Exercise 1

Extracts from

Taylors College COMPLEX NUMBERS Study Guide

+ ANSWERS

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COMPLEX NUMBERS

Q1 Write down some examples of
(i) real numbers **(ii)** integers **(iii)** rational numbers **(iv)** irrational numbers

Q2 Solve these quadratic equations and describe their roots.
(i) $x^2 = 4$ **(ii)** $x^2 = \frac{1}{4}$ **(iii)** $x^2 = 3$ **(iv)** $x^2 = -4$

Definition: $i =$

Q4 Simplify **(i)** $3i + 4i$ **(ii)** $3 + 4i$ **(iii)** $3 \times 4i$ **(iv)** $3i \times 4i$.

Powers of i

Q1 Simplify i^n. Hint: Find i^2, i^3, i^4, i^5.

Q2 Simplify i^{2002}.

Complex Numbers

Complex numbers are of the form $z = a + ib$ where a and b are real.
The real part of $z =$ _______ $=$ _______
The imaginary part of $z =$ _______ $=$ _______
If $a = 0$ then $z =$ _______. We say z is ______________________
If $b = 0$ then $z =$ _______. We say z is ______________________
Complex numbers are not ordered, i.e., $2 + 3i$ _______ $3 + 2i$ but they obey the usual number laws.

Q1 If $z = 3 + 4i$ then $\Re(z) =$ _______ and $\Im(z) =$ _______.

Q2 '94 HSC If $z = a + ib$ find $\Im(4i - z)$.

Q3 Simplify **(i)** $3 + 4i + 4 - 3i$ **(ii)** $(3 + 4i)(4 - 3i)$ **(iii)** $(3 + 4i)^2$ **(iv)** $(3 + 4i)(3 - 4i)$

Conjugate of z

If $z = a + ib$ then $\overline{z} =$ _______

Q1 Find $\overline{z}$ if $z =$
(i) $3 + 4i$ **(ii)** 3 **(iii)** $4i$.

Q2 '94 HSC Find $\overline{3iz}$ in the form $x + iy$ if $z = a + ib$.

Q3 Write $\frac{18+4i}{3-i}$ in the form $x + iy$. (This is called ____________________)

Q4 Simplify $(3 + 4i)^{-2}$

Q6 '89 HSC Given that a, b, x, y are real, express the following in the form $x + iy$.
(i) $(a + ib)\overline{(5 + i)}$ **(ii)** $\frac{a+ib}{3+4i}$

Quadratic Equations

Solve for the set of complex numbers,
(i) $z^2 = -12$ **(ii)** $z^2 - 2z + 5 = 0$ **(iii)** $z^2 - 2z + 3$ **(iv)** $2z^2 - 6iz - 3 = 0$

If the coefficients of a quadratic equation are ____________________, then the roots form a ____________________ pair.

QQ Solve **(a)** $z^2 + iz = 2$ **(b)** $z\overline{z} = \frac{1}{4} + i$

The Square Roots of Complex Numbers

1. Expand $(a + ib)^2$

2. Solve $z^2 = 5 - 12i$, i.e., find the square roots of $5 - 12i$. Let $z = a + ib$ where a and b are real.

3. Solve $z^2 = i$

4. Solve $z^2 - (1 - 4i)z - (5 - i) = 0$.

5. Solve $z^2 - (1 - i)z + 7i - 4 = 0$

6. '91 HSC (i) Find al pairs of integers x and y such that $(x + iy)^2 = -3 - 4i$.
(ii) Using **(i)** or otherwise, solve $z^2 - 3z + (3 + i) = 0$.

7. Prove that a non-zero complex number always has **two** square roots.
Let $\sqrt{a + ib} = x + iy$ where a, b, x and y are real.

Then $a+ib =$ ______________________
Equating the real and imaginary parts,
$a =$ ______________________ and $b =$ ______________________.
Now substitute into the identity $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$
$\therefore (x^2+y^2)^2 =$ ______________________
Since x and y are real, $x^2+y^2 \geq 0$ taking the square root then
$x^2+y^2 =$ ______________________...(1)
$x^2-y^2 =$ ______________________...(2)
(1)+(2): $2x^2 =$ ______________________
$\therefore\ x^2 =$ ______________________
Since $a^2+b^2 \geq a^2$ then $\sqrt{a^2+b^2} \geq a$ so RHS≥ 0
$\therefore x =$ ______________________
For each of these values of x there corresponds a unique value of $y = \frac{b}{2x}$.
Thus $a+ib$ has 2 square roots.

ANSWERS

COMPLEX NUMBERS

Q1 **(i)** 0.5 **(ii)** 1 **(iii)** 1/2 **(iv)** $\sqrt{2}$
Q2 **(i)** $\pm 2 \subset \mathbb{R}$ **(ii)** $\pm\frac{1}{2} \subset \mathbb{Q}$ **(iii)** $\pm\sqrt{3} \subset \mathbb{R}\backslash\mathbb{Q}$ **(iv)** $\pm 2i \subset i\mathbb{R} \subset \mathbb{C}$.
$i = \sqrt{-1}$
Q4 **(i)** $7i$ **(ii)** can't be simplified **(iii)** $12i$ **(iv)** -12.

Powers of i

Q1 $i^n = i^{n \mod 4}$; $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$
Q2 -1

The real part of $z = \Re(z) = a$
The imaginary part of $z = \Im(z) = b$
If $a = 0$ then $z = ib$. We say z is purely imaginary.
If $b = 0$ then $z = a$. We say that z is purely real.
Complex numbers are not ordered, i.e., $2+3i \not< 3+2i$ but they obey the usual number laws.

Q1 If $z = 3+4i$ then $\Re(z) = 3$ and $\Im(z) = 4$.
Q2 $4-b$
Q3 **(i)** $7+i$ **(ii)** $24+7i$ **(iii)** $-7+24i$ **(iv)** 25.

Conjugate of z

If $z = a+ib$ then $\overline{z} = a-ib$.

Q1 **(i)** $3-4i$ **(ii)** 3 **(iii)** $-4i$

Q2 $-3b-3ia$
Q3 $5+3i$ Realising the denominator.
Q4 $\frac{-7-24i}{625}$
Q6 **(i)** $(5a+b)+i(5b-a)$ **(ii)** $\frac{3a+4b}{25}+i(\frac{3b-4a}{25})$

Quadratic Equations

(i) $\pm 2\sqrt{3}i$ **(ii)** $1\pm 2i$ **(iii)** $1\pm\sqrt{2}i$ **(iv)** $\frac{3i\pm\sqrt{3}i}{2}$

If the coefficients of a quadratic equation are real, then the roots form a conjugate pair.

QQ **(a)** $\frac{-1\pm\sqrt{3}}{2}$ **(b)** $\frac{i}{2}$, $-2+\frac{i}{2}$.

The Square Roots of Complex Numbers

1. a^2-b^2+2iab **2.** $\pm(3-2i)$ **3.** $\pm\frac{1}{\sqrt{2}}(1+i)$ **4.** $2-3i$, $-1-i$.
5. $3-2i$, $-2+i$. **6.** **(i)** $x=1, y=-2$ or $x=-1, y=2$. **(ii)** $2-i$, $1+i$
7. Let $\sqrt{a+ib}=x+iy$ where a, b, x and y are real.
Then $a+ib=x^2-y^2+2ixy$.
Equating the real and imaginary parts,
$a=x^2-y^2$ and $b=2xy$.
Now substitute into the identity $(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2$
$\therefore (x^2+y^2)^2=a^2+b^2$.
Since x and y are real, $x^2+y^2\geq 0$.
Taking the square root then
$x^2+y^2=\sqrt{a^2+b^2}$...(1)
$x^2-y^2=a$.....(2)
(1)+(2): $2x^2=a+\sqrt{a^2+b^2}$
$x^2=\frac{1}{2}(a+\sqrt{a^2+b^2})$
Since $a^2+b^2\geq a^2$ then $\sqrt{a^2+b^2}\geq a$ so RHS≥ 0
$\therefore x=\pm\frac{1}{2}\sqrt{a+\sqrt{a^2+b^2}}$.
For each of these values of x there corresponds a unique value of $y=\frac{b}{2x}$.
Thus $a+ib$ has 2 square roots. $\square$