4 Unit Mathematics

Assignment 1

Question 1.

(i) Find $\frac{dy}{dx}$ if $x^2y + xy^3 = 1$

(ii) Without the use of calculus, sketch $y = x(x-3)(x+2)$ hence or otherwise sketch $y = \frac{1}{x(x-3)(x+2)}$.

(iii) If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$2x^4 - 5x^3 - 7x^2 - 1 = 0$$

find the value of

(a) $\alpha + \beta + \gamma + \delta$

(b) $\alpha\beta\gamma\delta$

(c) $\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}$

(d) $(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)$

(iv) factorise completely over the complex field

$$x^3 - x^2 - 4x - 6.$$

Question 2.

(i) Sketch, on separate diagrams for the domain $-\pi \leq x \leq \pi$

(a) $y = \sin x$

(b) $y = \sin |x|$

(c) $y = |\sin x|$

(d) $y = \sin^2 x$

(ii) $ABCD$ is a parallelogram. X and Y are two points on the diagonal AC such that $AX = CY$. Prove that $DXBY$ is a parallelogram.

(iii) Express in the form $a + ib$

(a) $(3 - 2i)^2$

(b) $\frac{3+2i}{5+2i}$

(c) $(-1 - i)^{10}$

(iv) Find the square roots of $7 + 6\sqrt{2}\ \ i$.

Question 3.

(i) Using De Moivre's theorem. or otherwise, express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin 3\theta$ in terms of $\sin\theta$. Hence, or otherwise, evaluate $\int_0^{\pi/2} \cos^3\theta\, d\theta$.

(ii) Find the cube roots of unity and express them in the form $r(\cos\theta + i\sin\theta)$. Show the roots on the argand diagram.
If ω is one of the complex roots, show that the other complex root is ω^2 and show that $1 + \omega + \omega^2 = 0$.

(iii) Sketch on the argand diagram

(a) the region bounded by

$$1 \leq |z| \leq 3 \ \text{ and } \ \Im(z) \geq 0.$$

(b) $|z - i| = |z - 1|$.

Question 4.

The hyperbola H has cartesian equation $5x^2 - 4y^2 = 20$.

(i) Write down its eccentricity, the co-ordinates of its foci S and S', the equation of the directrices and the equation of the asymptotes.
Sketch the curve, indicating all important features.

(ii) P is an arbitrary point $(2\sec\theta, \sqrt{5}\,\tan\theta)$. Show that the tangent to H at P has equation

$$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1$$

(iii) If this tangent cuts the asymptotes in L and M, prove that $LP = PM$ and the area of ΔOLM is independent of the position of P on H. (O is the origin.)

Question 5.

(i) For what value of c does the following equation represent a hyperbola

$$\frac{x^2}{9-c} + \frac{y^2}{5-c} = 1$$

(ii) Show that the circle on diameter the join of (x_1, y_1) and (x_2, y_2) has equation

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

(iii) Show that the tangents at the points $(cp, \frac{c}{p})$ and $(cq, \frac{c}{q})$ to the rectangular hyperbola $xy = c^2$ meet at the point

$$\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

(iv) Find the equation of the ellipse whose centre is the origin and which has foci at the points $(2, 0)$ and $(-2, 0)$ given that it passes the point $(2, \frac{5}{3})$.

4 Unit Mathematics

Assignment 2

Question 1.

(i) Find $\frac{dy}{dx}$ if $x^4y^2 = 3$.

(ii) Given $z_1 = 6 - i$ and $z_2 = 1 + 3i$ express the following in the form $a + ib$:

(a) z_1^2

(b) $\frac{z_1}{z_2}$

(iii) Find the square roots of $5 - 12i$.

(iv) Sketch the curve $y = \frac{4}{x^2-1}$ showing clearly the turning points and asymptotes.

Question 2.

(i) If $z = 1 + i\sqrt{3}$, plot the following points on the argand diagram:

$$z, \bar{z}, iz, -z, z^{\frac{1}{2}}, z^2, z^4, z + z^2.$$

(ii) Express $(\sin\frac{\pi}{4} + i\cos\frac{\pi}{4})^6$ in the form $a + ib$.

(iii) Find the roots of $z^5 = -1$ in modulus-argument form and plot them on the argand diagram.

Question 3.

(i) Draw a clear sketch to show the locus defined by

(a) $|z - A| = |z - B|$ where $A = 2 + i$ and $B = -3 + 2i$

(b) $1 \leq |z| \leq 4$ and $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$

(ii) Use De Moivre's Theorem to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.
Hence show that $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

Question 4.

(i) For what values of C, does the equation

$$\frac{x^2}{9-c} + \frac{y^2}{5-c} = 1$$

represent a hyperbola?

(ii) For the ellipse $4x^2 + 25y^2 = 100$ find

(a) eccentricity

(b) the co-ordinates of the foci

(c) the equation of the directrices

(d) sketch the ellipse, showing important features.

(iii) For the ellipse $4x^2 + 25y^2 = 100$ find the equation of the normal at the point $P(4, \frac{6}{5})$.
If this normal meets the major axis at T and the minor axis at Q find the area of triangle TOQ (O the origin).

Question 5.

(i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is given by $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

(ii) If the tngent in part **(i)** meets the x axis at T and the perpendicular from P to the x axis meets the x axis at N, show that $ON.OT = a^2$.

(iii) Sketch the curve $y = xe^{-x}$ showing clearly the turning points and points of inflexion.

4 Unit Mathematics

Assignment 3

Question 1.

(i) $\int \frac{2x\,dx}{(x+1)(x+3)}$

(ii) $\int \frac{dx}{x^2-4x+8}$

(iii) $\int \sin^4 x \cos^3 x\,dx$

Question 2.

(i) Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$.

(ii) Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ using the substitution $t = \tan\frac{\theta}{2}$.

Question 3.

If $I_n = \int \sin^n x\,dx$ show that

$$I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}.$$

Hence find $\int_0^{\frac{\pi}{4}} \sin^4 x\,dx$

Question 4.

The base of a certain solid is the region between the curves $y = x$ and $y = x^2$. Each plane perpendicular to the x axis has cross sections which are semi-circles with its diameter in the base of the solid. Find the volume of the solid.

4 Unit Mathematics

Assignment 4

Question 1.

(a) $\int x \ln x \, dx$ **(b)** $\int \frac{4 \, dx}{(x-3)(x-1)}$ **(c)** $\int_{-1}^{3} \frac{x^2 \, dx}{\sqrt{x^3+5}}$ **(d)** $\int \frac{2x^2-5x-11}{x^2-2x-3} \, dx$
(e) $\int e^{2x} \cos x \, dx$

Question 2.

The polynomial $x^4 - 2x^3 + 6x^2 - 8x + 8$ has $(x - 2i)$ as a factor.
Factorise the polynomial completely over the field of complex numbers.

Question 3.

The equation $mx^2 + nx + 3 = 0$ has a root of multiplicity 2. Find a relationship between m and n.

Question 4.

If ρ is a complex root of $x^5 - 1 = 0$:

(a) and if $\theta = \rho + \rho^4$ and $\gamma = \rho^2 + \rho^3$ find the value of $\theta + \gamma$ and $\theta.\gamma$;

(b) find the quadratic equation with roots θ and γ.

Question 5.

(a) Briefly explain four methods which could be used to evaluate the integral $\int_0^1 \sin^{-1} x \, dx$

(b) Evaluate this integral using two of the methods mentioned above.

4 Unit Mathematics

Assignment 5

1. Sketch on the argand diagram the curve described by

$$|z-2| = \Re(z) + 1$$

2. If $z_1 = 1 + 3i$ and $z_2 = 2 - i$ find the locus of z if $|z - z_1| = |z - z_2|$.

3. If $z = x + iy$ sketch the curves represented by

(a) $\Re(z) = 3$

(b) $\Im(\bar{z}) = 1$.

4. If $z_1 = 8 - 3i$ and $z_2 = 5i$ show that the locus of z, where $|z - z_1| = 3|z - z_2|$ is a circle with centre $(-1, 6)$ and radius $\sqrt{18}$.

4 Unit Mathematics

Assignment 6

1. Simplify $\frac{-1+3i}{2-i}$.

2. Find x, y if $\frac{(1+i)^2}{(1-i)^2} + \frac{1}{x+iy} = 1 + i$.

3. Find the square root of $45 + 28i$.

4. Express $(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3})^8$ in the form $a + ib$.

5. Prove by mathematical induction De Moivre's Theorem for positive values of n, and then extend the proof to include negtive values.

6. Using De Moivre's Theorem, find an expression for

(a) $\cos 3\theta$ in terms of $\cos\theta$,
(b) $\sin 3\theta$ in terms of $\sin\theta$.

7. Evaluate $\int_0^{\frac{\pi}{2}} \cos^5\theta \, d\theta$.

8. For the complex number z, define $\bar{z}, |z|, \arg z$.

9. If $z = \sqrt{3} + i$ plot $z, \bar{z}, -z, iz, z^{\frac{1}{2}}, z^2, z^4, z + z^2$ on an argand diagram.

10. Find the 4 roots of the equation $z^4 + 1 = 0$ in $\cos\theta + i\sin\theta$ and $a + ib$ form. Plot the roots on the argand diagram.

11. Factorise $z^4 + 1 = 0$ over the field of real numbers.

12. Indicate graphically the locus of the points given by:

(a) $\Im(z) < 1$,
(b) $|z| = 2$,
(c) $|z + 1| = |z - 1|$,
(d) $\Re(\bar{z} - i) = 2$,
(e) $1 < |z| \leq 4$ and $0 \leq \arg z \leq \frac{\pi}{2}$,
(f) $|z|^2 + 2\Re(z\,\overline{z_0}) + |z_0|^2 = 4$ when $z_0 = 2 + i$.

4 Unit Mathematics

Assignment 7

from Advanced Mathematical Publications

Question 1.

(a) Solve the following equations over the complex field.

(i) $x^2 + 5x + 10 = 0$

(ii) $x^3 + x^2 - 2 = 0$.

(b) Simplify, expressing each answer in the form $a + ib$:

(i) $(i-2)^2 + (i+3)^2$

(ii) $3 - 2i + \frac{1}{2+i}$

(c) Find the modulus and argument of each complex number

(i) $1 - 3i$

(ii) $1 + i\tan\alpha$

(d) If $z = 2 - 3i$ evaluate $\bar{z}, z + 4$ and $\bar{z} - 4$. Plot points, to represent these four complex numbers, in the Argand diagram. Interpret these results geometrically.

Question 2.

(a) Find the square roots of $7 - 24i$.

(b) $ABCD$ is a square described in an anticlockwise sense. If A and B respectively represent $4 - 2i$ and $3 + 2i$, find the complex numbers represented by C and D.

(c) Shade the region in the Argand diagram defined by the inequalities:

$$-\frac{\pi}{4} < \arg z < \frac{\pi}{4} \quad \text{and} \quad |z| \leq 2.$$

(d) If ω is a non-real cube root of unity, evaluate $(1+\omega)^3(1+2\omega+2\omega^2)$. (You may assume that $1+\omega+\omega^2=0$.)

(e) By expanding $(\cos\theta + i\sin\theta)^5$, show that $\sin 5\theta$ may be expressed in the form $a\sin^5\theta + b\sin^3\theta + c\sin\theta$, where a, b and c are constants and find a, b and c.

Question 3.

(a) Use De Moivre's theorem to solve $z^6 = 64$. Show that the points representing the six roots of this equation on an Argand diagram form the vertices of a regular hexagon. Find the area of this regular hexagon.

(b) Solve the equation $x^4 - 3x^3 - 6x^2 + 28x - 24 = 0$ given that it has a triple root.

(c) Use the factor theorem to show that $1+i$ is a zero of the polynomial $P(z) = 2z^3 - 5z^2 + 6z - 2$. Hence factorise the polynomial function over the complex field.

(d) If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

(i) Show that $|z_1 z_2| = |z_1|.|z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.

(ii) Hence deduce the result for $\left|\frac{z_1}{z_2}\right|$ and $\arg\left(\frac{z_1}{z_2}\right)$.

(iii) Using the above properties, find $\left|\frac{1-i\sqrt{3}}{z}\right|$ and $\arg\left(\frac{1-i\sqrt{3}}{z}\right)$.

(e) If $z = \cos\theta + i\sin\theta$,

(i) Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

(ii) Hence show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$.

4 Unit Mathematics

Assignment 8

1. Let $P(x)$ denote the polynomial

$\binom{2k+1}{1}(1-x^2)^k x - \binom{2k+1}{3}(1-x^2)^{k-1}x^3 + \binom{2k+1}{5}(1-x^2)^{k-2}x^5 - \cdots + (-1)^k\binom{2k+1}{2k+1}x^{2k+1}$

where $k \in \mathbb{Z}^+$.

Use de Moivre's theorem to show that $P(\sin\alpha) = \sin(2k+1)\alpha$.

Deduce that $P(x) =$

$(-1)^k 2^{2k} x(x^2 - \sin^2\frac{\pi}{2k+1})(x^2 - \sin^2\frac{2\pi}{2k+1})(x^2 - \sin^2\frac{3\pi}{2k+1})\cdots(x^2 - \sin^2\frac{k\pi}{2k+1})$.

2. Hence show that for any positive integer k,

(i) $\sin\frac{\pi}{2k+1} \cdot \sin\frac{2\pi}{2k+1} \cdot \sin\frac{3\pi}{2k+1} \cdot \cdots \cdot \sin\frac{k\pi}{2k+1} = \frac{\sqrt{2k+1}}{2^k}$

(ii) $\operatorname{cosec}^2\frac{\pi}{2k+1} + \operatorname{cosec}^2\frac{2\pi}{2k+1} + \operatorname{cosec}^2\frac{3\pi}{2k+1} + \cdots + \operatorname{cosec}^2\frac{k\pi}{2k+1} = \frac{2}{3}k(k+1)$

(iii) $\cot^2\frac{\pi}{2k+1} + \cot^2\frac{2\pi}{2k+1} + \cot^2\frac{3\pi}{2k+1} + \cdots + \cot^2\frac{k\pi}{2k+1} = \frac{1}{3}k(2k-1)$.

3. Deduce from these results that if $S_k = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2}$ then

$$\frac{\pi^2}{6}\left(1 - \frac{6k+1}{(2k+1)^2}\right) < S_k < \frac{\pi^2}{6}\left(1 - \frac{1}{(2k+1)^2}\right)$$

and thus deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.