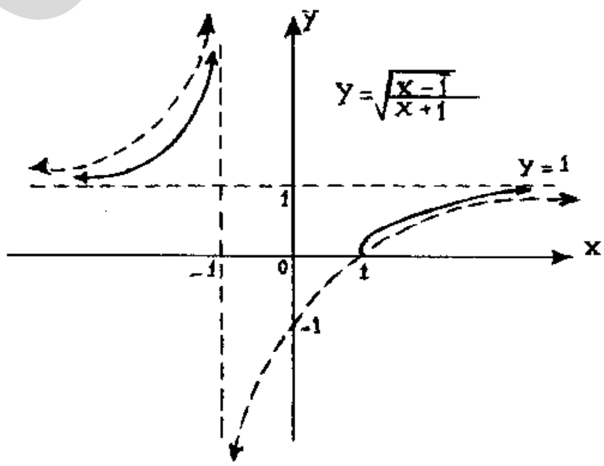
When $x \to \pm \infty$, $y \to 1$ $\therefore$ $y = 1$ is a horizontal asymptote.
When $x \to -1$, $y \to \pm \infty$ $\therefore$ $x = -1$ is a vertical asymptote.

$\dfrac{dy}{dx} = \dfrac{2}{(x + 1)^2}$ (gradient function)

$\therefore \dfrac{dy}{dx} > 0$ $\therefore$ there are no stationary turning points and the function is always increasing.

the graph of this function is as shown below (broken line).

$y = \sqrt{\dfrac{x - 1}{x + 1}}$ The function is defined for $x < -1$ or $x \ge 1$ as $y = (x-1)/(x+1)$ is positive in this domain.



Q.8) $y = \dfrac{x + 2}{x - 3}$ the function is defined for all x (real numbers) except $x = 3$

Asymptotic behaviour:

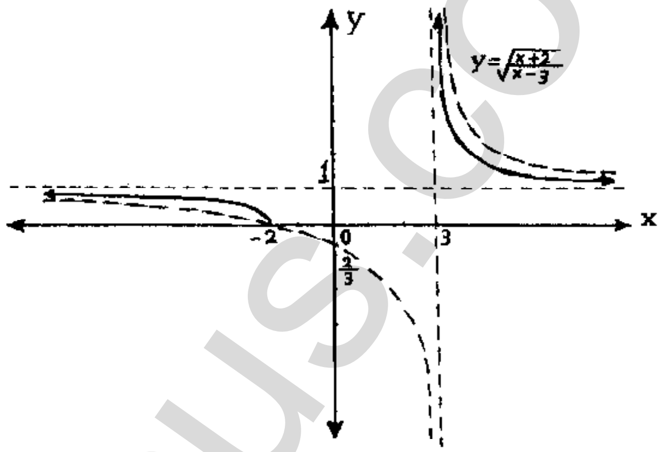
When $x \to \pm \infty$, $y \to 1$ $\therefore$ $y = 1$ is a horizontal asymptote.
When $x \to 3$, $y \to \pm \infty$ $\therefore$ $x = -1$ is a vertical asymptote.

$\frac{dy}{dx} = \frac{-5}{(x - 3)^2}$ (gradient function)

$\therefore \frac{dy}{dx} < 0$ $\therefore$ there are no stationary turning points and the function is always decreasing.

the graph of this function is as shown below (broken line).

$y = \sqrt{\frac{x + 2}{x - 3}}$ The function is defined for $x \le -2$ or $x > 3$ as $y = (x+2)/(x-3)$ is positive in this domain.



Q.9) $y = \frac{1 - x^2}{1 + x^2}$ the function is defined for all x (real numbers).

Asymptotic behaviour:

When $x \to \pm \infty$, $y \to -1$ $\therefore$ $y = -1$ is a horizontal asymptote.

$\frac{dy}{dx} = \frac{-2x(1+x^2)-2x(1-x^2)}{(1 + x^2)^2} = \frac{-4x}{(1 + x^2)^2}$ (gradient function)

Let $\frac{dy}{dx} = 0$ To get the stationary turning points.

$\therefore x = 0$, $y = 1$

When $x < 0$, $\frac{dy}{dx} > 0$

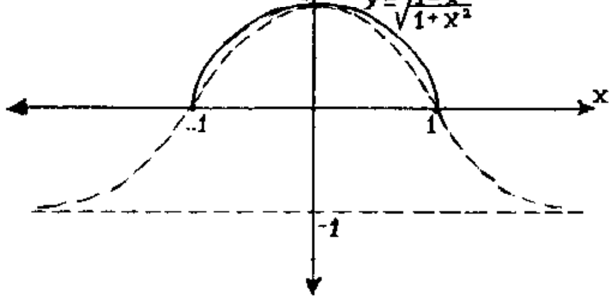
$x > 0$, $\frac{dy}{dx} < 0$

x		0	
$\frac{dy}{dx}$	+	0	−
y	↗	max 1	↘

$\therefore$ *At (0 ,1) there is a maximum turning point.*

The graph of this function is as shown below (broken line).

$y = \sqrt{\frac{1 - x^2}{1 + x^2}}$ The function is defined for $-1 \le x \le 1$,since $y = (1-x^2)/(1+x^2)$ is positive in this domain.



Q.10) a) $y = \dfrac{\sqrt{x-1}}{x}$ The function is defined for $x > 1$.

Asymptotic behaviour:

When $x \to +\infty$, $y \to 0^+$ $\therefore$ $y = 0$ is a horizontal asymptote.

Turning points:

$\dfrac{dy}{dx} = \dfrac{x/2\sqrt{(x-1)} - \sqrt{(x-1)}}{x^2} = \dfrac{2-x}{2x^2\sqrt{(x-1)}}$ (gradient function)

Let $\dfrac{dy}{dx} = 0$ To get the stationary turning points.

$\therefore$ $x = 2$, $y = 1/2$

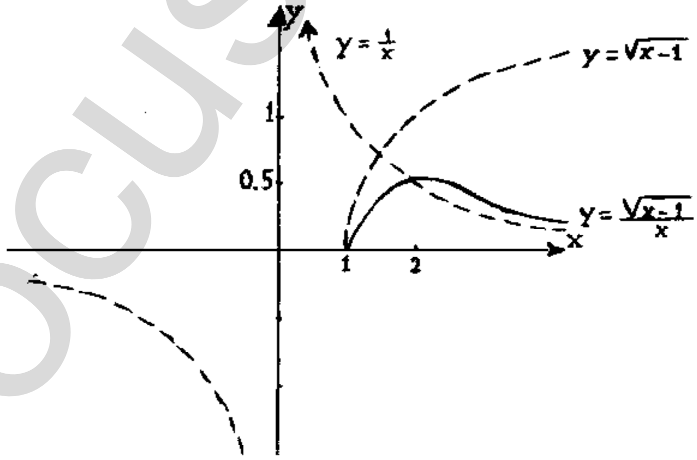
When $x < 2$, $\dfrac{dy}{dx} > 0$

$x > 2$, $\dfrac{dy}{dx} < 0$

x		2	
$\dfrac{dy}{dx}$	+	0	−
y	$\nearrow$	max $1/2$	$\searrow$

$\therefore$ At $(2, 1/2)$ there is a maximum turning point.

The graph of this function is as shown below .



b) $y = \dfrac{x+1}{\sqrt{x-1}}$ The function is defined for $x > 1$.

Asymptotic behaviour:

When $x \to +\infty$, $y \to \sqrt{x}$ $\therefore$ $y = \sqrt{x}$ is an asymptote.

$x \to 1$, $y \to +\infty$ $\therefore$ $x = 1$ is a vertical asymptote.

Turning points:

$\dfrac{dy}{dx} = \dfrac{\sqrt{(x-1)} - (x+1)/2\sqrt{(x-1)}}{(x-1)^2} = \dfrac{x-3}{2(x-1)^2\sqrt{(x-1)}}$ (gradient function)

Let $\dfrac{dy}{dx} = 0$ To get the stationary turning points.

$\therefore$ $x = 3$, $y = 2\sqrt{2}$

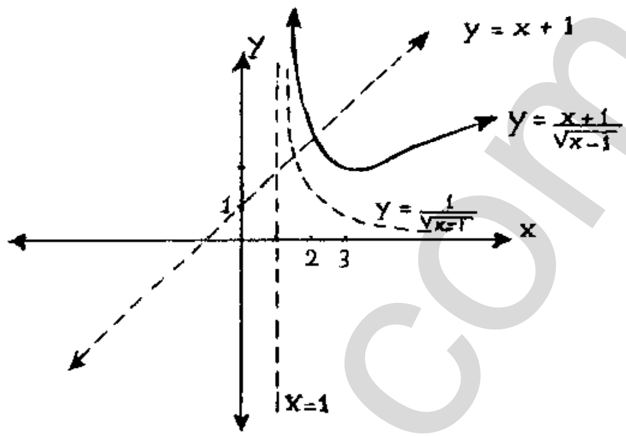
When $x < 3$, $\dfrac{dy}{dx} < 0$

$x > 3$, $\dfrac{dy}{dx} > 0$

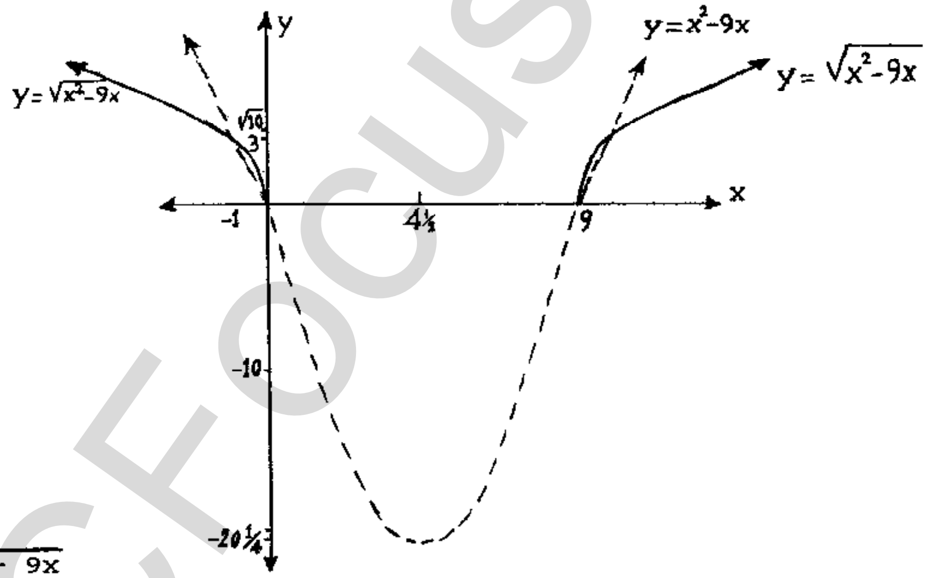
x		3	
$\dfrac{dy}{dx}$	−	0	+
y	$\searrow$	min $2\sqrt{2}$	$\nearrow$

$\therefore$ At $(3, 2\sqrt{2})$ there is a minimum turning point.

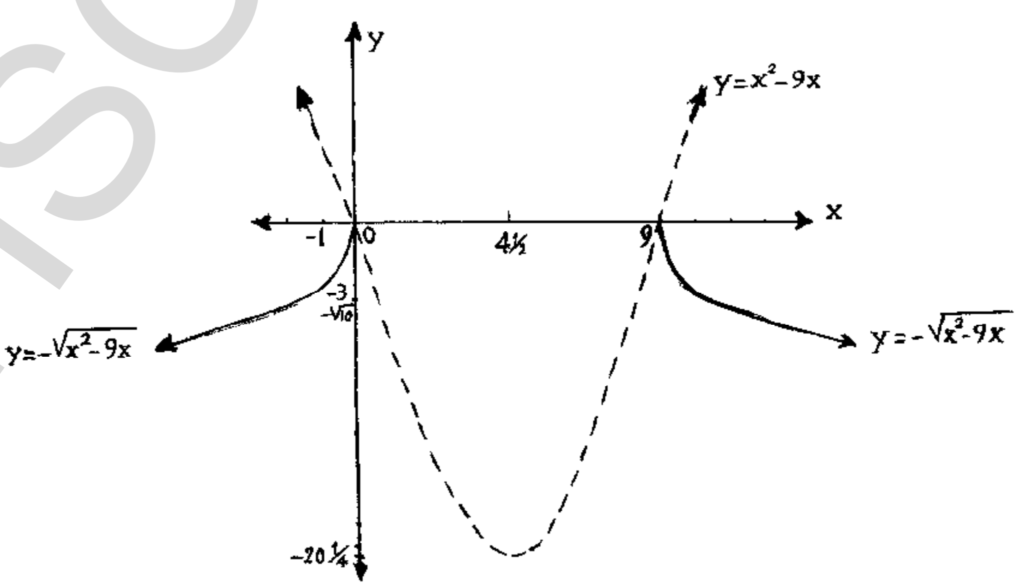
The graph of this function is as shown below .



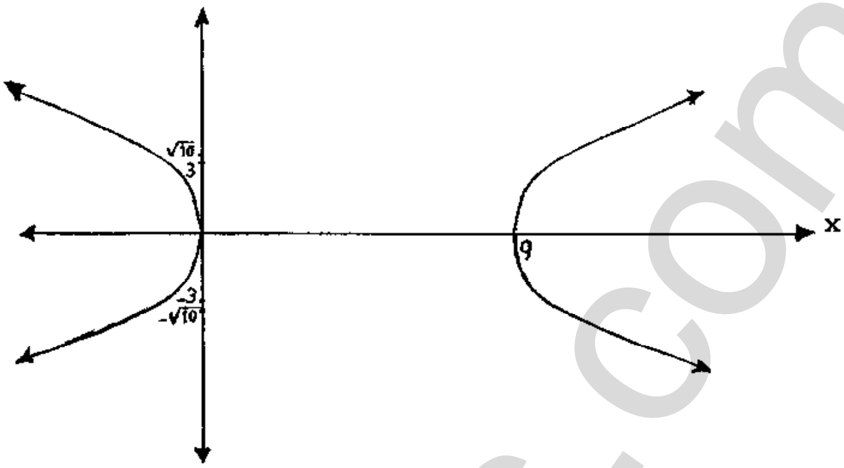
Q.11) a) $y = \sqrt{x^2 - 9x}$



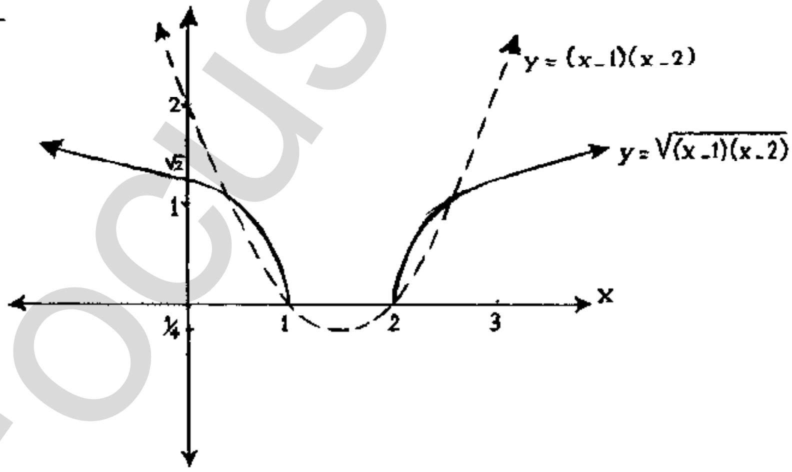
b) $y = -\sqrt{x^2 - 9x}$



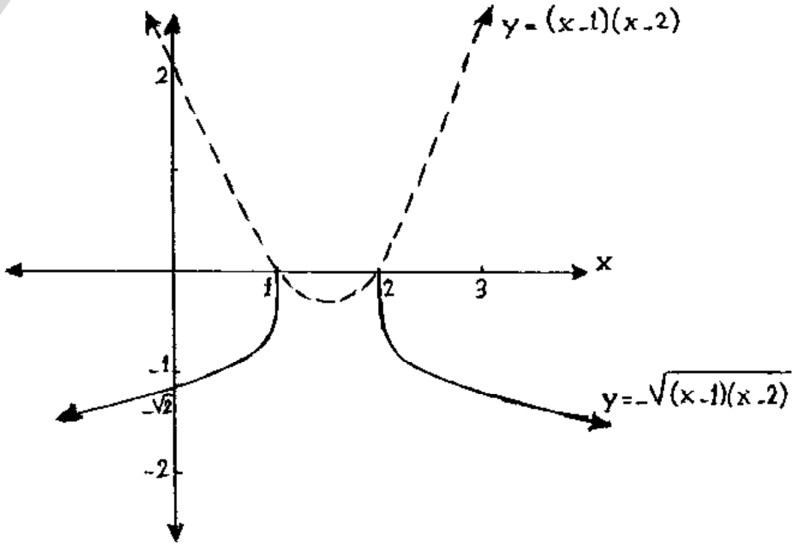
c) $y^2 = x^2 - 9x$



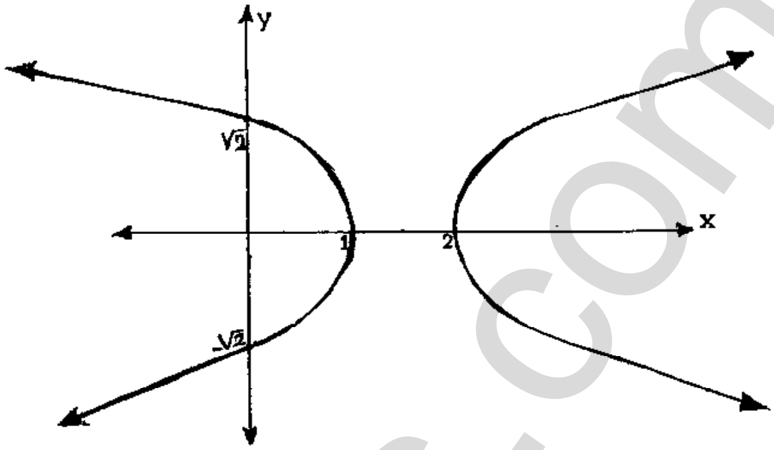
Q.12) a) $y = \sqrt{(x-1)(x-2)}$



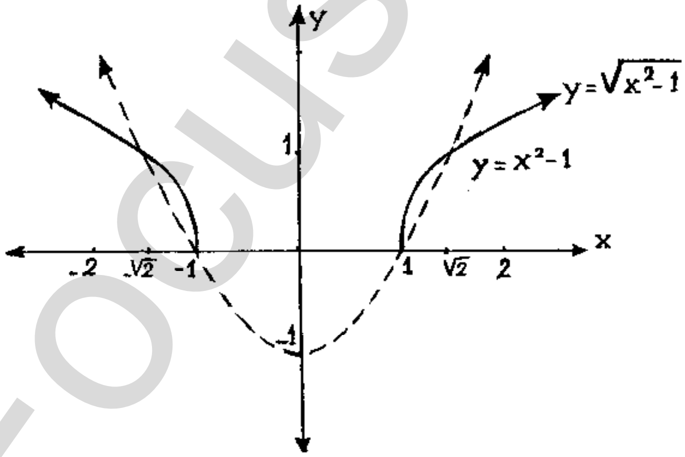
b) $y = -\sqrt{(x-1)(x-2)}$



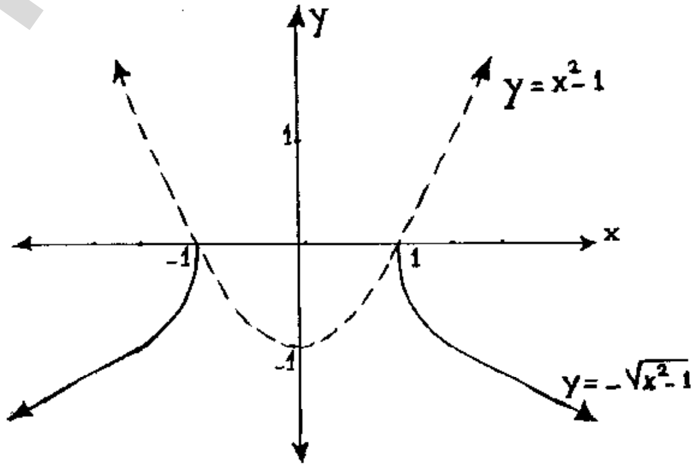
c) $y^2 = (x-1)(x-2)$



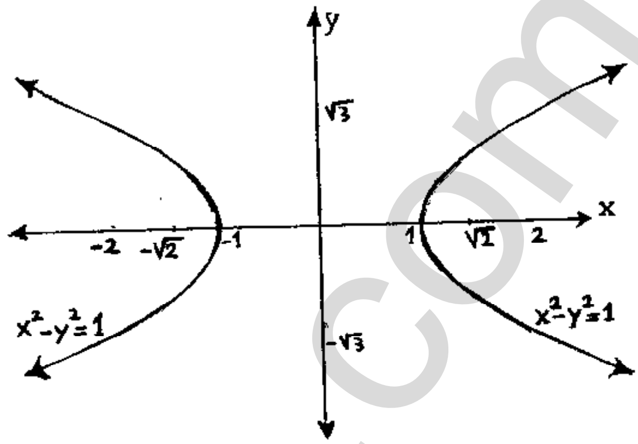
Q.13) a) $y = \sqrt{x^2 - 1}$



b) $y = -\sqrt{x^2 - 1}$



c) $x^2 - y^2 = 1$



Q.14 & Q.15)

a) $y = f(x) = \dfrac{(x-1)(x+1)}{(x-2)(x+2)} = \dfrac{x^2-1}{x^2-4}$

Domain: *the function is defined for all x (real numbers) except for $x = 2, -2$*

Note: $f(-x) = \dfrac{(-x)^2-1}{(-x)^2-4} = \dfrac{x^2-1}{x^2-4} = f(x)$

$\therefore f(-x) = f(x)$

$\therefore$ *The function is an even function.* $\therefore$ *It is symmetric about the y - axis.*

Asymptotic behaviour:

When $x \to \pm\infty$, $y \to 1$ $\therefore y = 1$ is a horizontal asymptote.

$x \to \pm 2$, $y \to \pm\infty$ $\therefore x = \pm 2$ are vertical asymptotes.

Turning points:

$\dfrac{dy}{dx} = \dfrac{2x(x^2-4)-2x(x^2-1)}{(x^2-4)^2} = \dfrac{-6x}{(x^2-4)^2}$ (gradient function)

Let $\dfrac{dy}{dx} = 0$ To get the stationary turning points.

$\therefore x = 0$, $y = 1/4$

When $x < 0$, $\dfrac{dy}{dx} > 0$

$x > 0$, $\dfrac{dy}{dx} < 0$

x		0	
$\frac{dy}{dx}$	+	0	-
y	↗	max 1/4	↘

$\therefore$ *At (0 ,1/4) there is a maximum turning point.*

The graph of the function f(x) is as shown below (solid line).

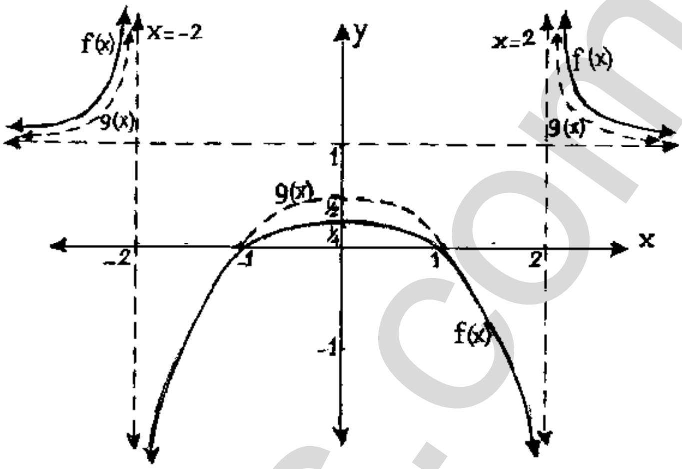
$y = g(x) = \sqrt{\dfrac{x^2-1}{x^2-4}}$

The function is defined for :

$x < -2$, $-1 \le x \le 1$ or $x > 2$

as $y = f(x) = \dfrac{(x+1)(x-1)}{(x+2)(x-2)}$ is positive in this domain.

$\therefore$ The graph of g(x) is as shown below (broken line).



b) $y = f(x) = \dfrac{x}{x^2 - 1}$

Domain: *the function is defined for all x (real numbers) except for $x = 1$, -1*

Note: $f(-x) = \dfrac{x}{(-x)^2 - 1} = \dfrac{-x}{x^2 - 1} = -f(x)$

$\therefore$ ***The function is an odd function.*** $\therefore$ ***It is symmetric about the origin.***

Asymptotic behaviour:

When $x \to \pm \infty$, $y \to 0$ $\therefore$ $y = 0$ is a horizontal asymptote.
$x \to \pm 1$, $y \to \pm \infty$ $\therefore$ $x = \pm 1$ are vertical asymptotes.

Turning points:

$\dfrac{dy}{dx} = \dfrac{-1-x^2}{(x^2 - 4)^2}$ (gradient function)

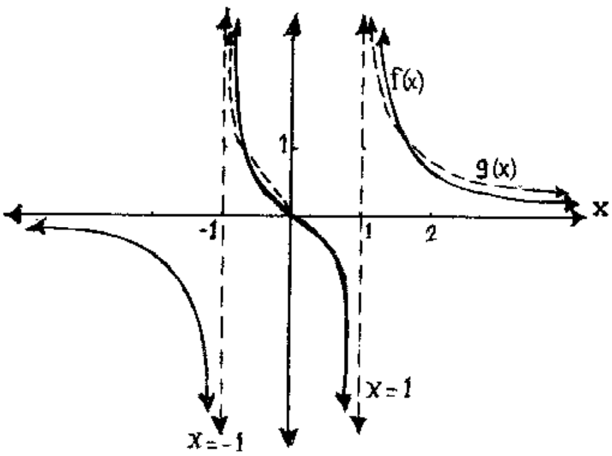
Since $-1 - x^2 < 0$ for all values of x $\therefore$ $dy/dx < 0$ and the function is always decreasing.

The graph of this function is as shown below(solid line).

$y = g(x) = \sqrt{\dfrac{x}{x^2 - 1}}$

The function is defined for :
$-1 < x \le 0$ or $x > 1$
as $y = f(x) = \dfrac{x}{x^2 - 1}$ is positive in this domain.

$\therefore$ The graph of g(x) is as shown below (broken line).



c) $y = f(x) = \dfrac{(x+2)(x-3)}{(x+1)(x-6)} = \dfrac{x^2 - x - 6}{x^2 - 5x - 6}$

Domain:the function is defined for all x (real numbers) except for x = -1 , 6

Asymptotic behaviour:

When $x \to \pm\infty$, $y \to 1$ $\therefore y = 1$ is a horizontal asymptote.
$x \to -1$, $y \to \pm\infty$ $\therefore x = -1$ is a vertical asymptote.
$x \to 6$, $y \to \pm\infty$ $\therefore x = 6$ is a vertical asymptote.

Turning points:

$$\frac{dy}{dx} = \frac{(2x-1)(x^2-5x-6)-(2x-5)(x^2-x-6)}{(x^2-5x-6)^2} = \frac{-4x^2-24}{(x^2-5x-6)^2}$$

Since $-4x^2 -24 < 0$ for all values of x $\therefore dy/dx < 0$ and the function is always decreasing.

The graph of this function is as shown below(solid line).

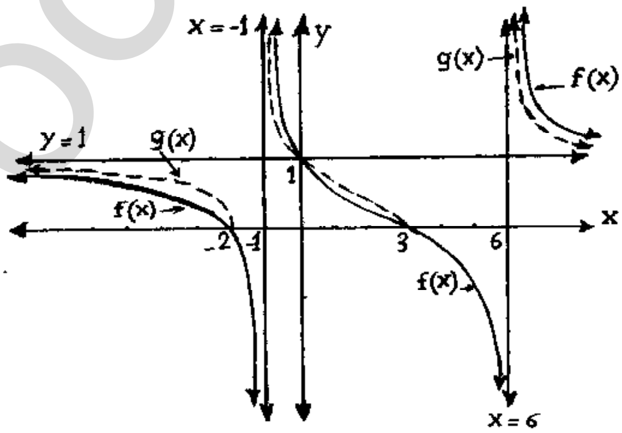
$y = g(x) = \sqrt{\dfrac{(x+2)(x-3)}{(x+1)(x-6)}}$

The function is defined for :

$x > -2$, $-1 < x \le 3$ or $x > 6$

as $y = f(x) = \dfrac{(x+2)(x-3)}{(x+1)(x-6)}$ is positive in this domain.

$\therefore$ The graph of g(x) is as shown below (broken line).



d) $y = \dfrac{x^2+3}{x(x-3)} = \dfrac{x^2+3}{x^2-3x}$

Domain:the function is defined for all x (real numbers) except for x = 0 , 3

Asymptotic behaviour:

When $x \to \pm\infty$, $y \to 1$ $\therefore y = 1$ is a horizontal asymptote.
$x \to 0$, $y \to \pm\infty$ $\therefore x = 0$ is a vertical asymptote.
$x \to 3$, $y \to \pm\infty$ $x = 3$ is a vertical asymptote.

Turning points:

$$\frac{dy}{dx} = \frac{2x(x^2 - 3x) - (2x - 3)(x^2 + 3)}{(x^2 - 3x)^2} = \frac{-3(x^2 + 2x - 3)}{(x^2 - 3x)^2}$$

Let $\frac{dy}{dx} = 0$ To get the stationary turning points.

$\therefore x^2 + 2x - 3 = 0 \quad \therefore x = -3$ or $x = 1$

x		-3		0		1		3	
$\frac{dy}{dx}$	$-$	0	$+$		$+$	0	$-$		$-$
y	$\searrow$	min $2/3$	$\nearrow$		$\nearrow$	max -2	$\searrow$		$\searrow$

When $x < -3$, $\frac{dy}{dx} < 0$

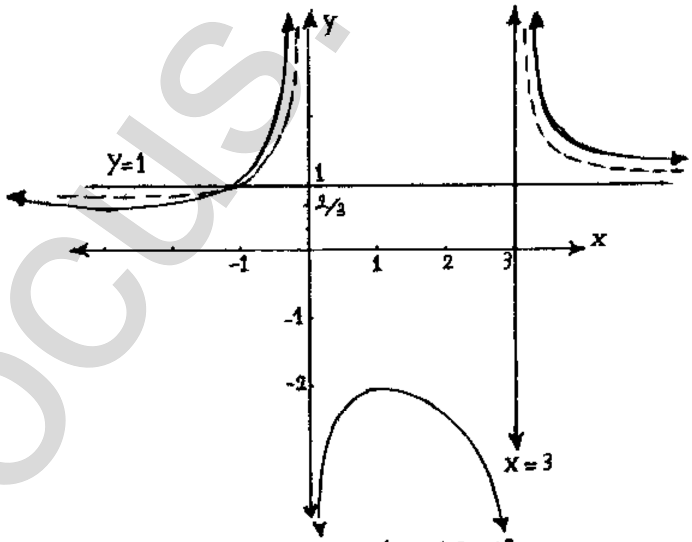
$x > -3$, $\frac{dy}{dx} > 0$

$\therefore$ *At* $(-3\ ,\ 2/3)$ *there is a minimum turning point.*

When $x < 1$, $\frac{dy}{dx} > 0$

$x > 1$, $\frac{dy}{dx} < 0$

$\therefore$ *At* $(1\ ,\ -2)$ *there is a maximum turning point. The graph of this function is as shown below (solid line).*



$y = g(x) = \sqrt{\frac{x^2 + 3}{x^2 - 3x}}$

The function is defined for :

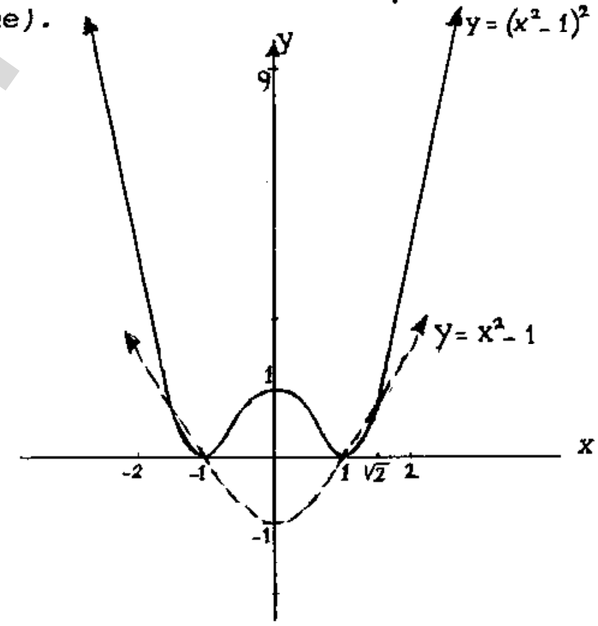
$x < 0$ *or* $x > 3$

as $y = f(x) = \frac{x^2 + 3}{x^2 - 3x}$

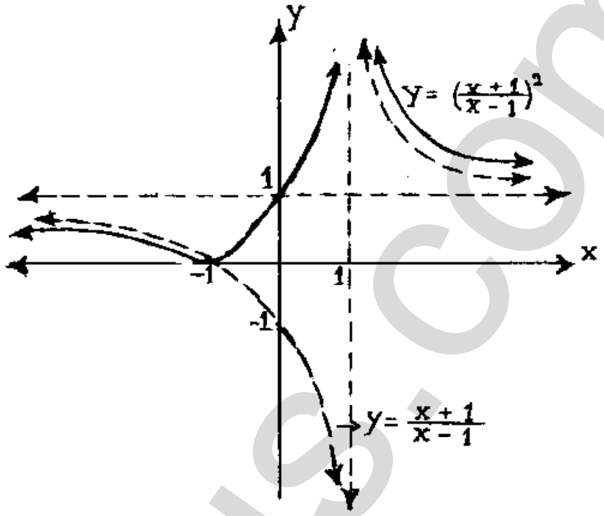
is positive in this domain.

$\therefore$ *The graph of* $g(x)$ *is as shown below (broken line).*

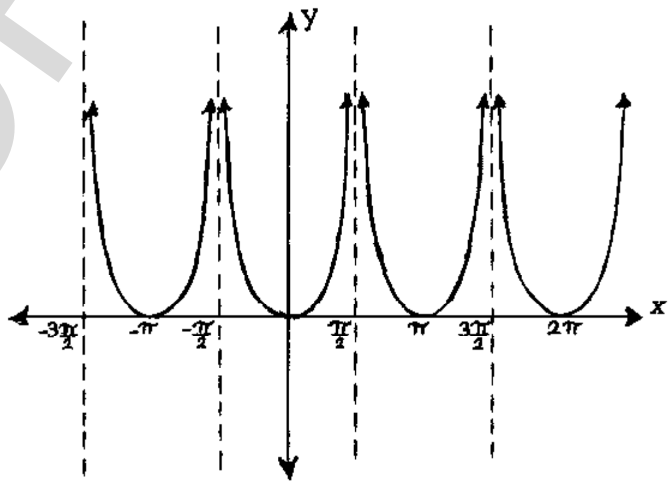
Q.16) a) $y = (x^2 - 1)^2$



b) $y = \left(\frac{x + 1}{x - 1}\right)^2$



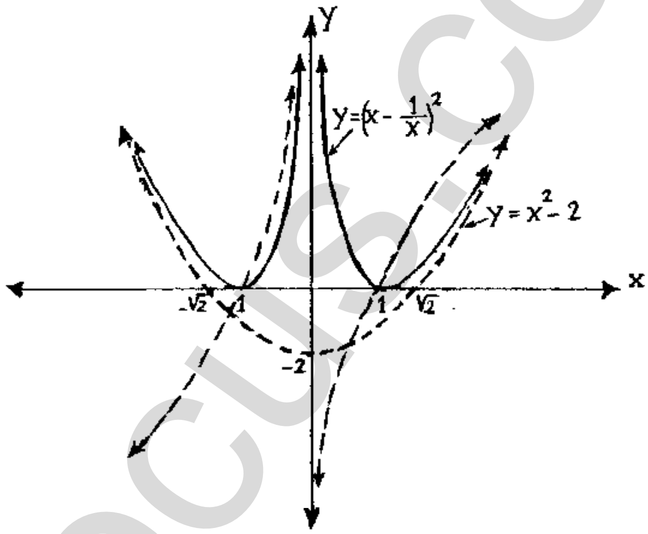
c) $y = \tan^2 x$



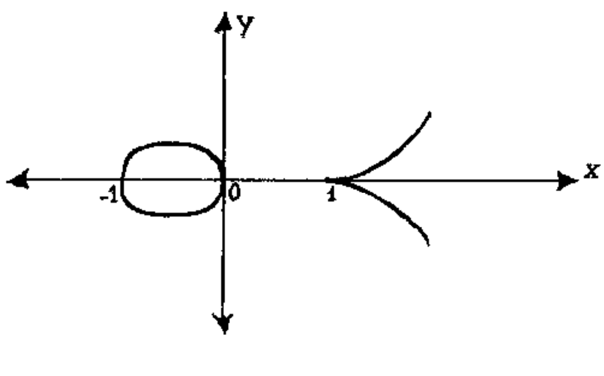
$$y = \left(x - \frac{1}{x}\right)^2 = x^2 - 2 + \frac{1}{x^2}$$

Asymptotic behaviour:

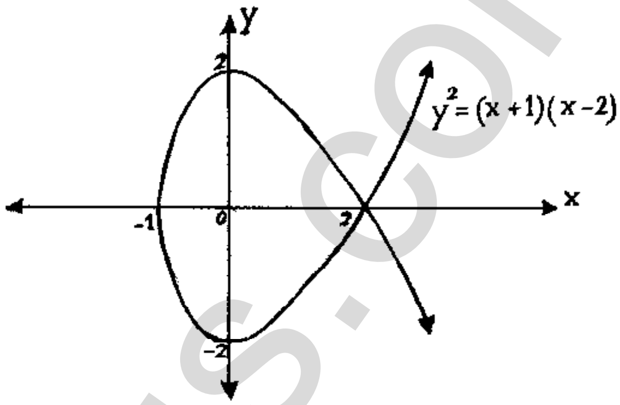
When $x \to \pm \infty$, $y - (x^2 - 2) \to 0$ $\therefore$ $y = x^2 - 2$ is an asymptote.



Q.17) a) $y^2 = x(x-1)(x+1)$



b) $y^2 = (x+1)(x-2)^2$



c) $y^2 = \sin x \quad , \qquad -2\pi \le x \le 2\pi$