NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 13/67

by James Coroneos*

1. If the points L_1, L_2, L_3 represent the complex numbers z_1, z_2, z_3 on the Argand diagram, and if $z_1/z_2 = z_2/z_3$, show that OL_2 bisects the angle L_1OL_3.

2. If $\alpha, \beta, \gamma, \delta$ are complex numbers, then if $\frac{\alpha-\beta}{\gamma-\delta}$ is imaginary, show that the join of α, β is perpendicular to the join of γ, δ. Hence if $\frac{\alpha-\beta}{\gamma-\delta}$ and $\frac{\alpha-\gamma}{\delta-\beta}$ are both purely imaginary, show that $\frac{\alpha-\delta}{\beta-\gamma}$ is also purely imaginary. [Hint: a careful figure is needed; note concurrence property of altitudes of a triangle.]

3. Prove that the three points z_1, z_2, z_3 are collinear, *if and only if* the ratio $(z_3 - z_1)/(z_2 - z_1)$ is real. Hence show that the points $5+8i, 13+20i, 19+29i$ are collinear.

4. Show that if the complex numbers $z, 1/z$ are represented by the points P, Q in an Argand diagram, then $|OQ| = 1/|OP|$ and $Q\hat{O}X = X\hat{O}P$. The points which represent z_1, z_2, z_3 lie on a circle through the origin. Show that the points which represent $1/z_1, 1/z_2, 1/z_3$ are collinear.

5. A complex number whose principal argument lies between 0 and π, is represented by the point P. If P lies on the circle $|z| = 1$, show that
(i) $\arg(z+1) = \frac{1}{2}\arg z$ **(ii)** $\arg(z-1) = \arg(z+1) + \frac{\pi}{2}$ **(iii)** $|\frac{z-1}{z+1}| = \tan(\frac{1}{2}\arg z)$

6. If $\arg(+1) = \pi/6$ and $\arg(z-1) = 2\pi/3$, show the points $z, z+1, z-1$ on the Argand diagram and show that $z = \frac{1}{2}(1 + i\sqrt{3})$. [Study geometry of figure.]

*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{AMS}$-TeX.

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7. In the Argand diagram, the points Z_1, Z_2 represent the complex numbers z_1, z_2 respectively. Show that if the triangle OZ_1Z_2 is isosceles and right angled at O, then $z_1^2 + z_2^2 = 0$. [Hint: consider arg and mod of z_1^2/z_2^2]

8. P, Q, R, S are points on the Argand diagram corresponding to the complex numbers z_1, z_2, z_3, z_4 respectively. If O is the origin, prove that the triangles PQO, RSO are directly similar if $z_1/z_2 = z_3/z_4$.

9. If $\alpha, \beta, \gamma;\ \lambda, \mu, \nu$ are six complex numbers and $\frac{\alpha-\gamma}{\alpha-\beta} = \frac{\lambda-\nu}{\lambda-\mu}$; show that the triangles represented by these numbers are similar.

10. If Z_1, Z_2, Z_3 represent the complex numbers z_1, z_2, z_3 prove that $\arg(\frac{z_1-z_3}{z_2-z_3}) = Z_1\hat{Z}_3Z_2$. Prove that the points in the Argand diagram representing the points $az_1 + b, az_2 + b, az_3 + b$ form a triangle similar to the triangle formed by the points representing the complex numbers z_1, z_2, z_3. [a, b are complex numbers]

11. If α, β, γ are 3 complex numbers represented by the points A, B, C on the Argand diagram and $\frac{\alpha-\gamma}{\beta-\gamma} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, prove that the triangle ABC is equilateral, and that $\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \gamma\alpha$. If $\alpha = 1, \beta = i$, find the two possible positions for γ.

12. Prove that if $\frac{z_3-z_2}{z_3-z_1} = \frac{z_3-z_1}{z_2-z_1}$, then the points which represent the complex numbers z_1, z_2, z_3 form an equilateral triangle.

13. A, B, C are the vertices of a triangle in the Argand diagram. If these points represent the complex numbers α, β, γ; what are the points which represent **(i)** $\frac{1}{2}(\alpha+\beta)$ **(ii)** $\frac{1}{3}(\alpha+\beta+\gamma)$ **(iii)** $\frac{1}{4}(3\alpha+\beta)$

14. A, B, C, D represent the complex numbers $\alpha, \beta, \gamma, \delta$. E, F, G, H, P, Q are the midpoints of AB, BC, CD, DA, AC, BD. What numbers are represented by E, G and show that EG, FH, PQ have a common midpoint.

15. The angular points of a triangle are represented by $z = \alpha, z = \beta, z = \gamma$. Show that the centroid is $z = \frac{1}{3}(\alpha+\beta+\gamma)$ and the circumcentre is given by $|z-\alpha| = |z-\beta| = |z-\gamma|$.

16. Show that the points z_1, z_2, z_3 which satisfy the relations $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$ determine an equilateral triangle with vertices on the unit circle.

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17. O, D, P represent the complex numbers $0, 1, z$ respectively. Show that PO is perpendicular to PD if the real part of $\frac{z-1}{z}$ is zero. Deduce that if $z = \frac{1}{1+ir}$, where r is a variable real number, then the point representing z describes a circle of unit diameter.

18. If $z_1 = 2(3+i)$ and $z_2 = 2(1+i)$ and $\arg(\frac{z-z_1}{z-z_2}) = \frac{\pi}{6}$, show that $|z - 2\{2 + i(1+\sqrt{3})\}| = 4$. [Study figure carefully]

19. If z_1, z_2, z_3, z_4 are complex numbers represented by A, B, C, D on the Argand diagram, show that the points A, B, C, D are concyclic or collinear if $\frac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)}$ is real. Show that the points

(i) $1, -1, z, 1/z$ (where z is any complex number) are concyclic.

(ii) $-4+2i, 1-3i, 5+5i, -2+6i$ are concyclic.

20. Prove that on the Argand diagram a general circle can be represented by $|z-a| = r$. What is the locus represented by $\arg(z-k) = \alpha$? State clearly which of the constants a, k, r, α are real and which complex, and what restrictions there are on their ranges of values. The two circles $|z-a| = r$ and $|z-b| = s$ are given. What geometrical relations have the circles **(i)** when $|a-b| = r+s$ **(ii)** when $|a-b| = \sqrt{r^2+s^2}$? **(iii)** Find an inequality which is necessary and sufficient to ensure that the circle $|z-a| = r$ lies entirely inside the circle $|z-b| = s$.

21. If 'a' be a complex number and r, θ are real, show that the point representing z, where r is constant, and $z = a + re^{i\theta}$, lies on a fixed circle, whose centre is 'a', for all values of θ. If T is the length of the tangent from the point representing Z to this circle, and if $Z = a + Re^{i\alpha}$, where R, α are real and $R > r$, show that $\sqrt{T^2+r^2} = |Z-a|$, and that this is independent of α.

22. P, Q, R, S is a parallelogram. H is the point of intersection of the diagonals. If the points P, R, S represent complex numbers $1+3i, 2+6i, 5+7i$ respectively, find the complex numbers giving the points Q, H.

23. E is the centre of a square $ABCD$, lettered anticlockwise. E, A are the points $-2+i, 1+5i$ respectively. Find complex numbers giving the vertices B, C, D.

24. The centroid of an equilateral triangle is at the origin, and one vertex is at the point $\sqrt{3}+i$. Find the complex numbers giving the other vertices.

25. **(i)** If $|z| \leq 1$, find the least and greatest values of **(a)** $|z-1|$ **(b)** $|z-2i|$

(ii) If $|z-1| = 2$, find the least and greatest values of $|z|$

(iii) Find the greatest value of $\arg z$ when $|z-i| = \frac{1}{2}$.

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26. The complex numbers that correspond to the points A, B, C, D are z_1, z_2, z_3, z_4. Interpret geometrically each of the following relations
(i) $z_1 - z_2 + z_3 - z_4 = 0$ **(ii)** $z_1 + 2iz_2 - z_3 - 2iz_4 = 0$
What is the nature of the quadrilateral $ABCD$ if *both* relations are satisfied?

27. If $\frac{|z_1|}{3} = \frac{|z_2|}{4} = \frac{|z_1 - z_2|}{5}$ show that $16z_1^2 + 9z_2^2 = 0$.

28. If the complex numbers z_1, z_2, z_3 are represented in the Argand diagram by the points P, Q, R respectively and the angles of $\triangle PQR$ at Q, R are each $\frac{1}{2}(\pi - \alpha)$, prove that $(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$.

29.
- **(i)** Express $1, \omega, \omega^2$ (the cube roots of unity; see **Exercise 1/67** question **4**) in mod-arg form. Verify that $1.\omega.\omega^2 = 1$ and $1 + \omega + \omega^2 = 0$.
- **(ii)** Show that $1, \omega, \omega^2$ are equally spaced around the unit circle. Hence prove that $|1 - \omega| = |\omega - \omega^2| = |\omega^2 - 1|$.
- **(iii)** If $x = a+b, y = a+\omega b, z = a+\omega^2 b$, prove that $|y-z| = |z-x| = |x-y|$.
- **(iv)** If the points representing z_1, z_2, z_3 form an equilateral triangle (taken in anticlockwise order) prove that $z_1 + \omega z_2 + \omega^2 z_3 = 0$.

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