New South Wales

Higher School Certificate

4 Unit

Mathematics

Specimen Papers

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

PAPER 1

Question 1

(i) *Find the derivatives, in simplest form, of:*

(a) $\sqrt{\frac{1+x^2}{1-x^2}}$; (b) $(\tan^{-1} x)^x$; (c) $\tan^3(\sqrt{6x+7})$.

(ii) *Find* dy/dx *if:*

(a) $x = e^{-t}\cos t, y = e^{-t}\sin t,$ *and* $t = 0$;

(b) $2x^2 - 3xy + y^2 = 6$ *and* $x = 1$.

(iii) *If* $y = \sin(\log x)$, *show that* $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ky = 0$,*where* k *is a constant, and find* k.

Question 2

(i) *Find the indefinite integrals*:

(a) $\int \frac{2x\ dx}{(x+1)(x+3)}$; (b) $\int x^2 e^{x^3+1}\ dx$; (c) $\int \frac{x+1}{\sqrt{4-9x^2}}\ dx$.

(ii) *Evaluate* (a) $\int_0^{\pi/2} \sin^2 x \cos^2 x\ dx$; (b) $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}$.

(iii) *If* $u_n = \int_0^{\pi/2} x^n \sin x\ dx$, *prove that, for* $n \geq 2, u_n + n(n-1)u_{n-2} = n(\pi/2)^{n-1}$. *Calculate* u_1 *directly, and deduce that* $u_3 = \frac{3}{4}\pi^2 - 6$.

Question 3

(i) *If* z *represents a variable complex number, show the region of the Argand diagram in which* $2 < |z| < 3$ *and* $\frac{\pi}{3} \leq \arg z \leq \frac{2\pi}{3}$.

(ii) *State and prove De Moivre's Theorem for an integral index (both positive and negative).*

(a) *Simplify* $\frac{(\cos\theta+i\sin\theta)^9.(\cos 3\theta+i\sin 3\theta)^{-5}}{(\cos 2\theta-i\sin 2\theta)^4}$.

(b) *Express* $-1+i\sqrt{3}$ *in mod-arg form and hence evaluate* $(-1+i\sqrt{3})^{-6}$.

(*iii*) *On an Argand Diagram, the points P, Q represent the complex numbers z, $1/z$ respectively. If P moves on the staight line $x = 1$, show that Q lies on a certain circle, and find its centre and radius.*

Question 4

(*i*) *Show that the circle on diameter the join of (x_1, y_1) and (x_2, y_2) has equation $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.*

(*ii*) *Prove that the tangent at the point $P(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$ has equation $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.*

(*a*) *This ellipse meets the y-axis at B, B'. The tangents at B, B' to the ellipse meet the tangent at P at the points Q, Q' respectively. Prove that $BQ.B'Q' = a^2$.*

(*b*) *The circle on QQ' as diameter meets the x-axis at the points R, R'. Prove that $OR.OR' = a^2 - b^2$, where O is the origin.*

Question 5

(*i*) *Sketch the curve $9y^2 = x(3 - x)^2$ and show that it forms a loop. Find:*

(*a*) *the maximum width of loop measured parallel to the y-axis;*

(*b*) *the area enclosed by the loop.*

(*ii*) (*a*) *The base of a certain solid S_1 is the region bounded by the parabola $y^2 = 4ax$ and the latus rectum. Each section of the solid by planes parallel to the y-axis is an equilateral triangle. Find the volume of S_1.*

(*b*) *The area bounded by the parabola $y^2 = 4ax$ and the latus rectum is rotated through four right angles about the latus rectum. Calculate the volume of the solid S_2 so generated.*

Question 6

Prove, from first principles, that the acceleration of a point moving with constant angular velocity ω in a circle of radius r is $r\omega^2$ towards the centre.

(*i*) *A particle is fastened to one end of a light inextensible string of length l, the other end of which is fastened to a fixed point O. The particle rotates with uniform angular velocity ω about the vertical through O. Show that if α is the inclination of the string to the* <u>*downward*</u> *vertical, then $\alpha = \cos^{-1}(g/l\omega^2)$ and* *deduce that steady circular motion is impossible if* $l\omega^2 < g$. *What is the effect on the inclination of the string to the vertical by an increase in* ω?

(*ii*) *If the point* O, *instead of being fixed, is descending with uniform acceleration* f, *the particle still rotating with uniform angular velocity* ω, *find* f *in order that the string may make an angle* α *with the* <u>*upward*</u> *vertical.*

Question 7

(*i*) *Show that* $1 + i$ *is a root of the polynomial* $p(x) = x^3 + x^2 - 4x + 6$ *and hence resolve* $p(x)$ *into irreducible factors over the field of:*

(*a*) *complex numbers*; (*b*) *real numbers.*

(*ii*) *If* α, β, γ *are the roots of the cubic equation* $x^3+qx+r = 0$, <u>*find*</u> *the value of* $\Sigma\alpha, \Sigma\alpha\beta, \Sigma\alpha\beta\gamma$ *in terms of* q, r. *Hence prove that* $(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2 = -6q$.

(*iii*) *Verify, without use of tables, calculator or slide rule, that* $\theta = 18°$ *satisfies the equation* $\sin 2\theta = \cos 3\theta$. *Deduce that* $\sin 18°$ *is a root of the equation* $4x^2 + 2x - 1 = 0$, *and hence find the value of* $\sin 18°$ *in simplest surd form. What does the other root represent*?

Question 8

(*i*) *Assuming* a, b, c, d *to be real, write down the condition that the roots of the equation* $(a^2 + b^2)x^2 + 2(ac + bd)x + (c^2 + d^2) = 0$ *should be real. Then prove that if this condition holds, the roots must be equal; and show that they are equal to* $-c/a$.

(*ii*) *State the binomial theorem for* $(1 + x)^n$ *where* n *is a positive integer.*

(*a*) *If* k *is a positive integer, show that* $(1 + \frac{1}{n})^k \to 1$ *as* $n \to \infty$.

(*b*) *Prove that* $(1 + \frac{1}{n})^n$ *approaches the sum of an infinite series as* $n \to \infty$.

(*c*) *Show that* $\frac{1}{n!} < \frac{1}{2^{n-1}}$ *for all positive integral* $n \geq 3$ *and use this to deduce that* $\lim\limits_{n\to\infty} (1 + \frac{1}{n})^n = l$ *where* $2 < l < 3$.

PAPER 2

Question 1

(i) *Evaluate*:

(a) $\int_0^\pi x\cos x\ dx$; (b) $\int_3^6 \frac{dx}{x^2-6x+18}$; (c) $\int_0^{\ln 2} \frac{e^{2x}dx}{1+e^x}$.

(ii) *Find* $\int_1^N \frac{dx}{x(1+x)}$ *and hence find the area bounded by the curve* $y = x^{-1}(1+x)^{-1}$ *and the* x*-axis, to the right of the* $x = 1$.

(iii) *Differentiate* $\cos^{n-1}\theta\sin\theta$ *and express the result in terms of* $\cos\theta$ *only. Deduce that* $\int_0^{\pi/2}\cos^n\theta d\theta = \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2}\theta d\theta$, *and hence evaluate* $\int_0^{\pi/2}\cos^6\theta d\theta$.

Question 2

(i) *Sketch the graph of* $y = x^2e^{-x}$. *Find* dy/dx *and* d^2y/dx^2, *and find the values of* x *which correspond to the turning points and points of inflection on the curve. Mark these points on your graph.*

(ii) *Show that the curve* $y = 3x^4 - 4x^3 + 6x^2 - 12x + k$ *has only one stationary point and find its coordinates. Hence prove that the equation* $3x^4 - 4x^3 + 6x^2 - 12x + k = 0$ *has no real root if* $k > 7$ *and state the number of real roots if* $k \le 7$.

Question 3

(i) *The rectangular hyperbola* H *has cartesian* (x, y) *equation* $x^2 - y^2 = 8$. *Write down its eccentricity, the coordinates of its foci* S *and* S', *the equations of each directrix and of each asymptote. Sketch the curve, indicating on your diagram, the foci, directrices and asymptotes.*

(ii) *If this curve is rotated through* $45°$ *in an anti-clockwise direction, the equation of the curve takes the form* $xy = 4$. *Prove that the equation of the normal to the rectangular hyperbola* $xy = 4$ *at the point* $(2p, 2/p)$ *is* $py - p^3x = 2(1-p^4)$. *If this normal meets the hyperbola again at* $(2q, 2/q)$ *prove that* $q = -1/p^3$. *Hence, or otherwise, show that there exists only one chord of the hyperbola which is a normal at both ends of the chord, and find its equation.*

Question 4

(i) *If* z *is a complex number so that* $|z| = 3$ *and* $\arg z = \pi/6$, *mark clearly (using geometry where possible) on the same Argand diagram the points representing the complex numbers*:

z; $\quad iz$; $\quad \bar{z}$; $\quad \frac{1}{z}$; $\quad z\bar{z}$; $\quad z^2$; $\quad z^2+z$; $\quad z^2-z$.

(ii) Find $\sqrt{6i-8}$ *and hence solve the equation* $2z^2-(3+i)z+2=0$, *expressing values of* z *in the form* $x+iy$.

(iii) Sketch on an Argand diagram, the locus of the point P *which satisfies the equation* $|z-3|=3$. *From the figure, using Euclidean properties of the locus, show that* P *also satisfies the condition* $\arg(z-3)=\arg z^2$. *Determine the complete locus of points satisfying this condition.*

Question 5

(i) Use the method of integration by parts to evaluate $\int_0^1 \sqrt{1+t^2}\,dt$.

(ii) PQ, QR *are two straight roads meeting at right angles;* $PQ = a$ *km,* $QR = 3a$ *km and* S *is a point on* QR. *A girl malks across country from* P *to* S *at* $3km/h$ *and then along the road from* S *to* R *at* $5km/h$. *Find the distance of* S *from* Q *in order that the time for the whole journey should be a minimum.*

(iii) The area bounded by the curve $y = x(2-x)$ *and the* x*-axis is rotated about the* y*-axis. By considering cylindrical shells with generators parallel to the* y*-axis, show that the volume* V *unit*^{3} *of the solid so generated is given by* $V = \int_0^2 2\pi xy\,dx$. *Hence determine the volume of this solid.*

Question 6

A particle moves in a straight line. Prove that its acceleration at any instant is $v\frac{dv}{dx}$ *where* x *denotes its position coordinate and* v *its velocity. A particle of mass* m *is projected vertically upwards with initial speed* U. *If the air resistance at any instant is proportional to the velocity at that instant (say resistance* $=-mkv$, *where* k *is a constant), prove that the particle will reach its highest point in time* T *given by* $kT = \log(1+kU/g)$, *where* g *is the acceleration due to gravity (assuming constant). Also show that the particle will ascend to height* H, *where* $kH = U - gT$.

Question 7

(i) Show that if the polynomial $P(x)$ *has a root* α *of multiplicity* m, *then* $P'(x)$ *has the root* α *with multiplicity* $(m-1)$. *Given that the polynomial* $P(x) = x^4+x^3-3x^2-5x-2$ *has a 3-fold root, find all the roots of* $P(x)$.

(ii) If $p^2 = 1+p$, *prove that* $p^3 = 1+2p$. *Express in the form* $a+bp$, *where* a, b *are integers:* $\quad$ *(a)* p^5; $\quad$ *(b)* $1/p^5$.

(iii) Assuming the results for $\sin(A \pm B)$, *prove that* $\sin C + \sin D = 2\sin\frac{1}{2}(C+D)\cos\frac{1}{2}(C-D)$. *Find the solutions of* $\sin 3x + \sin x = \cos x$ *for* $0 \le x \le 2\pi$.

Question 8

(i) Given a, b, c, d *are real positive numbers, prove that:*

(a) $a^4 + b^4 \ge 2a^2b^2$;

(b) $a^4 + b^4 + c^4 + d^4 \ge 4abcd$;

(c) If $a^4+b^4+c^4+d^4 \le 4$ *then* $a^{-4}+b^{-1}+c^{-1}+d^{-1} \ge 4$. *If* $a^4+b^4+c^4+d^4 > 4$, *what statement (if any) can be made about* $a^{-4}+b^{-4}+c^{-4}+d^{-4}$?

(ii) *If* ω *is a complex root of the equation* $x^3 = 1$, *show that the other complex root is* ω^2.

(a) *Show that* $1 + \omega + \omega^2 = 0$.

(b) *Find in its simplest form, the cubic equation whose roots are* $a+b, a\omega + b\omega^2, a\omega^2 + b\omega$ *where* a, b *are real numbers.*

PAPER 3

Question 1

(i) Decompose $\frac{8}{(x+2)(x^2+4)}$ *into partial fractions, and use this result to evaluate* $\int_0^2 \frac{8\ dx}{(x+2)(x^2+4)}$.

(ii) Evaluate , giving numerical answers to 3 *significant figures,*

(a) $\int_0^{1/3} \frac{dx}{\sqrt{1-9x^2}}$; (b) $\int_{-1}^{1} 2^x dx$; (c) $\int_1^e x \log_e x\ dx$.

(iii) Show that $\frac{d}{dx}\log(x+\sqrt{x^2+4}\ ) = \frac{1}{\sqrt{x^2+4}}$. *Hence, or otherwise, prove that* $\int_{-\pi/4}^{\pi/4} \frac{\sec^2 x\ dx}{\sqrt{\tan^2 x+4}} = 2\log\{(\sqrt{5}+1)/2\}$.

Question 2

(i) Show that the curve $y = \frac{x^3+4}{x^2}$ *has one stationary point, but no point of inflexion and is always concave up. For large* x, *show that the equation of the curve approximates to that of a straight line. What does this suggest is an asymptote? Sketch the curve. Use the graph to show that the equation* $x^3 - kx^2 + 4 = 0$ *may have* 1, 2 *or* 3 *real roots. State the values of* k *corresponding to each case.*

(ii) By means of substituting $y = a - x$ *or otherwise, prove that* $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. *Hence evaluate* $\int_0^\pi \frac{x \sin x\ dx}{1+\cos^2 x}$.

Question 3

The ellipse E *has cartesian equation* $\frac{x^2}{25} + \frac{y^2}{16} = 1$. *Write down the eccentricity, the coordinates of the foci* S *and* S', *and the equations of the directrices. Sketch the curve, and indicate the foci and directrices on the diagram.*

(i) Show that any point P *on* E *can be represented by the coordinates* $(5\cos\theta, 4\sin\theta)$ *and prove that* $PS + PS'$ *is independent of the position of* P *on the curve.*

(ii) Find the equation of the normal n *at the point* P *on the ellipse. If this normal meets the major axis of the ellipse in* M *and the minor axis in* N, *prove that* $PM/PN = 16/25$. *Also show that the line* n *bisects the angle* $S'PS$.

Question 4

(i) If $Z = (4+3i)/(1-2i)$ *determine*

(a) $|Z|$ (b) $\Re(Z)$ (b) $\Im(Z)$ (d) $\overline{Z}$

(e) $Z+\overline{Z}$ (f) $Z\cdot\overline{Z}$ (g) $\arg Z$ *(to the nearest minute)*

(ii) If the complex numbers z, ω *are related by* $\omega = z^2$. *Find the locus of* ω *if* z *describes*

(a) *the locus* $|z| = 3$ (b) *the locus* $\arg z = \pi/3$

(c) *the* y*-axis* (d) *the line* $x = 3$

Sketch the loci of z *and* ω *on an Argand diagram, in each case.*

Question 5

Show that $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

If m, n *are unequal integers, prove that*

(a) $\int_0^\pi \sin mx \sin nx\ dx$ (b) $\int_0^\pi \sin^2 mx\ dx$

From graphical considerations, or otherwise, prove that if $0 \leq x \leq \pi$, *then* $\sin x + \frac{1}{3}\sin 3x \geq 0$.

Find the area A *between the curve* $y = a(\sin x + \frac{1}{3}\sin 3x)$ *and the* x*-axis between the limits 0 and* π, $(a > 0)$.

Find also the volume V *of the solid obtained by rotating this area about the* x*-axis, and prove that* $V = \frac{1}{4}\pi^2 aA$.

Question 6

A particle is projected from O *with initial speed* V *at an angle* α *with the horizontal, where* $\alpha = \tan^{-1}(3/4)$. *The particle passes through a point* P *whose coordinates relative to horizontal and vertical axes through* O *are* $(8a, 4a)$. *Show that* $V = 5\sqrt{ag}$ *(where* g *is the acceleration due to gravity, assumed constant), and find the time of flight from* O *to* P. *It is possible for this particle to be projected from* O*, at a different angle* β *to the horizontal, with the same speed* V *so as to still pass through* P. *Find the cartesian equation of its path, and hence determine* $\tan\beta$.

Question 7

(i) Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, *given that the roots are in arithmetic progression.*

(ii) *If* α, β, γ *are the roots of the equation* $x^3 - 7x^2 + 18x - 7 = 0$ *find the value of* $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)$.

(iii) *By considering the value of* $\int_0^1 \sqrt{x(1-x)}$ *as the limit of a sum, show that* $\lim\limits_{n \to \infty} \sum_{r=1}^{n-1} \frac{1}{n^2} \sqrt{r(n-r)} = \frac{\pi}{8}$.

Question 8

(i) *Using the letters A,A,B,C,D,E,E find how many 'code-words' can be formed, if the code-word contains*

(a) *all* 7 *letters* (b) *exactly* 4 *of the* 7 *letters.*

$($*Solutions to* (b) *should contain sufficient explanation to make the method clear.*$)$

(ii) *Find* $($*as trigonometrical quantities*$)$ *the five fifth roots of unity, and represent them on an Argand diagram. If* α *is one of the complex roots, show that the other complex roots can be expressed as* $\alpha^2, \alpha^{-1}, \alpha^{-2}$ *and find the value of* $\alpha^2 + \alpha^{-2}$. *Factorise* $x^5 - 1$ *as the product of a linear factor and two real quadratic factors.*

PAPER 4

Question 1

(i) *Given* $\frac{d}{dx}(\sin x) = \cos x$, *find* $\frac{d}{dx}(\sin^{-1} x)$. *If* $y = \sin^{-1}(\frac{1}{a}x) + \sin^{-1}(\frac{1}{a}\sqrt{a^2 - x^2})$, *find* $\frac{dy}{dx}$ *and hence determine the exact value of* $y, (a, x > 0)$.

(ii) *Find indefinite integrals for*

(a) $\int \frac{dx}{x(1+\log x)}$ (b) $\int \frac{dx}{2+\sin x+\cos x}$ (c) $\int r^m \log r \, dr, \ m \neq -1$

Question 2

(i) *Evaluate*

(a) $\int_0^1 \frac{(1+x)dx}{\sqrt{1-x^2}}$ (b) $\int_0^{\pi/4} x \sec^2 x \, dx$ (c) $\int_0^1 \sqrt{4 - x^2} \, dx$

(ii) *Express* $\frac{2-x^2}{(x^2+1)(x^2+4)}$ *in partial fractions, and hence show that* $\int \frac{(2-x^2)dx}{(x^2+1)(x^2+4)}$ $= \tan^{-1}(\frac{1}{3})$

Question 3

(i) *Determine the greatest and least values of* $\arg z$, *when* $|z - 4i| = 2$.

(ii) *Describe in geometric terms the locus of the point in the Argand diagram representing the complex number* z, *such that*

(a) $|z - i| = |z + 2 - 3i|$ (b) $\frac{1}{2}(z + \overline{z}) = |z| - 2$

(iii) *If* $z = x + iy$ *and* $w = \frac{z+1}{z}$, *express* w *in the form* $u + iv$, *where* u, v *are real. If* Z, W *are the points in the Argand diagram representing the complex numbers* z, w *and* Z *moves on the unit circle centre the origin, find the locus of* W.

Question 4

The hyperbola H *has cartesian* (x, y) *equation* $\frac{x^2}{4} - \frac{y^2}{5} = 1$. *Write down its eccentricity, the coordinates of its foci* S *and* S', *the equations of the directrices, and the equations of the asymptotes. Sketch the curve and indicate on your diagram, the foci, the directrices and the asymptotes.* P *is an arbitrary point* $(\sec\theta, \sqrt{5}\tan\theta$. *Show that* P *lies on* H *and prove that the tangent to* H *at* P *has equation* $\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1$. *This tangent cuts the asymptotes in* L, M. *Prove that* $LP = PM$ *and the area of* $\triangle OLM$ *is independent of the position of* P *on* H. *(*O *is the origin.)*

Question 5

(i) *Sketch the curve* $y = e^{-\frac{1}{2}x^2}$, *showing clearly any stationary points and points of infexion. The area between this curve and the positive parts of the coordinate axes is rotated about the* y*-axis. Find the volume of the solid of revolution so formed.*

(ii) *A variable line is drawn through the fixed point* (a, b) *inj the first quadrant, to cut the positive parts of the axes in* A, B. *If* θ *is the acute angle between this line and the* x*-axis, show that* $AB = a\sec\theta + b\operatorname{cosec}\theta$. *Hence find the minimum value of the length* AB.

Question 6

A point is moving in a circular path about the centre O. *Define the angular velocity of the point with respect to* O *at time* t, *and derive expressions for the tangential and normal accelerations of the point at time* t.

(i) *A light inextensible string* AP *is fixed at the end* A, *and is attached at the other end* P *to a particle which is moving uniformly in a horizontal circle whose centre* O *is vertically below and distant* h *from* A. *Prove that the period of this motion is* $2\pi\sqrt{h/g}$, *where* g *is the acceleration at the Earth's surface. What is the effect (if any) on the motion of the particle as a result of doubling its* (a) *mass* (b) *speed?*

(ii) *A simple pendulum consists of a small bob* B *which is suspended from a fixed point* C *by a light inextensible string of length* l. *The pendulum is displaced through a small angle and released from rest. Show that the motion approximates to simple harmonic motion of period* $2\pi\sqrt{l/g}$. *(Hint: Resolve forces in the direction of the motion only.) What is the effect (if any) on the motion of the bob if*

(a) *the length of the string is doubled;*

(b) *the pendulum is taken to the top of a mountain?*

Question 7

(i) *Find the equation whose roots are the* (a) *negatives* (b) *squares, of those of* $f(x) \equiv x^4 + qx^2 + rx + s = 0$.

If $\alpha, \beta, \gamma, \delta$ *are the roots of* $f(x) = 0$, *show that* $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2) = (1 - q + s)^2 + r^2$

(ii) The polymomial $P(x)$ is given by $P(x) = 2x^3 - 9x^2 + 12x - k$ where k is real.

By considering the turning points, find the range of values of k for which $P(x) = 0$ has

(a) just one real root *(b) three real roots.*

Solve completely the equations obtained with the values of k at the extremeties of the range in (b).

Question 8

(i) If ${}^nC_{r+1} = {}^nC_r = k \cdot {}^nC_{r-1}$ prove that n is odd, and express n and r in terms of k.

(ii) A sequence $u_1, u_2, u_3, \ldots$ is defined by the relations $u_1 = 1, u_2 = 5, u_n = 5u_{n-1} - 6u_{n-2}$ for $n = 3, 4, 5, \ldots$ Use the method of induction to prove that $u_n = 3^n - 2^n$.

(iii) Show that $1 + \cos 2\theta + i \sin 2\theta = 2\cos\theta(\cos\theta + i\sin\theta)$ and hence prove that $(1+\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^n = -2^n \cos^n \frac{\pi}{n}$, where n is integral. Simplify $(1+\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^n - (1+\cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n})^n$

PAPER 5

Question 1

(i) If $y = \frac{1}{2}(e^x - e^{-x})$, *show that* $x = \log(y + \sqrt{y^2+1}\ )$. *Also find* $(\frac{dy}{dx})^2 - y^2$ *and hence deduce that* $\int \frac{dy}{\sqrt{y^2+1}} = \log(y + \sqrt{y^2+1}\ ) + C$.

(ii) Evaluate

$(a)\ \int_0^1 \frac{dx}{x^2-x+1}$ $\qquad (b)\ \int_0^{\log 2} te^{-t}\ dt$ $\qquad (c)\ \int_{-1}^{1} \frac{9+x^2}{9-x^2}\ dx$

Question 2

(i) Find indefinite integrals for

$(a)\ \int \frac{9-x}{1+x+x^2+x^3}\ dx$ $\qquad (b)\ \int \frac{e^{-2x}\ dx}{e^{-x}+3}$ $\qquad (c)\ \int \sin^3\theta\ d\theta$

(ii) Sketch the curve $y = x^2(6-x^2)$ *and determine the greatest value of* y *for* $x > 0$. *The area in the first quadrant bounded by the curve and the* x*-axis is rotated about the* y*-axis.*

(a) If a plane parallel to the x*-axis and distant* y *from it cuts the solid formed, show that the area of section* A *is given by* $A = 2\pi\sqrt{9-y}$. *Hence determine the volume of the solid.*

(b) Check your answer to (a) by using the method of cylindrical shells, with generations parallel to the y*-axis.*

Question 3

Show that the equation of the chord through $P_1(cp_1, c/p_1)$ *and* $P_2(cp_2, c/p_2)$ *on the curve* $xy = c^2$ *has equation* $x + p_1p_2y = c(p_1+p_2)$, *and hence deduce the equation of the tangent at* P_1. *The tangents to the hyperbola at* P_1, P_2 *meet one asymptote at* A_1, A_2 *and the other asymptote at* B_1, B_2 *respectively. The line* P_1, P_2 *cuts the first asymptote in* K *and the second at* M.

(a) Show that $A_1K = KA_2$ *and* $B_1M = MB_2$

(b) Prove that MK, P_1P_2 *have the same midpoint* R *and hence that* $MP_1 = P_2K$.

(c) If the tangents at P_1, P_2 *meet at* Q, *find the coordinates of* Q *and show that the line* OQ *passes through* R.

Question 4

(*i*) *Sketch the region of the complex (Argand) plane whose points z satisfy*

(*a*) $3 < |z - 2 + i| < 4$ (*b*) $-\pi/3 \leq \arg z \leq \pi/3$

(*ii*) *If z is a complex number such that $z + \frac{1}{z}$ is real, prove that either $\Im(z) = 0$ or* $\mod(z) = 1$.

(*iii*) *Z_1, Z_2 are points in the Argand diagram representing the complex numbers z_1, z_2 so that $z_2 = \frac{1}{z_1 - 3} + 4$. Determine the locus of Z_2 if Z_1 describes the circle centre 3 and radius 2 units.*

(*iv*) *Show, by means of a diagram, that if z, w are complex numbers then $|z + w| \leq |z| + |w|$. If A, B, C, D represent the complex numbers $z_1, z_2 z_3, z_4$ use the identity $(z_1 - z_4)(z_2 - z_3) + (z_2 - z_4)(z_3 - z_1) + (z_3 - z_4)(z_1 - z_2) = 0$ to show that $AD \cdot BC \leq BD \cdot CA + CD \cdot AB$.*

Question 5

(*i*) *Sketch the curve $y = 1/u$ for $u > 0$. From your diagram, show that $\int_1^{\sqrt{x}} \frac{du}{u} < \sqrt{x} - 1$ for $x > 1$. Use this result to show that $0 < \log x < 2(\sqrt{x} - 1)$ and hence that $\frac{\log x}{x} \to 0$ as $x \to \infty$. Deduce that $x \log x \to 0$ as $x \to 0$.*

(*ii*) *Sketch (on separate diagrams), the curves*

(*a*) $y = (\frac{\log x}{x})^2$ (*b*) $y = (x \log x)^2$

paying attention to their slope and turning points.

Question 6

A particle of mass m falls vertically from rest, from a point O, in a medium whose resistance is mkv, where k is a positive constant and v its velocity. Obtain an expression for its velocity after t seconds. An equal particle is projected vertically upwards from O with initial velocity u, in the same medium. (This particle is released simultaneously with the first particle.) Show that the velocity of the first particle when the second particle is momentarily at rest, is given by $Vu/(V + u)$ where V is the terminal velocity of the first particle.

Question 7

(*i*) *Solve for x*

(*a*) $\tan^{-1} 3x - \tan^{-1} 2x = \tan^{-1} \frac{1}{5}$ (*b*) $\cos x = \sin 5x$, *for* $0 < x < \pi$

(*ii*) *State, without proof, the 'fundamental theorem of algebra'. What can we deduce from this about the factors of polynomials with real coefficients? Obtain the respective numbers of linear and quadratic factors in the case* $f(x) = x^5 + 2x^3 - x^2 - 2$*. For this particular polynomial, find also its complete factorisation into factors having integral coefficients.*

Question 8

(*i*) *Sketch the following curves, for* $-2\pi \le x \le 2\pi$

(*a*) $y = |\sin x|$ (*b*) $y = \sin|x|$ (*c*) $y = \sin^2 \frac{x}{2}$

(*ii*) *Show that* $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$*.*
Hence, use the principle of induction, to establish the result
$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)$
Deduce De Moivre's theorem for a positive integer n*.*

(*a*) *Find the four fourth roots of* -1 *and show them on an Argand diagram.*

(*b*) *If* $z = \cos\theta + i\sin\theta$*, by expanding* $(z - \frac{1}{z})^4$ *show that* $8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$*. Hence find the value of* $\int_0^a \frac{x^{3/2}\,dx}{\sqrt{a-x}}$ $\{$*Put* $x = a\sin^2\theta\}$*.*

PAPER 6

Question 1

(i) *Assuming y is a function of x such that $xy^2 = ce^{y/x}$, where c is a positive constant, show that* $x(2x-y)\frac{dy}{dx} + y(x+y) = 0$.

(ii) *The tangent at the point $P(x_0, y_0)$ on the curve $y = \frac{1}{2}(e^x + e^{-x})$ meets the x-axis in R. Find the coordinates of R in terms of x_0. If M is the foot of the perpendicular from P to the x-axis, determine the limiting value of the length RM as $x_0 \to \infty$.*

(iii) *Find indefinite integrals for*

(a) $\int \frac{du}{\sqrt{u}+1}$ (b) $\int \sqrt{\frac{1-x}{x}}\ dx$ (c) $\int \cos^{-1} x\ dx$

Question 2

(i) *Show that* $\int_{-a}^{a} f(x)\ dx = \int_0^a \{f(x) + f(-x)\}\ dx$. *Hence prove that* $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\sin x} = 2\int_0^{\pi/4} \sec^2 x\ dx$, *and evaluate the integral. Check your result by using the substitution* $t = \tan x/2$.

(ii) *Evaluate, correct to 3 significant figures,*

(a) $\int_0^{\pi/4} \tan^4 x\ dx$ (b) $\int_1^2 x^2 \log x\ dx$ (c) $\int_0^{1/2} \frac{x^2\ dx}{\sqrt{1-x^2}}$

Question 3

P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the equation of the tangent at P, and an expression for the perpendicular distance of the point (x_1, y_1) from the line. Write down the equation of that diameter of the ellipse parallel to this tangent and hence determine the coordinates of the ends of this diameter. If the distance of P from the focus $S(ae, 0)$ is k, the perpendicular distance of S from the tangent at P is q, and the length of the diameter parallel to the tangent at P is $2h$, prove that $hq = kb$.

Question 4

(i) *The point Z representing the complex number z is rotated anti-clockwise about the origin O through $\pi/2$ to the point Z' representing the complex number z'. Prove that $z' = iz$.*

(ii) *Define $|z|$ and $\arg z$ and show that every complex number z can be written in the form $r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg z$. Express $z = \frac{1+2i}{1-i} + \frac{1}{i}$ in this form.*

(iii) *Describe in geometric terms, the loci in the Argand plane represented by the equations*

(a) $|z-1|+|z+1|=4$ (b) $|z-1|=\Re(z)$

Give the equations of these loci in cartesian form.

(iv) Z_1, Z_2, Z_3 *represent the complex numbers* z_1, z_2, z_3 *respectively, where* $z_1 z_3 = z_2^2$. *Show geometrically that* OZ_2 *bisects the angle* Z_1OZ_3.

Question 5

Sketch on the same diagram the curves $y=\frac{1}{x^2+1}, y=\frac{x^2}{x^2+1}$ *showing any turning points, and the shapes of the curves as* $x \to \pm\infty$. *Determine the area bounded by the curves. This area is rotated about the* (a) *x-axis* (b) *y-axis. Find the volume of the solid generated in each case.*

Question 6

A particle moves in a circular path of radius a, *with angular speed* ω. *State, without prrof, the tangential and normal components of the acceleration of the particle. What effect, if any, is there on these components if* ω *is constant?*

(i) *The ends of a light string of length* $18a$ *are attached to two fixed points* L, M *in the same vertical line,* M *being at a depth* $12a$ *below* L. *A small smooth ring* R *of mass* m *is threaded on the string. The ring revolves with constant speed in a horizontal circle with* M *as centre and the string taut. Show that* $MR = 5a$ *and that the tension in the string is* $13 \text{ mg}/12$. *Find the period of the motion.*

(ii) *The ring* R *is now* <u>*tied*</u> *to the string so that* $MR = 5a$, *and the ring revolves with constant speed in a horizontal circle with* M *as centre, the string remaining taut. If the period of motion is* $\sqrt{2}$ *times the period of motion when the ring was free to move on the string, find the ratio of the tensions in the string.*

Question 7

(i) *Sketch the curves*

(a) $y=\log|x|$ (b) $y=|\log x|$ (c) $y=(\log x)^2$

(ii) *If* n *is a positive integer, and* a, b *are constants then it is known that* $(1+ax)^n = 1-8x+\frac{88}{3}x^2+bx^3+\cdots$. *Find the values of* n, a, b.

(iii) The equation $x^3 + ax^2 + bx + 15 = 0$ *(where* a, b *are real) has* $2 + i$ *as one root. Find* a, b *and solve the equation.*

Question 8

(i) Show that (a) $\cos\theta + \cos(\theta + \frac{2\pi}{3}) + \cos(\theta + \frac{4\pi}{3}) = 0$

(b) $\cos^2\theta + \cos^2(\theta + \frac{2\pi}{3}) + \cos^2(\theta + \frac{4\pi}{3})$ *is independent of* θ *and find its value.*

(ii) A set of 12 *coloured plastic discs consist of* 5 *white,* 4 *black and* 3 *yellow discs. (These discs are indistinguishable apart from colour.) Three of the discs are chosen at random. Find*

(a) the probability that only one yellow disc is chosen

(b) the expected number of yellow discs chosen.

(iii) Express $32(16 - u^4)^{-1}$ *in partial fractions and hence show that*
$\int_{3/2}^{2} \frac{dx}{16-(2x-3)^4} = \frac{1}{64}\{\log 3 + 2\tan^{-1}(\frac{1}{2})\}$.

PAPER 7

Question 1

(i) *If* $y = e^{-x}(A\sin 2x + B\cos 2x)$ *prove that* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

(ii) *Express* $\frac{3x+7}{(x+1)(x+2)(x+3)}$ *in partial fractions and hence prove that* $\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\, dx = \log_e 2$.

(iii) *Sketch the curve* $y = e^{px}$ *where* p *is a positive constant. Find the equation of the tangent to this curve at the point* (x_1, y_1) *on it, and hence determine the equation of that tangent to the curve which passes through the origin. Determine the relations between* p, e *so that the equation* $e^{px} = 2x$ *should have* $0, 1$ *or* 2 *real roots.*

Question 2

(i) *Evaluate*

(a) $\int_0^2 \frac{1+x}{4+x^2}\, dx$ (b) $\int_0^5 \frac{x\ dx}{\sqrt{x+4}}$ (c) $\int_0^{\pi/2} \sin 5x \cos 3x\ dx$

(ii) *Explain the method of integration by parts. If* $u_n = \frac{1}{n!}\int_0^1 x^n e^{-x} dx$, *where* $n \geq 0$, *show that* $\frac{1}{n!} = e(u_{n-1} - u_n)$. *Use this result to find the value of* u_4.

Question 3

F is the point $(4, 0)$ and d the line $x = 1$. M is the foot of the perpendicular from a variable point P to d, and P moves so that $FP = 2PM$. Find the equation of the locus of P. Draw a sketch showing clearly the principal axes of the curve, the loci, the directrices and the asymptotes, marking on each of them its equation or coordinates. State the angle between the asymptotes and the eccentricity of the curve.

(a) *A line, parallel to one asymptote, is drawm through the focus F of this hyperbola. It meets the hyperbola in H, the directrix corresponding to F in D, the other asymptote in K and the conjugate axis in R. Prove that $FH = HD$ and $FK = KR$.*

(b) *Find the equation of the tangent to the hyperbola at the point $T(x_1, y_1)$ on it. This tangent meets the line d in the point M. Prove that FM is perpendicular to FT.*

Question 4

(i) *Shade in the region in the complex (Argand) plane, for which*

(a) $\pi/6 \leq \arg z \leq \pi/3$ *<u>and</u>* $1 \leq \Im(z) \leq 2$.

(b) $|z| \geq 2$ *<u>and</u>* $2 \leq z + \overline{z} \leq 6$.

(ii) Define the modulus of a complex number z and hence prove for 2 complex numbers z_1, z_2 that $|z_1 z_2| = |z_1| \cdot |z_2|$. If z_1, z_2 are the complex numbers $a + ib, c + id$ respectively where a, b, c, d are real, write down $z_1 z_2$. Hence express $(ac-bd)^2+(bc+ad)^2$ as the product of real quadratic factors. Noting $34 = 9+25$ and $185 = 64 + 121$, express 34×185 as the sum of two squares.

(iii) If $w = \frac{z}{1+z}$, where $z = x + iy, w = u + iv$ express u and v as (real) functions of x and y. The point Z in a complex plane representing the number z moves along the y-axis from the negative end to the positive end. Show that the point W in another plane, representing the complex number w, describes the circumference of a certain circle. Draw this circle and indicate the sense in which W describes the circle.

<u>Question 5</u>

(i) Evaluate $\int_0^a \sqrt{a^2 - x^2}\ dx$ and $\int_0^a (a^2 - x^2)\ dx$

(ii) Find the area R enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(a) This region is rotated about the x-axis. Find the volume of the solid so generated.

(b) The region R is now rotated about the line $y = b$. A plane perpendicular to the line $y = b$ through the point (x, y) on the ellipse cuts the solid of revolutin formed in a section of area A. Prove that $A = 4\pi by$ and use this to find the volume of the solid.

<u>Question 6</u>

A particle moves on the x-axis so that at time t, its acceleration is given by $\frac{d^2x}{dt^2} = -n^2x$, where n is a positive constant. Initially, the particle is released from rest where $x = a$. Derive expressions for its

(a) velocity v in position x *(b) position x at time t*

Determine the period of motion of the particle.

On a certain day high water for a harbour occurs at 5 a.m. and low water at 11.20 a.m., the corresponding depths being 30 metres and 10 metres. If the tidal motion is assumed to be simple harmonic prove that, to the nearest minute, the *latest time before noon that a ship, drawing* 25 *metres, can enter the harbour is* 7.06 *a.m.*

Question 7

(i) (a) There are 6 *persons from whom a game of tennis is to be made up, two on each side. How many different matches could be arranged; a change in either pair giving a different match? How many different matches are possible if two particular persons are to both play in the match.*

(b) In the morse telegraphic code, a letter of the alphabet is denoted by a succession of signs, each of which is a dot or a dash. Show that for an alphabet of 26 *letters, not more than* 4 *signs need be used for any letter, and find how many different arrangements of dots and dashes there can be made when not more than* 4 *signs are used.*

(ii) If α, β, γ *are the roots of the equation* $x^3 - px^2 + qx - r = 0$*, find* $\alpha + \beta + \gamma, \alpha^2 + \beta^2 + \gamma^2, \alpha^3 + \beta^3 + \gamma^3$ *in terms of* p, q, r*. Hence, find a solution of the set of equations* $X + Y + Z = -1, X^2 + Y^2 + Z^2 = 5, X^3 + Y^3 + Z^3 = -7$

Question 8

(i) Find the cube roots of unity and express them in the form $r(\cos\theta + i\sin\theta)$*. Show the roots on an Argand diagram. If* ω *is one of the complex roots, prove the other root is* ω^2*, and show that* $1 + \omega + \omega^2 = 0$.

(a) Prove that if n *is a positive integer, then* $1 + \omega^n + \omega^{2n} = 3$ *or* 0*, depending on whether* n *is or is not a multiple of* 3.

(b) If $x = a + b, y = a\omega + b\omega^2, z = a\omega^2 + b\omega$ *show* $x^2 + y^2 + z^2 = 6ab$.

(ii) A function $f(x)$ *is called even if* $f(-x) = f(x)$ *and it is called odd if* $f(-x) = -f(x)$.

(a) State whether the following functions are odd or even or neither:

$F(x) = 1 + \cos 2x;\quad G(x) = e^{-x^2}/(1 + x);\quad (x) = 4x^2 \sin^3 6x$

(b) If $f(x)$ *is even and* $g(x)$ *is odd, and both are continuous and differentiable for all* x*, show that*

(α) the curve $y = g(x)$ *passes through the origin*

(β) if $b > a > 0$*, then* $\int_{-a}^{b} g(x)\ dx = \int_a^b g(x)\ dx$

(γ) $f(g(x))$ is an even function whilst $f'(x)$ is an odd function.

PAPER 8

Question 1

(i) *If* $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ *prove that* $1 + (\frac{dy}{dx})^2 = \frac{y^2}{a^2}$. *Hence find the value of* $\int_0^{a \log 2} \sqrt{1 + (\frac{dy}{dx})^2}\ .dx$

(ii) *For the curve given by the parametric equations* $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$, *where 'a' is a positive constant, show that the normal at the point* θ *has equation* $x \sin\frac{\theta}{2} + y \cos\frac{\theta}{2} = a\theta \sin\frac{\theta}{2}$.

(iii) *Evaluate* $\int_0^1 \tan^{-1} y\ dy$. *Sketch the curve* $y = \tan x$ *for* $0 \le x \le 2\pi$. *Find the area bounded by the curve and the lines* $x = 0, y = 1$. *Check your result by another method.*

Question 2

(i) *Evaluate*

(a) $\int_1^2 (\log x)^2 dx$ (b) $\int_0^1 \frac{x^2}{x+1} dx$ (c) $\int_{-1/4}^{1/4} \frac{dx}{\sqrt{1-4x^2}}$

(ii) *Sketch the curve* $y = 1 - \frac{1}{4-x^2}$ *showing any stationary points and asymptotes clearly. Show that the area bounded by the curve and the* x*-axis is of magnitude* $\{2\sqrt{3} - \log(2+\sqrt{3})\}$ unit^2, *and find the volume of the solid obtained by rotating this area about the* y*-axis.*

Question 3

Show that the equation of the tangent to the rectangular hyperbola $xy = c^2$ *at the point* $P(cp, c/p)$ *on it is* $x + p^2 y = 2cp$.

(a) *If the tangents at* $P(cp, c/p)$ *and* $Q(cq, c/q)$ *meet at the point* $R(x_0, y_0)$ *prove that* $pq = x_0/y_0$ *and* $p + q = 2c/y_0$. *If the length of the chord* PQ *id* d units, *find an expression for* d^2 *in terms of* c, p, q *(in factorised form). If* d *is fixed, deduce that the locus of* R *has equation* $4c^2(x^2 + y^2)(c^2 - xy) = x^2 y^2 d^2$.

(b) *The tangent and normal at* P *to the rectangular hyperbola meet the* x*-axis at* A, B *respectively. Show that the area of the triangle* PAB *is* $\frac{1}{2}c^2(1+p^{-4})$ unit^2.

Question 4

(i) *If* w *is the complex number* $2\sqrt{3}i - 2$ *find* $|w$ *and* $\arg w$.

(a) *Indicate on an Argand diagram the complex numbers* $w, \overline{w}, iw, \frac{1}{w}, -w$.

(b) *Show that* $w^2 = 4\overline{w}$.

(c) *Prove that* w *is a root of the equation* $z^3 - 64 = 0$.

(ii) *Draw neat labelled sketches (not on graph paper) to indicate the regions of the Argand plane defined by*

(a) $|z| \leq 2$ *and* $0 \leq \arg z \leq \frac{\pi}{4}$ $\qquad$ (b) $|z - \overline{z}| \leq 6$ *and* $0 \leq \Re(2z) \leq 4$.

(iii) *The points* P, Q *represent the complex numbers* z, w *on an Argand diagram, where* $w = \frac{2z+3}{z-1}$. *If* P *moves on the circle* $|z| = 1$, *find the locus of* w.

Question 5

(i) *Show that* $y = \frac{1}{2+\sin x}$ *is a decreasing function for* $0 \leq x \leq \pi/2$. *Use a sketch to show that* $\frac{\pi}{6} < \int_0^{\pi/2} \frac{dx}{2+\sin x} < \frac{\pi}{4}$.

(ii) (a) *Show that* $y = \frac{1}{2+\sin x}$ *is a decreasing function for* $0 \leq x \leq \pi/2$. *Use a sketch to show that* $\frac{\pi}{6} < \int_0^{\pi/2} \frac{dx}{2+\sin x} < \frac{\pi}{4}$

(ii) (a) *Use the substitution* $t = \tan \frac{x}{2}$ *to prove* $\int_0^{\pi/2} \frac{dx}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$.

(b) *Show that* $\int_0^{2a} f(x)\ dx = \int_0^{a} \{f(x) + f(2a - x)\}\ dx$. *Hence, or otherwise, evaluate* $\int_0^{\pi} \frac{x\ dx}{2+\sin x}$.

(c) *Evaluate* $\int_0^{\pi/2} \frac{\sin x\ dx}{2+\sin x}$ *and* $\int_0^{\pi/2} \frac{\cos x\ dx}{2+\sin x}$.

Question 6

(i) *A light inextensible string* ABC *is such that* $AB = 5a/3$ *and* $BC = 5a/4$. *A particle of mass* m *is attached to the string at* C *and one of mass* $7m$ *is attached at* B. *The end* A *is tied to a fixed point and the whole system rotates steadily about the vertical through* A *in such a way that* B *and* C *describe horizontal circles of radii* a *and* $2a$ *respectively. Show that the tension in* BC *is* $5mg/3$. *Also find the tension in* AB *and the speeds of* B *and* C.

(ii) *The governor of a gramophone consists essentially of a particle on a light wire* 5 cm *long which rotates as a conical pendulum (i.e., the particle describes a horizontal circle of constant radius with uniform speed in a horizontal plane below the fixed point to which the wire is attached). The govenor is geared to the turntable so that the angular velocity of the govenor is twice that of the turntable. Find the semi-vertical angle of the conical pendulum when the turntable is rotating at* 78 rev/min. *(Answer to nearest degree; take* $g = 9.8\ \mathrm{m/s^2}$.

Question 7

(i) *State the Remainder Theorem for polynomials. If a polynomial $P(x)$ is divided by the polynomial $Q(x)$, what can you say about the degree of the remainder $R(x)$? When a certain polynomial is divided by $x+1, x-3$ the respective remainders are $6, -2$. Find the remainder when this polynomial is divided by x^2-2x-3.*

(ii) *The cubic equation $x^3+px+q=0$ has 3 non-zero roots α, β, γ.*

(a) *Find, in terms of the constants p, q, the values of $\alpha^2+\beta^2+\gamma^2$ and $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$. If α, β, γ are all real, what can you deduce about p?*

(b) *Form the cubic equation with roots $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}$.*

(c) *Show that the condition for the roots α, β, γ to be in geometric progression is that $p=0$, and prove that these roots cannot be in arithmetic progression.*

Question 8

(i) *If $x+\frac{1}{x}=u$, find an expression for $x^3+\frac{1}{x^3}$ in terms of u, and prove that $x^5+\frac{1}{x^5}=u^5-5u^3+5u$. By letting $x=\cos\theta+i\sin\theta$, show that $1+\cos 10\theta = 2(16\cos^5\theta-20\cos^3\theta+5\cos\theta)^2$*

(ii) *Prove that ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$ and $\sum_{r=0}^{n}(-1)^r \cdot {}^nC_r = 0$.*

(iii) *Evaluate $\int_0^{\pi/4} \sec\theta \; d\theta$ and hence show that $\int_0^{\pi/4} \sec^3\theta \; d\theta = \frac{1}{2}\{\sqrt{2}+\log(\sqrt{2}+1)\}$*

PAPER 9

Question 1

(i) *If* $y = \sin(n\sin^{-1} x)$ *prove that* $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^y = 0$.

(ii) *A right circular cone has base radius* r*, curved surface area* S *and volume* V*. Show that* $9V^2 = r^2(S^2 - \pi^2 r^4)$*. Hence show that the maximum volume of the cone for fixed curved surface area* S *is* $2^{1/2}.S^{3/2}.\pi^{-1/2}.3^{-7/4}$.

(iii) *If* m *and* n *are positive integers and* $m \neq n$*, show that* $\int_0^\pi \cos mx \cos nx \, dx = 0$*. What is the value of this integral when* $m = n$*?*

Question 2

(i) *A solid has as its base the circle* $x^2 + y^2 = a^2$ *in the xy-plane. Find the volume of the solid such that every cross-section by a plane parallel to the* x*-axis is a square with one side in the xy-plane.*

(ii) *If* $f(x) = 2e^{-4x} - e^{-2x}$*, find* x_1, x_2, x_3 *so that respectively* $f(x_1) = 0, f'(x_2) = 0, f''(x_3) = 0$ *and show that* x_1, x_2, x_3 *are in arithmetic progression. Sketch the graph of* $y = f(x)$ *and show clearly any stationary points and points of inflexion. Show that the magnitude of the area bounded by the curve from* $x = 0$ *to* $x = x_1$*, and the coordinate axes is rotational.*

Question 3

(i) *Find the square root of* $-8 - 8i\sqrt{3}$ *and hence solve the quadratic equation* $z^2 - i2\sqrt{2}z + i2\sqrt{3} = 0$*, expressing the roots in the form* $x + iy$.

(ii) *On the Argand diagram, how clearly the regions defined by the sets*

(a) $\{z : |z - 3i| \leq 2\}$ (b) $\{z : |z - 2| < |z + 2|\}$

(iii) *If* $z \neq 0$*, show that* $u = z + \frac{|z|^2}{z}$ *is always real.*

(iv) *The complex number* z *and its conjugate* $\overline{z}$ *satisfy the relation* $(z - \overline{z})^2 + 8(z + \overline{z}) = 16$*. Prove that the point in the Argand diagram representing* z *lies on a parabola. Show that when* z *satisfies the above relation, its principal amplitude (argument) lies between* $-\pi/4$ *and* $\pi/4$.

Question 4

The point P *moves so that the sum of its distances from the points* $(-2, 0)$ *and* $(2, 0)$ *is* 6 units*. Prove that the equation of the locus of* P *is* $x^2/9 + y^2/5 = 1$.

(i) Sketch the curve E represented by this equation, showing clearly the foci, directrices and auxiliary circle A. What is the equation of the auxiliary circle, and the eccentricity of E?

(ii) A line parallel to the postive y-axis meets the x-axis at N and the curves E, A at P, Q respectively. If N has coordinates $(3\cos\theta, 0)$ determine the coordinates of P, Q.

(a) Find the equations of the tangents at P, Q to the curves E, A respectively.

(b) Show that the point of intersection R of these tangents lies on the major axis of E, and prove that $ON.OR$ is independent of the positions of P, Q on the curves.

Question 5

(i) Find indefinite integrals for

(a) $\int \frac{dx}{9-4x^2}$ (b) $\int \frac{dx}{\sqrt{9-4x^2}}$ (c) $\int \frac{dx}{(9-4x^2)^{3/2}}$

(ii) Evaluate

(a) $\int_0^{1/2} \sin^{-1} t \; dt$ (b) $\int_0^c x \sin(c-x) \; dx$ (c) $\int_0^2 \frac{x^2 \; dx}{x^6+64}$

(iii) Use the substitution $y = \frac{1}{x}$ to evaluate $\int_1^\infty \frac{dx}{x\sqrt{x^2+2x-1}}$

Question 6

A particle is projected vertically upwards from the surface of the Earth with initial velocity u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth. Prove that the speed v in any position x is given by $v^2 = u^2 - 2gR + \frac{2gR^2}{x}$ where R is the radius of the Earth, and g is the acceleration due to gravity at the surface of the Earth.

(i) Prove that the greatest height H, above the Earth's surface, reached by the particle is given by $H = u^2R(2gR - u^2)^{-1}$. Show that the particle will escape from Earth if $u \geq \sqrt{2gR}$ and find the least speed in km/s (to the nearest whole nimber) with which the particle must be initially prrojected from the surface of the Earth so as to never return. (Take $R = 6400$ km, $g = 9.8$ m/s^2.)

(ii) If $u = \sqrt{2gR}$, prove that the time taken to reach a height $3R$ above the surface of the Earth is $\frac{1}{3}.7\sqrt{2}.\sqrt{R/g}$.

Question 7

(i) *Decompose* $\frac{1}{(x^2+1)(x^2-x+1)}$ *into partial functions, and hence evaluate* $\int_0^1 \frac{dx}{(x^2+1)(x^2-x+1)}$.

(ii) *Find (as trigonometrical quantities) the seventh roots of unity, and represent them on an Argand diagram. If* ϕ *is one of the complex roots show that the quadratic equation whose roots are* $\phi+\phi^2+\phi^4$ *and* $\phi^3+\phi^5+\phi^6$ *is* $x^2+x+2=0$*. Find also the cubic equation whose roots are* $\phi+\phi^6, \phi^2+\phi^5$ *and* $\phi^3+\phi^4$.

Question 8

(i) *If* a, b, c *are real and unequal show that* $a^2+b^2>2ab$ *and hence deduce that* $a^2+b^2+c^2>ab+bc+ca$*. If* $a+b+c=6$*, show that* $ab+bc+ca<12$.

(ii) *On the same axes, sketch the curves* $y=x, y=\sin x$ *for* $0 \leq x \leq 4\pi$*. Hence, sketch the curves*

(a) $y=x+\sin x$ $\qquad$ (b) $y=\frac{\sin x}{x}$

(iii) *Show that there are* 365 *different ways of choosing a cricket eleven out of* 15 *members of a club, if of two brothers only one may be chosen, and of three other brothers not more than two may be chosen.* {*Explain your method carefully.*}

PAPER 10

Question 1

(i) If $y = \frac{\cos mx}{1+x}$*, show that* $\frac{d^2y}{dx^2} + \frac{2}{1+x}\frac{dy}{dx} + m^2y = 0$.

(ii) Evaluate

(a) $\int_1^2 \frac{dx}{x^2(1+x)}$ *(b)* $\int_0^1 x\sin^{-1}x\ dx$ *(c)* $\int_0^{\pi/2} \frac{dx}{\cos^2 x + 2\sin^2 x}$

(iii) Sketch the curve $y = e^{x^2-4x+3}$*, showing any stationary points and points of inflexion. Show that* $3e^{-1} < \int_0^3 e^{x^2-4x+3}dx < 3e^3$.

Question 2

(i) By considering turning points on the curve $y = x^3 - 3ux + v$*, (where* u, v *are real) show that the equation* $x^3 - 3ux + v = 0$ *has*

(a) one and only one real root if $u < 0$;

(b) three distinct real roots if $u > 0$*, provided that* $u > (v/2)^{2/3}$.

(ii) Show that the curve $y = \frac{x^2}{x^3+1}$ *consists of two branches. Find the coordinates of the turning points and sketch the curve. Use the sketch to find the number of real roots of the equation* $x^3 - 5x^2 + 1 = 0$.

Question 3

(i) Define the modulus of a complex number z*. Use this definition to prove that if* z_1, z_2 *are complex numbers then* $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$*. If the points* A, B, C *represent the complex numbers* $z_1, z_2, -z_2$ *in an Argand diagram and* O *is the origin, write this result as a geometric theorem involving lengths in the triangle* ABC.

(ii) A point z *on the Argand diagram is given by* $z = w^2 + 2iw$*, where* $w = u + iv$ *and* u, v *are real. Find the locus of* z *when*

(a) $v = 0$ *and* u *varies* *(b)* $v = 1$ *and* u *varies*

(c) $u = 0$ *and* v *varies* *(d)* $u = 1$ *and* v *varies*

Sketch the four loci on the same diagram.

Question 4

Prove that the condition for the line $y = mx + c$ to touch the ellipse E : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = b^2 + a^2m^2$.

If S, S' are the foci of E, and $SZ, S'Z'$ are perpendiculars to any tangent meeting it in Z, Z', show that $SZ.S'Z' = b^2$.

(ii) Tangents are drawn from the variable point $T(X, Y)$ to the ellipse E, show that $(Y - mX)^2 = b^2 + a^2m^2$ and hence deduce that if these tangents are at right-angles to one another, then T lies on the circle $x^2 + y^2 = a^2 + b^2$

(iii) Show that the tangent to E making an angle α with the positive x-axis has equation $x \sin\alpha - y \cos\alpha = \pm\sqrt{a^2 \sin^2\alpha + b^2 \cos^2\alpha}$. If p is the length of the perpendiculars from the origin to this tangent, prove that $p = a\sqrt{1 - e^2 \cos^2\alpha}$.

Question 5

(i) A solid 2 m high rests on a horizontal surface. Every section of the solid by a horizontal plane x m above this surface is a square of side $(3x + 1)^{-1/2}$ m. Find the volume of the solid, correct to 3 significant figures.

(ii) If > 0 and $u_n = \int_1^e (\log x)^n \, dx$, prove that $u_n = e - nu_{n-1}$. Hence find the value of u_4.

(iii) Sketch the curve $x^{2/3} + y^{2/3} = a^{2/3}$ where $a > 0$ for $x \geq 0, y \geq 0$. Show the curve touches the x, y axes. Prove the tangent to the curve at the point $P(x_0, y_0)$ on it, has equation $y_0^{1/3}x + x_0^{1/3}y = x_0^{1/3}y_0^{1/3}a^{2/3}$, and show that the segment intercepted on this tangent by the coordinate axes is independent of the position of P on the curve.

Question 6

A body of mass m in falling from rest experiences air resistance of magnitude kv^2 per unit mass, where k is a positive constant.

(i) Write down the equation of motion of the body and state the value of the terminal velocity V of the body in terms of k, g. If W is the velocity of the body when it reaches the ground, show that the distance S fallen is given by $S = -\frac{1}{2k}\log(1 - \frac{W^2}{V^2})$.

(ii) With air resistance as above, prove that if the body is projected vertically upwards from the ground with initial velocity U, then it will attain its greatest height H where $H = \frac{1}{2k}\log(1 + \frac{U^2}{V^2})$, and return to the ground with speed W given by $W^{-2} = U^{-2} + V^{-2}$.

Question 7

(i) *Express* $-1+i\sqrt{3}$ *and* $-1-i\sqrt{3}$ *in mod-arg form, and hence prove that if* n *is any integer, positive or negative, then* $(-1+i\sqrt{3})^n+(-1-i\sqrt{3})^n$ *has either the value* 2^{n+1} *or* -2^n.

(ii) *By expanding* $(\cos\theta+i\sin\theta)^5$, *show that* $\sin 5\theta$ *can be written in the form* $a\sin^5\theta+b\sin^3\theta+c\sin\theta$, *where* a, b, c *are constants and find* a, b, c.

(iii) *Sketch the curves (not on graph paper)*

(a) $y=e^{-x^2}$ (b) $(x+1)(y-2)=1$ (c) $y=\max(x, 1-x)$

where $\max(p,q)$ *denotes the greater of the two numbers* p, q.

Question 8

(i) *It the polynomial* $P(x)=x^4-4x^3+11x^2-14x+10$ *has roots* $a+ib, a-2ib$ *(where* a, b *are real), find the values of* a *and* b. *Hence find the zeros of* $P(x)$ *over the complex field* $\mathbb{C}$, *and express* $P(x)$ *as the product of* 2 *quadratic factors, each with rational coefficients.*

(ii) *A container holds* 4 *red and* 3 *green counters. Two players* A, B *draw one counter at a time from the container, the first player to draw a red counter wins. If* A *draws first, find the probability of each player winning*

(a) *without replacement* (b) *with replacement.*

PAPER 11

Question 1

(i) *Evaluate, to three significant figures,*

(a) $\int_{1/4}^{3/4} \frac{dx}{\sqrt{x(1-x)}}$ (b) $\int_{\pi/4}^{\pi/2} \frac{1-\cot x}{1+\cot x}\ dx$ (c) $\int_1^4 \frac{dx}{x+\sqrt{x}}$

(ii)(a) *If* $v_n = \int \sec^n x\ dx$, *show that* $v_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} v_{n-2}$ *and find the indefinite integral* $\int \sec^6 \theta\ d\theta$.

(b) *If* $u_n = \int \tan^n x\ dx$, *show that* $u_n = \frac{\tan^{n-1} x}{n-1} - u_{n-2}$, *and find* $\int \tan^6 \theta\ d\theta$.

Question 2

Find the turning points on the curve $y = 2x(1 + x^2)^{-1}$. *If the curve* $y = ax^3+bx^2+cx+d$ *has the same turning points as those of the previous curve, find the constants* a, b, c, d. *Show that the first curve has three points of inflexion, one of which is the same as that of the second curve. Find the equations of the inflexional tngents to the two curves at this common point. Sketch the two curves on the same set of axes. (Graph paper not to be used.)*

Question 3

(i) *If* $y = \log_e(\cos\theta + i\sin\theta)$, *show that* $\frac{dy}{d\theta} = i$, *and hence deduce that* $\cos\theta + i\sin\theta = e^{i\theta}$. *Prove that if* u, v, x, y *are real numbers connected by the relation* $u + iv = e^{x+iy}$, *then* $u = e^x \cos y, v = e^x \sin y$. *The variable point* (x, y) *describes, in a counter clockwise sense, the boundary of the semi-infinite strip defined by the three straight lines:*

$$\left.\begin{array}{l} y = \frac{1}{4}\pi, x \leq 0 \\ x = 0 \\ y = \frac{1}{2}\pi, x \leq 0 \end{array}\right\}$$

Illustrate the corresponding locus of the point (u, v) *by a sketch, giving a careful explanation.*

(ii) *Define the modulus and conjugate of the complex number* $z = x + iy$ *(where* x, y *are real) and show that* $|z|^2 = z\,\overline{z}$.

(a) *Express the complex numberr* $\overline{z}.|z|^{-2}$ *in terms of* z.

(b) *If* z_1, z_2 *are* 2 *complex numbers, prove that* $\overline{z_1 z_2} = \overline{z_1}\ .\ \overline{z_2}$ *and hence deduce that* $|z_1 z_2| = |z_1|.|z_2|$.

Question 4

Show that the tangents at the points $(cp, c/p)$ and $(cq, c/q)$ to the rectangular hyperbola $xy = c^2$ meet at the point $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$.

(i) The circle $x^2 + y^2 + 2gx + 2fy + k = 0$ meets the curve $x = ct, y = c/t$ at the points P_1, P_2, P_3, P_4 whose parameters are t_1, t_2, t_3, t_4 respectively. Prove that $t_1t_2t_3t_4 = 1$. Also prove that the centroid (or mean point) of the 4 points P_1, P_2, P_3, P_4 is midway between the centres of the two curves.

(ii) Show that the tangents at P_1, P_2 meet on the diameter of the hyperbola perpendicular to the chord P_3P_4.

Question 5

(i) Sketch the curve $y^2 = x^2(4 - x^2)$ and show that it enclosed an area of magnitude $32/3$.

(ii) If $f(x), g(x), h(x)$ are distinct non-negative continuous functions of x in the interval $a \le x \le b$ and $f(x) \le g(x) \le h(x)$, show by means of a diagram, that $\int_a^b f(x)\ dx < \int_a^b g(x)\ dx < \int_a^b h(x)\ dx$. Prove that $\frac{1}{2}\int_0^1 x^3(1-x)^3\ dx < \int_0^1 \frac{x^3(1-x)^3}{1+x}\ dx < \int_0^1 x^3(1-x)^3\ dx$, and hence deduce that $\frac{1552}{2240} < \log_e 2 < \frac{1553}{2240}$.

Question 6

A body is moving with steady angular velocity ω in a circular path of radius a. State without proof, results for the normal acceleration and the period of motion of the body.

(i) The two ends of a light inextensible string of length $5a$ are attached to two fixed points A and B (B is vertically below A) where $AB = 2a$. A smooth ring R of mass m is threaded on the string, and the system rotates about AB with constant angular velocity. The ring moves in a horizontal circle (whose plane is below the level of B), so that $AR = 3a$ and $RB = 2a$. Prove that the tension in the string is $8\ \mathrm{mg}/7$ and find the angular velocity of the system.

(ii) A particle P of mass m is attached by two equal light inextensible strings (of length l) to two points Q, S distant a apart in the same vertical line; S is below Q. P rotates in a horizontal circle with uniform angular velocity ω. Prove that the tension in the string PQ is $ml(\frac{1}{2}\omega^2 + \frac{g}{a})$ and find the tension in the string PS. Show that for both strings to remain stretched, then ω must exceed $\sqrt{2g/a}$. If the tensions in the string are in the ratio $2:1$, find the peroiod of motion of the particle.

Question 7

(i) *Express* (a) $\frac{3}{(x-1)(4-x)}$ *and* (b) $\frac{9}{(x-1)^2(4-x)^2}$ *in partial fractions.*

(ii) *Sketch the curve* $y = \frac{3}{(x-1)(4-x)}$ *showing clearly any stationary points and asymptotes. Use the results of* (i) *to find the*

(a) *area enclosed by the curve, the* x*-axis and the ordinates* $x = 2, x = 3$.

(b) *volume of the solid of revolution obtained by rotating this area about the* x*-axis.*

Question 8

(i) *The probability that a certain type of missile hits a target is* 0.8. *Find the minimum number of these missiles which must be fired in order that the probability of hitting the target at least once is greater than* 99.9%. *Calculate the probability that at least two of these missiles are on target.*

(ii) *By graphical considerations, show that the equation* $z^6 + 64 = 0$ *has no real roots. By considering the equation obtained by substituting* $z = iy$ *show that two of the values of* z *satisfying the equation* $z^6 + 64 = 0$ *are pure imaginary. Find the six roots of the equation* $z^6 + 64 = 0$, *and plot their positions on an Argand diagram.*

PAPER 12

Question 1

(i) *Evaluate, correct to* 3 *significant figures,*

(a) $\int_{\pi/6}^{\pi/2} \frac{\cos^3 x}{\sin^2 x}\ dx$ (b) $\int_0^1 \frac{dx}{e^x+1}$ (c) $\int_0^1 x^2 e^{-x}\ dx$

(a) *The length* s *of the arc of curve given in parametric coordinates* θ*, is given by* $s = \displaystyle\int_{\theta=\theta_1}^{\theta_2} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\ .d\theta$. *Find the length of the arc of the curve given by* $x = a\cos^3\theta, y = a\sin^3\theta$ *from* $\theta = 0$ *to* $\theta = \pi/2$.

(iii) *It is known that* $(1-x^2)\frac{dy}{dx} = 2$ *and when* $x = 0, y = 0$. *Find an expression for* x *in terms of* y*, and* <u>*hence*</u> *find* x *when* $y = 1$*, giving the answer to* 3 *significant figures.*

Question 2

(i) *Find any stationary points on the curve* $y = 5x^6 - 12x^2 + k$*, and show that the equation* $5x^6 - 12x^5 + k = 0$ *has a real root between* $x = 0, x = 2$ *providing* $0 < k < 64$. *What is the condition on* k *for the equation to have no real roots?*

(ii) *Show that* $f(x) = \frac{\sec x + \tan x}{2\sec x + 3\tan x}$ *is a monotonic decreasing function of* x *if* $0 < x < \pi/2$. *Deduce that* $\frac{\pi}{28} > \int_{\pi/6}^{\pi/4} \frac{\sec x+\tan x}{2\sec x+3\tan x}\ dx > (\sqrt{2}-1)\frac{\pi}{12}$.

(iii) *By substituting* $x = a - y$*, show that* $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$. *Use this result to evaluate*

(a) $\int_0^1 x(1-x)^{12}\ dx$ $\int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin 2x}\ dx$.

Question 3

(i) *Explain how complex numbers are represented on the Argand diagram. On an Argand diagram, what is the locus represented by each of the equations*

(a) $|z-a| = r$ (b) $\arg(z-k) = \alpha$*?*

State clearly which of the constants a, k, r, α *are real and which are complex, and what restrictions (if any) there are on their range of values.*

(ii) *If* z_1, z_2 *are complex numbers prove that* $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$. *The complex numbers* $z = \frac{c+2i}{c-2i}$ *where* c *is real. Find* $|z|$ *and hence describe the exact locus of* z*, if it is given that* c *varies from* -1 *to* $+1$.

(iii) If $z = x + iy$, express $w = \frac{1-z}{1+z}$ in the form $X + iY$. Hence find the path of w if z moves along the line $x = 1$.

Question 4

P is the point $(3\cos\theta, 2\sin\theta)$ and Q is the point $(3\sec\theta, 2\tan\theta)$. Sketch the curves which are the loci, as θ varies, of P and Q, marking on them the range of positions occupied by P and Q respectively as θ varies from $\pi/2$ to π.

(a) Prove that for any value of θ, the line PQ passes through one of the common points of the two curves.

(b) Show also that the tangent at P to the first curve meets the tangent at Q to the second curve in a point on the common tangent to the two curves at their other common point.

(c) Prove that the 2 curves have the same length latus rectum.

Question 5

(i) If $f(x) = 2\cos x + 2x\sin x - x^2\cos x$ is defined for $-\pi/2 \le x \le 5\pi/2$, find any stationary points and sketch $y = f(x)$. What are the greatest and least values of $f(x)$?

(ii) Give the formula for integration by parts and evaluate $\int_0^{\pi/2} x^2\cos x\ dx$. The portion of the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ revolves about the y-axis. Prove the volume of the solid described by the area between the curve, the y-axis and the line $y = 1$ is $\frac{1}{4}\pi(\pi^2 - 8)$. The area bounded by the curve, the x-axis and the line $x = \pi/2$ is rotated about the y-axis. What is the volume of the solid so formed?

Question 6

A particle is projected from O with initial speed V inclined at an angle θ to the horizontal. If the acceleration due to gravity is g (and assumed constant) show that the particle describes a parabola with equation $y = x\tan\theta - \frac{1}{2}gx^2\sec^2\theta/V^2$ and find the focal length of the parabola. Also determine the horizontal range R of the particle.

(a) If the horizontal range is equal in length to the latus rectum of the parabolic path, find the angle of projection with the horizontal.

(b) If $V = \sqrt{2gc}$ and the particle is projected in order to pass through a point P at a horizontal distance b from 0 and a vertical distance h above 0, prove that θ satisfies the equation $b^2\tan^\theta - 4bc\tan\theta + b^2 + 4ch = 0$. Deduce that the

particle cannot pass through P if $b^2 > 4c(c-h)$. If α and β are the two possible angles of projection when $b^2 < 4c(c-h)$, prove that $\tan(\alpha+\beta) = -b/h$.

Question 7

(*i*) *Given that the equation $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$ has a treble root, find all the roots of this equation.*

(*ii*) *If α, β, γ are the roots of the cubic equation $x^3 + px + q = 0$, find in terms of the constants p, q, the values of*

(*a*) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (*b*) $\alpha^3 + \beta^3 + \gamma^3$ (*c*) $\alpha^4 + \beta^4 + \gamma^4$

Determine the cubic equation whose roots are the squares of the roots of the given equation.

(*iii*) *Assuming results for $\cos(\alpha \pm \beta)$, find an expression for $\cos S + \cos T$ as a product of two trigonometric functions. Solve the equation $\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta$, for $0 \leq \theta \leq 2\pi$.*

Question 8

If ω is one of the complex cube roots of unity, simplify

(*a*) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

(*b*) $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$ *where a, b, c are real.*

(*ii*) *If n is a positive odd integers, show by induction that $1^3 - 2^3 + 3^3 - 4^3 + \cdots + n^3 = \frac{1}{4}(2n-1)(n+1)^2$*

(*iii*) *If $(1+x)^2 = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n$, where n is a positive integer, express in terms of n*

(*a*) $c_0 + c_1 + c_2 + \cdots + c_n$ (*b*) $c_1 + 2c_2 + 3c_3 + \cdots + nc_n$

(*c*) $c_2 + 2c_3 + 3c_4 + \cdots + (n-1)c_n$ (*d*) $c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \cdots + \frac{1}{n+1}c_n$.
