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NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 48/67

by James Coroneos*

1. Determine the three cube roots of unity, expressing each in both forms $r(\cos\theta + i\sin\theta)$ and $a + ib$.

(i) Show the points representing these roots on an Argand diagram form an equilateral triangle.

(ii) If ω is one of the complex roots, show the other complex root is ω^2. Prove that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

2. Find the five fifth roots unity in the form $r(\cos\theta + i\sin\theta)$, and show them on an Argand diagram.

(i) Factorise $z^5 - 1$ over the **(a)** complex field $\mathbb{C}$ **(b)** real field $\mathbb{R}$.

(ii) Show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

(iii) If α is one of the complex fifth roots of unity, form the quadratic equation with roots $\alpha + \alpha^4$ and $\alpha^2 + \alpha^3$.

3. Find all the roots of $z^6 = 1$, expressing each in the form $a + ib$. Show these on an Argand diagram.

(i) Express $z^6 - 1$ as the product of 2 linear factors and 2 real quadratic factors.

(ii) Show that if α is one of the complex roots, then the other complex roots are $\alpha^2, \alpha^{-1}, \alpha^{-2}$.

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4. Find the roots of the equation $z^5 + 1 = 0$, and prove that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

(i) Show the points representing these roots on an Argand diagram form a regular pentagon of side $2\sin\frac{\pi}{5}$ units.

(ii) If α is a complex root, show that the other complex roots can be written as $\alpha^3, \alpha^7, \alpha^9$. Prove that $\alpha + \alpha^3 + \alpha^7 + \alpha^9 = 1$ and $\alpha^5 = -1$.

(iii) Express $z^5 + 1$ as the product of three real factors, and hence show that $z^4 - z^3 + z^2 - z + 1 = (z^2 - 2z\cos\frac{\pi}{5} + 1)(z^2 + 2z\cos\frac{2\pi}{5} + 1)$

5. Find the 7 seventh roots of -1. Show these roots on an Argand diagram.

(i) Find the factors of $z^7 + 1$ over the **(a)** complex field $\mathbb{C}$ **(b)** real field $\mathbb{R}$.

(ii) By considering the equation $z^7+1 = 0$, prove $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7} = \frac{1}{2}$

(iii) If α is one of the complex seventh roots of -1

(a) show that the other complex roots can be expressed either as $-\alpha^2, \alpha^3, -\alpha^4, \alpha^5, -\alpha^6$ or as $\alpha^{-5}, \alpha^{-3}, \alpha^{-1}, \alpha^3, \alpha^5$

(b) form the cubic equation with roots $\alpha - \alpha^6, \alpha^3 - \alpha^4, \alpha^5 - \alpha^2$

6. Find the 8 eighth roots of unity, giving results in the form $a + ib$, and show these roots on an Argand diagram. Hence express $z^8 - 1$ as the product of 2 real linear factors and 3 real quadratic factors. Check these factors by another method.

7. Find the n-th roots of unity, and show they are in geometric progression. If $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}$ are the n-th roots of unity, prove that $1^p + \alpha^p + (\alpha^2)^p + (\alpha^3)^p + \cdots + (\alpha^{n-1})^p = 0$, unless p is a multiple of n, in which case it is equal to n.

8. Find, as trigonometric quantities, the seventh roots of unity and indicate them on an Argand diagram.

(i) If α is one of the complex roots, find the equation of which the roots are $\alpha+\alpha^2+\alpha^4$ and $\alpha^6+\alpha^5+\alpha^3$. Explain why it does not matter which of the complex roots is selected as α. Mark the points representing $\alpha + \alpha^2 + \alpha^4$ and $\alpha^6 + \alpha^5 + \alpha^3$ on the Argand diagram, indicating their actual coordinates, and show a construction for them from the points representing the seventh roots of unity.

(ii) Show that $\frac{z^7-1}{z-1} = (z^2+2z\cos\frac{\pi}{7}+1)(z^2-2z\cos\frac{2\pi}{7}+1)(z^2+2z\cos\frac{3\pi}{7}+1)$

9. Show that the roots of the equation $y^4 + y^3 + y^2 + y + 1 = 0$ are $y = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}$, $(r = 1, 2, 3, 4)$.

- **(i)** By putting $x = y + \frac{1}{y}$, show that the roots of the equation $x^2 + x - 1 = 0$ are $2\cos\frac{2r\pi}{5}$ $(r = 1, 2)$ and deduce that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$
- **(ii)** Prove that $y^4 + y^3 + y^2 + y + 1 = (y^2 + 2y\cos\frac{\pi}{5} + 1)(y^2 - 2y\cos\frac{2\pi}{5} + 1)$. By equating the coefficients of y in this identity, show that $\cos 36° = \frac{1}{2} + \cos 72°$ and deduce that $\cos 36° = \frac{1}{4}(1 + \sqrt{5})$.

10. If $\alpha = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, show that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$. Also prove $\alpha^{5r}, \alpha^{5r+1}, \alpha^{5r+2}, \alpha^{5r+3}, \alpha^{5r+4}$ take the values $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ where r is a positive integer. Express $x^5 - 1$ as the product of 3 factors, each containing terms with real coefficients, and prove that $(1 - \cos\frac{2\pi}{5})(1 - \cos\frac{4\pi}{5}) = \frac{5}{4}$.

11. If α is a complex root of $z^7 - 1 = 0$, find the cubic equation whose roots are $\alpha + \alpha^{-1}, \alpha^2 + \alpha^{-2}, \alpha^3 + \alpha^{-3}$. Deduce that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$.

12. Given that $\alpha^5 = 1$, $\alpha \neq 1$, prove that

- **(i)** $\sum_{r=0}^{4} \alpha^{rs} x^r = \frac{1-x^5}{1-\alpha^s x}$, for any fixed integer value of s.
- **(ii)** $(1+x+x^2+x^3+x^4)(1+\alpha x+\alpha^2x^2+\alpha^3x^3+\alpha^4x^4)(1+\alpha^2x+\alpha^4x^2+\alpha x^3+\alpha^3x^4)(1+\alpha^3x+\alpha x^2+\alpha^4x^3+\alpha^2x^4)(1+\alpha^4x+\alpha^3x^2+\alpha^2x^3+\alpha x^4) = (1-x^5)^4$
- **(iii)** $(a+b+c+d+e)(a+\alpha b+\alpha^2c+\alpha^3d+\alpha^4e)(a+\alpha^2b+\alpha^4c+\alpha d+\alpha^3e)(a+\alpha^3b+\alpha c+\alpha^4d+\alpha^2e)(a+\alpha^4b+\alpha^3c+\alpha^2d+\alpha e)$ contains no term in a^4b.

13. Show that

- **(i)** The roots of the equation $z^{11} = 1$ are $\cos\frac{2k\pi}{11} + i\sin\frac{2k\pi}{11}$ $(k = 0, 1, 2, \ldots, 10)$ and hence find the value of $\sum_{k=1}^{5}\cos\frac{2k\pi}{11}$
- **(ii)** $\frac{x^{10}-1}{x^2-1} = (x^2 - 2x\cos\frac{\pi}{5} + 1)(x^2 - 2x\cos\frac{2\pi}{5} + 1)(x^2 - 2x\cos\frac{3\pi}{5} + 1)(x^2 - 2x\cos\frac{4\pi}{5} + 1)$

14.

- **(i)** If $w = \sqrt{3} + i$, find $w^{\frac{1}{2}}$ and show the roots on an Argand diagram.
- **(ii)** Show that $-1 + i$ is a cube root of $2 + 2i$, and find the other cube roots in the form $r(\cos\theta + i\sin\theta)$.
- **(iii)** Find the roots of the equation $z^5 = \frac{1}{2}(1 - i\sqrt{3})$, expressing each root in mod-arg form. Show these 5 roots and z^5 on an Argand diagram.

15. Find the solutions of the equation $x^3 = \frac{1}{2}(1 + i\sqrt{3})$ expressing roots in the form $r(\cos\theta + i\sin\theta)$. Express each of the roots of the equation $x^6 - x^3 + 1 = 0$ in the form $\cos\theta + i\sin\theta$, and find the real quadratic factors of $x^6 - x^3 + 1$. If α is one of these roots, show that the other roots may be written as $\alpha^{-1}, -\alpha^2, -\alpha^{-2}, -\alpha^4, -\alpha^{-4}$.

16. Without solving the equation $z^{10} + 1024 = 0$ show that it has no real roots. By considering the equation obtained by substituting $z = iy$ or otherwise, show that two of the values of z satisfying the equation $z^{10} + 1024 = 0$ are pure imaginary. Find the ten roots of the equation $z^{10} + 1024 = 0$ and plot their positions on an Argand diagram.

17. Solve the equation $(\frac{x}{a})^8 = -1$, giving roots in $r(\cos\theta + i\sin\theta)$ form. Hence show that $x^8 + a^8 = (x^2 + 2ax\cos\frac{\pi}{8} + a^2)(x^2 + 2ax\cos\frac{3\pi}{8} + a^2)(x^2 + 2ax\cos\frac{5\pi}{8} + a^2)(x^2 + 2ax\cos\frac{7\pi}{8} + a^2)$. Deduce that $(x+1)^8 + (x-1)^8 = 256\cos^2\frac{\pi}{16}\cos^2\frac{3\pi}{16}\cos^2\frac{5\pi}{16}\cos^2\frac{7\pi}{16}(x^2 + \tan^2\frac{\pi}{16})(x^2 + \tan^2\frac{3\pi}{16})(x^2 + \tan^2\frac{5\pi}{16}) \cdot (x^2 + \tan^2\frac{7\pi}{16})$, and state the value of $\cos^2\frac{\pi}{16}\cos^2\frac{3\pi}{16}\cos^2\frac{5\pi}{16}\cos^2\frac{7\pi}{16}$.