NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 52/67

by James Coroneos*

1. For the ellipse $E: \ 5x^2+9y^2=45$, state the lengths of the semi-axes. Find the eccentricity, the coordinates of the foci S, S' and the equations of the directrices. Sketch the ellipse.

(i) What is the equation of the auxiliary circle? State parameter coordinates for E in terms of the eccentric angle α and show how to obtain the points on E corresponding to $\alpha=\pi/3$, $\alpha=5\pi/4$.

(ii) Determine the equation of the tangent to E at the point $(2, 5/3)$ on it. If perpendiculars are drawn from the foci to this tangent, show that the

(a) product of these perpendicular distances is the square of the semi-minor axis;

(b) feet of these perpendiculars both lie on the auxiliary circle.

2. For the ellipse $E: \ x^2/25+y^2/9=1$ determine the length of the latus rectum.

(i) Find the equation of the normal to E at the point $P(3, 12/5)$. This normal intersects the major axis of the ellipse at G and N is the foot of the perpendicular from P to this axis. Calculate the length of GN.

(ii) Find the equation of that diameter of the ellipse which passes through P and verify that the other end of the diameter has coordinates $(-3, -12/5)$.

3. **(i)** The straight line $x-2y+3=0$ intersects the ellipse $x^2+2y^2=9$ at P, Q. Determine the coordinates of P, Q and find the coordinates of R, the point of intersection of the tangents at P, Q.

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(ii) Find the equation of the tangent at $A(4,-1)$ to the ellipse $E:\ 9x^2 + 25y^2 = 169$. Prove that the circle $x^2 + y^2 + 28x - 23y = 152$ and E touch at A.

4. Find the equation of the tangent and normal to the ellipse $E:\ x^2+4y^2 = 100$ at the point $P(6,4)$.

(i) If this normal meets the major axis in G, and OZ is the perpendicular from the centre O to the tangent at P, prove that $PG.OZ = 25$.

(ii) The tangent at P meets the tangent at $Q(-8,3)$ to E in the point T, whilst the normals at P, Q meet in R. Find the coordinates of T, R and show that the diameter through R is perpendicular to PQ.

5. **(i)** Determine the equations of those tangents to the ellipse $x^2 + 2y^2 = 8$ which are parallel to the line $y = 2x$.

(ii) Find the equations of the tangents to the ellipse $9x^2+16y^2 = 36$ which are perpendicular to the line $2x + 2y = 7$. Find also the coordinates of the points of contact of these two tangents with the ellipse.

6. Find the equation of the ellipse whose centre is the origin and

(i) which has foci at the points $(-5\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$, given that it passes through the point $(8,3)$.

(ii) whose latus rectum is 10 units and whose minor axis is equal to the distance between the foci; (the axes of the ellipse lie along the coordinate axes).

7. **(i)** Find the locus of a point which moves so that the sum of its distance from two fixed points 6 units apart is always 10 units. (*Your answer should be in the form of an equation referred to axes of symmetry*).

(ii) Except for minor perturbations, the orbit of the Earth is an ellipse having the Sun at a focus. Find the eccentricity of the orbit, given that the least and greatest distances from the Earth to the Sun are in the ratio $29 : 30$.

8. A carriage spring is in the shape of part of an ellipse. Referred to the principal axes the coordinates of the ends of the spring are $(16,9)$ and $(-16,9)$, and the tangents to the ellipse at these points are at right angles. Find the lengths of the axes of the ellipse.

9. PQ is a diameter of the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$ and makes an angle $\pi/4$ with the major axis. Find the length of PQ and prove that the tangent of the angle between OP and the normal at P is $(a^2-b^2)/(a^2+b^2)$ {*O is the origin.*}

10. **(i)** Prove that the line $2x-2y+3=0$ is a tangent to the ellipse $2x^2+4y^2=3$, and find the point of contact.

(ii) Show that $2x+y=2$ is a normal to the ellipse $x^2+2y^2=12$ and find the foot of the normal.

11. Show that the line $y=mx\pm\sqrt{9m^2+1}$ touches the ellipse $\frac{x^2}{9}+y^2=1$ for all values of m. Hence find the equations of the tangents to this ellipse from the external point $(2,1)$. If θ is the acute angle between these tangents, find $\tan\theta$.