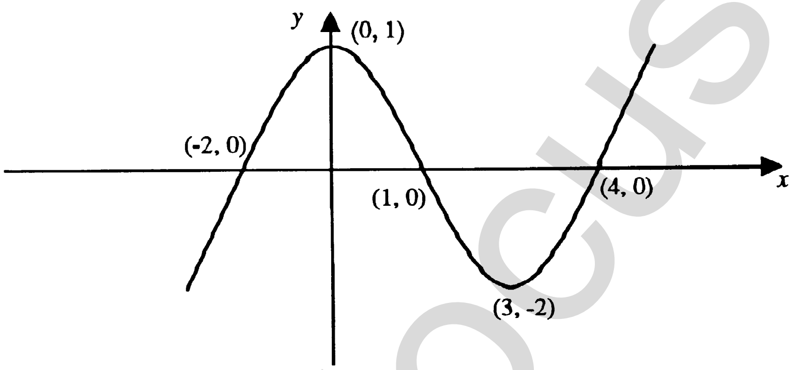
NSW HSC

Mathematics Extension 2 Specimen Paper*

1. **(a)** Find $\int \sin x \cos x \, dx$.
(b) **(i)** Write $\frac{4}{x^2-1}$ as the sum of two fractions. **(ii)** Hence find $\int \frac{4}{x^2-1} \, dx$
(c) Find $\int_0^2 \frac{x \, dx}{\sqrt{x^2+4}}$. **(d)** Find the exact value for $\int_{-2}^{2} \sqrt{4-x^2} \, dx$.
(e) Use integration by parts to evaluate $\int_0^{\pi/2} e^x \cos x \, dx$.

2. **(a)** If $f(x) = x^2 - 4$ sketch the following graphs on separate axes showing all relevant points. **(i)** $y = f(x)$ **(ii)** $y = \frac{1}{f(x)}$ **(iii)** $y = -\frac{4}{f(x)}$
(iv) $y = |f(x)| + 4$ **(v)** $y = \ln f(x)$
(b) The graph below represents $y = g(x)$.



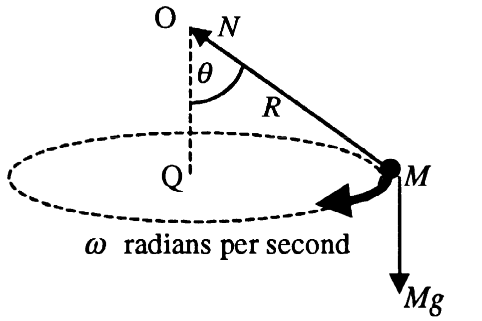
(i) Sketch the graph of $y = g'(x)$
(ii) Sketch the graph of $y = G(x)$ where $G(x) = \int g(x) \, dx$.
(c) Sketch the graph of $y = \frac{x}{\sqrt{x-1}}$ showing all relevant points.

3. **(a)** $16x^2 - 9y^2 - 96x + 128 = 0$ is the equation of an hyperbola. For this hyperbola find the
(i) centre **(ii)** eccentricity **(iii)** equations of the asymptotes **(iv)** foci.
(b) Find the Cartesian equation of the curve whose parametric equations are $x = 2\sin\theta, y = 4\cos\theta, 0 \le \theta \le 2\pi$.
(c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the equation of a hyperbola.
(i) Show that the tangent to this hyperbola at an arbitrary point $P(x_0, y_0)$ is $\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$.
(ii) From an arbitrary point P on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, a vertical line is drawn to cut the asymptote with a positive gradient at A. The normal to the hyperbola at P cuts the X axis at B. Show that $\angle BAO$ is a right angle.

4. (a) If $Z = 1 - 2i$, express the following in the form $a + ib$:
(i) $Z\overline{Z}$ **(ii)** Z^{-1} **(iii)** Z^5
(b) (i) Express $2 - 2\sqrt{3}i$ in the form $r\text{cis}\,\theta$.
(ii) If $Z^4 - 2 + 2\sqrt{3}i = 0$, find all possible values of Z. Give your answers in the form $r\text{cis}\,\theta$.
(c) If $\arg Z_1 \neq \arg Z_2$, show that $|Z_1| + |Z_2| > |Z_1 + Z_2|$
(d) Sketch the region in the Argand diagram where $|Z - 3 + 2i| \leq 2$ and $-\frac{\pi}{3} \leq \arg Z \leq 0$
(e) (i) Find the locus of the complex number Z if $\arg(Z - 1) - \arg(Z + 1) = \frac{\pi}{6}$
(ii) Sketch the locus on an Argand diagram.

5. (a) (i) Evaluate $\tan\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$ **(ii)** Find the domain of $\cos^{-1}(2x - 1)$.
(iii) Sketch the graph of the function $y = \cos^{-1}(2x - 1)$
(iv) Solve $\cos^{-1}(2x - 1) = \sin^{-1} x$
(b) In a cricket game between India and Australia, the probability that India wins the toss is 0.58. When India wins the toss, the team has a probability of 0.6 of winning the game. When Austalia wins the toss, the team has a probabilty of 0.8 of winning the game. (Assume that there are only wins and losses - no draws)
(i) What is the probability that Australia will win the game?
(ii) In a five game series, what is the probability that Australia wins 3 - 2 given that each game begins with a toss? (Assume that there are only wins and losses - no draws)
(iii) If draws are allowed, the probability that Australia and India draw when Australia wins the toss, is 0.05. What is the probability that Australia wins the first 3 games and loses the last 2 games given Austalia wins the toss for each game and the probability that Australia loses a game after winning the toss is 0.2.
(c) The region bounded by the graph $y = x^3$ and the line $y = 8$, is rotated about the Y-axis. Use the method of cylindrical shells to find the volume thus generated.

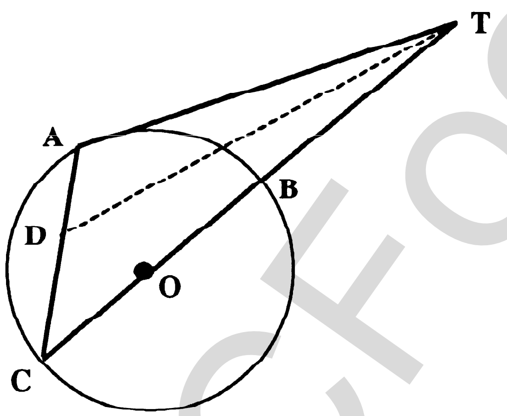
6. (a) For the polynomial $P(x) = (x - a)^r Q(x)$, show that $\frac{dp}{dx}$ has a factor of multiplicity $r - 1$.
(b) (i) The roots of $x^3 + 7x^2 + 3 = 0$ are α, β and γ. Find the polynomial equation whose roots are α^2, β^2 and γ^2.
(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$.
(c) A particle moves with a velocity given by $v = 9\sqrt{1 - x^2}$.
(i) Find the acceleration in terms of x.
(ii) Find the position x at any time t given that $x = 1$ when $t = 0$.
(d) A mass, M, is attached to the end of a light inextensible string of length R. The mass rotates about a vertical line, OQ, with an angular velocity, ω radians per second as shown in the diagram below. $\angle QOM = \theta$ with $\theta \neq 0, \pi$. The tension in the string is N and the force of gravity on M is Mg.



(i) Write an equation for the vertical forces acting on M.
(ii) Find the force acting on M towards the centre Q in terms of ω.
(iii) Show that $\omega^2 = \frac{g}{R\cos\theta}$ **(iv)** Show that $\omega > \sqrt{\frac{g}{R}}$.

7. **(a)** Prove by the method of mathematical induction that $x^3 - 4x^2 + 4x + 3 \geq 0$ where x is a positive integer.
(b) If $y = 3^x$ find $\frac{dy}{dx}$ by letting $3^x = e^a$.
(c) The rate of cooling of a hot object is proportional to the excess of its temperature above that of its surroundings. If the original temperature of the hot object is 100°C and the room temperature is 20°C, find the temperature at the end of 15 minutes if the object cools to 60°C in 10 minutes.
(d) Find all values of θ such that $3\tan^3\theta + 1 = 3\tan^2\theta + \tan\theta$.

8. A circle with centre O has tangent TA touching the circle at A. TO produced cuts the circle at B and C. D is a point on AC such that TD bisects angle ATC. This is shown in the diagram below.



(a) Prove that $\angle TDC = 135°$
(b) A particle of mass m kg is dropped from the top of a building and the force due to air resistance is $0.2mv$ where v is the velocity of the particle in m/sec.
(i) Write an equation for the acceleration of the particle in terms of v.
(ii) Show that the velocity at any time, t seconds, can be given by the equation $v = 5g\left(1 - e^{-\frac{t}{5}}\right)$.
(iii) What is the limiting velocity?
(iv) Find the distance fallen by the particle when $t = 4$ seconds. Give your answer to one decimal place.

Mathematics Plus
Extension 2 Mathematics
HSC Specimen Paper*

Question 1.

(a) Find $\int \sin^3 x \, dx$.

(b) Using the substitution $t = \tan \frac{\theta}{2}$, or otherwise, show that $\int_0^{\pi/2} \frac{1}{1+\sin\theta} \, d\theta = 1$.

(c) Evaluate $\int_0^1 \tan^{-1} x \, dx$

(d) **(i)** Express $\frac{3-x}{(1+2x^2)(1+6x)}$ in partial fractions.

(ii) Show that $\int_0^2 \frac{3-x}{(1+2x^2)(1+6x)} \, dx = \frac{1}{2} \ln \frac{13}{3}$.

Question 2.

(a) Given that $(2+3i)p - q = 1 + 2i$, find p and q if

(i) p and q are real

(ii) p and q are complex conjugate numbers

(b) If $z = \cos\theta + i \sin\theta$, show that $\frac{1}{1+z} = \frac{1}{2}(1 - i \tan \frac{\theta}{2})$.

(c) **(i)** On an Argand diagram, shade in the region for which $0 \leq |z| \leq 2$ and $1 \leq \Im(z) \leq 2$.

(ii) Write down the complex number with largest argument that satisfies the inequalities of **(i)**. Express your answer in the form $a + ib$.

(d) **(i)** Find the two square roots of $5 - 12i$ in the form $x + iy$ where x and y are real.

(ii) Show the points P and Q representing the square roots on an Argand diagram. Find the complex numbers represented by points R_1, R_2 such that the triangles PQR_1 and PQR_2 are equilateral.

Question 3.

(a) The rate of change, with respect to x, of the gradient of a curve is constant and the curve passes through the points $(1, 2)$ and $(-3, 0)$, the gradient at the former point being $-\frac{1}{2}$. Find the equation of the curve and sketch the curve.

(b) For the ellipse $x^2 + 4y^2 = 100$,

(i) Write down the eccentricity, the co-ordinates of the foci and the equations of the directrices.

(ii) Sketch a graph of the ellipse showing the above features.

(iii) Find the equation of the tangent and normal to the ellipse at the point $P(8, 3)$.

(iv) If the normal at P meets the major axis at G and the perpendicular from the centre O to the tangent at P meets the tangent at K, prove that $PG.OK$ is equal to the square of the minor semi-axis.

Question 4.

(a) (i) If $P(x) = x^3 - 9x^2 + 24x + c$ for some real number c, find the values of x for which $P'(x) = 0$. Hence find the two values of c for which the equation $P(x) = 0$ has a repeated root.

(ii) Sketch the graphs of $y = P(x)$ for these values of c. Hence write down the values of c for which the equation $P(x) = 0$ has three distinct real roots.

(b) Let $f(x) = x - 2 + \frac{3}{x+2}$.

(i) Find the points at which $f(x) = 0$.

(ii) Find the turning points of $f(x)$, if any, and identify them.

(iii) Find the asymptotes.

(iv) Sketch the curve, marking all the features you have found in parts **(i)-(iii)** above.

(c) The polynomial $x^3 + x^2 + 3x - 2 = 0$ has roots α, β and γ. Find the equation with roots $\alpha^2\beta\gamma, \alpha\beta^2\gamma$ and $\alpha\beta\gamma^2$.

Question 5.

A particle of mass m is projected vertically upwards under gravity in a medium which exerts a resisting force of magnitude $mg(v/k)^2$, where v is the speed of the particle and k is a constant.

(i) For the upward motion of the particle, draw a diagram showing the forces acting on the particle and write the equation of motion.

(ii) If U is the speed of projection, show that the greatest height of the particle above the point of projection is $\frac{k^2}{2g} \ln(\frac{k^2+U^2}{k^2})$.

(iii) Repeat part **(i)** for the downward motion of the particle and hence write down the particle's terminal velocity.

(iv) If V is the speed of the particle on returning to the point of projection, show that $\frac{1}{V^2} - \frac{1}{U^2} = \frac{1}{k^2}$.

Question 6.

(a) Let $\min(a, b)$ denote the minimum of the numbers a and b. Sketch the function $y = \min(2, x)$ over the interval $0 \leq x \leq 3$ and evaluate $\int_0^3 \min(2, x)\ dx$.

(b) Find the area enclosed between the curves $y = x^3$ and $y^3 = 16x$.

(c) **(i)** Sketch the curves $y = \tan x$ and $y = 2\cos(x + \frac{\pi}{12})$ between $x = 0$ and $x = \frac{\pi}{2}$.

(ii) Verify that $x = \frac{\pi}{4}$ is a solution of the equation $\tan x - 2\cos(x + \frac{\pi}{12}) = 0$.

(iii) Find the area enclosed by these curves and the y-axis.

(iv) If this area is rotated through one revolution about the x-axis, find the volume of the solid formed.

Question 7.

(a) Two circles intersect at A and B. The tangents from a point on BA produced meet the circles at P and Q. If P, A and Q are collinear,

(i) Draw a diagram showing this information.

(ii) Prove that $\triangle TAP ||| \triangle TBP$ and $\triangle TAQ ||| \triangle TBQ$.

(iii) Prove that T, Q, B, P are concyclic.

(iv) Prove that $TP = TQ$.

(b) For a given integer $n \geq 1$, let the positive integers $c_0, c_1, \ldots, c_n$ be defined by the equation $(1+z)^n = c_0 + c_1 z + \cdots + c_n z^n$, valid for all (real and) complex numbers z. (You are **not** required to establish this identity.) Prove that

(i) $c_0 = 1$,

(ii) $c_0 - c_1 + c_2 - c_3 + \cdots + (-1)^n c_n = 0$,

(iii) if n is odd then $c_1 + c_3 + \cdots + c_{n-2} + c_n = 2^{n-1}$,

(iv) if n is divisible by 4 then $c_0 - c_2 + c_4 - \cdots - c_{n-2} + c_n = (-1)^{n/4} 2^{n/2}$.

Question 8.

(a) If the functions $f(x)$ and $g(x)$ are such that $f(x) > g(x) \geq 0$ for $a \leq x \leq b$, by using a sketch (or otherwise) explain why $\int_a^b f(x)\ dx > \int_a^b g(x)\ dx$.

(b) Let $u_n = \int_0^1 (1 - t^2)^{(n-1)/2}\ dt$ where n is a non-negative integer.

(i) Using integration by parts, or otherwise, show that $nu_n = (n-1)u_{n-2}$ if $n \geq 2$.

(ii) Let $v_n = nu_n u_{n-1}, n \geq 1$. Show that $v_n = \frac{1}{2}\pi$, for all values of $n \geq 1$.

(iii) Using part **(a)**, or otherwise, show that $0 < u_n < u_{n-1}$.
Prove that $\sqrt{\frac{\pi}{2n+2}} < u_n < \sqrt{\frac{\pi}{2n}}$.

NSW HSC

Mathematics Extension 2

Two Specimen Papers

Paper 1

1. (a) Consider the function $f(x) = \frac{3x}{(x-1)(4-x)}$.
(i) Express $f(x)$ in partial fractions.
(ii) Find the coordinates and nature of any turning points on the graph $y = f(x)$.
(iii) Sketch the graph $y = f(x)$ showing clearly the coordinates of any turning points and the equations of any asymptotes.
(iv) Find the area of the region bounded by the curve $y = f(x)$ and the x axis between the lines $x = 2$ and $x = 3$.
(b) Consider the function $f(x) = \frac{x-1}{x}$.
(i) Sketch the graph $y = f(x)$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes.
(ii) Use the graph $y = f(x)$ to sketch on separate axes the graphs
(α) $y = |f(x)|$ **(β)** $y = f(|x|)$ **(γ)** $y = \{f(x)\}^2$

2. (a) (i) Find $\int \sqrt{e^x}\,dx$ **(ii)** Find $\int (\cos^2 x - \sin^2 x)\,dx$
(b) Find $\int \frac{x+1}{x-1}\,dx$ using the substitution $u = x - 1$.
(c) Evaluate $\int_1^2 \frac{1}{e^x - 1}\,dx$
(d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + 2}\,dx$ using the substitution $t = \tan\frac{x}{2}$
(e) (i) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, $n = 0, 1, 2, \ldots$ show that $I_n + I_{n-2} = \frac{1}{n-1}$, $n = 2, 3, 4, \ldots$
(ii) Hence find the value of $\int_0^{\frac{\pi}{4}} \tan^5 x\,dx$

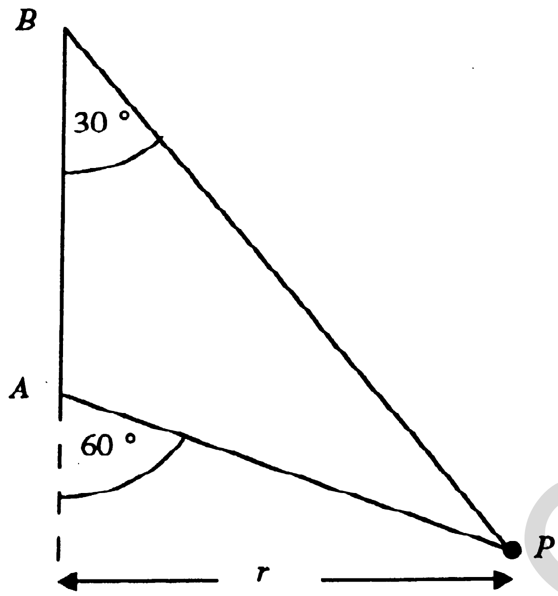
3. (a) (i) Show that $z = 1 + i$ is a root of the equation $z^2 - (3 - 2i)z + (5 - i) = 0$
(ii) Find the other root of the equation.
(b) (i) Expand $(1 + ic)^5$ in ascending powers of c.
(ii) Find the real values of c for which $(1 + ic)^5$ is real.
(c) (i) Express each of $z_1 = -\sqrt{2} + \sqrt{2}i$ and $z_2 = \sqrt{3} + i$ in modulus/argument form.
(ii) Find the exact value of $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1 + z_2)$.
(d) (i) On an Argand diagram sketch the graph of $|z - (\sqrt{2} + \sqrt{2}i)| = 1$.
(ii) If z satisfies $|z - (\sqrt{2} + \sqrt{2}i)| = 1$, find the set of possible values of $|z|$ and $\arg z$.

4. **(a)** **(i)** Show that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ in the first quadrant has equation $bx\cos\theta + ay\sin\theta - ab = 0$.
(ii) This tangent cuts the x axis at point A and the y axis at point B. Find the minimum area of $\triangle AOB$ and show that when this occurs P is the midpoint of AB.
(b) **(i)** Show that the tangent to the rectangular hyperbola $xy = 4$ at the point $T\big(2t, \frac{2}{t}\big)$ has equation $x + t^2y = 4t$.
(ii) This tangent cuts the x axis at point Q. Show that the line through Q which is perpendicular to the tangent at T has equation $t^2x - y = 4t^3$.
(iii) This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has coordinates $M(2t, -2t^3)$.
(iv) Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply.

5. **(a)** **(i)** If $x = \alpha$ is a double root of the equation $P(x) = 0$, show that $x = \alpha$ is a root of the equation $P'(x) = 0$.
(ii) If the equation $x^4 - (p+q)x^3 + (p-q)x - 1 = 0$ has a double root, show that $p^{\frac{2}{3}} - q^{\frac{2}{3}} = 2^{\frac{2}{3}}$.
(b) **(i)** Verify that $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a root of the equation $z^5 + z - 1 = 0$.
(ii) Find the monic cubic equation with real coefficients whose roots are also roots of $z^5 + z - 1 = 0$ but do not include α.

6. **(a)** **(i)** Sketch the graph of the curve $y = x + e^{-x}$ showing clearly the coordinates of any turning points and the equations of any asymptotes.
(ii) The region in the first quadrant between the curve $y = x + e^{-x}$ and the line $y = x$ and bounded by the line $x = 1$ is rotated through one complete revolution about the y axis. Use the method of cylindrical shells to show that the volume V of the solid of revolution is given by $V = 2\pi \int_0^1 xe^{-x}\, dx$ and hence find the volume of the solid.

(b)



A 1 kg mass P is attached by two strings to points A and B where B is vertically above A. The mass describes uniform circular motion with angular velocity ω in a horizontal circle of radius r with both strings taut.

(i) Draw a diagram showing the forces acting on P.

(ii) Find the tension in each string and hence find the set of possible values of ω.

7. (a) (i) Find the value of $1 + 10 + 10^2 + \cdots + 10^n$.

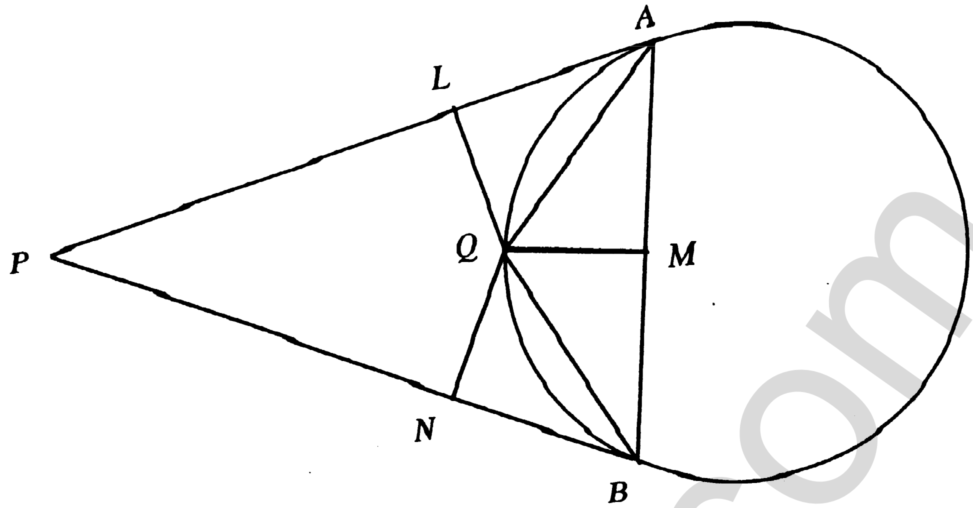
(ii) Use the method mathematical induction to show that
$(1 \times 9^2) + (11 \times 9^2) + (111 \times 9^2) + \cdots + (\underbrace{111 \cdots 1}_{n \text{ ones}} \times 9^2) = 10^{n+1} - 9n - 10$ for positive integers $n \geq 1$.

(b) (i) Show that for alol values of a and b, $\sin a \sin b \leq [\sin \frac{1}{2}(a+b)]^2$.

(ii) Hence show that if $\sin a > 0, \sin > 0, \sin c > 0, \sin d > 0$, then $\sin a \sin b \sin c \sin d \leq [\sin \frac{1}{4}(a+b+c+d)]^4$

(iii) By choosing a suitable value for d show that $\sin a \sin b \sin c \leq [\sin \frac{1}{3}(a+b+c)]^3$

8. (a)



In the diagram PA and PB ate tangents from P to the circle. Q is a point on the minor arc AB. QL, QM and QN are the perpendiculars from Q to PA, AB and PB respectively.

(i) Copy the diagram showing thye above information.

(ii) Show that $\triangle ALQ|||\triangle BMQ$ and $\triangle BNQ|||\triangle AMQ$

(iii) Hence show that QL. QM and QN are consecutive terms in a geometric sequence.

(b) A driver travels to work each day on a route that has five trafic lights. Each traffic light is green to the driver one quarter of the time and red to the driver three quarters of the time. If all of the traffic lights are green the journey takes ten minutes. Each time a traffic light is red it adds two minutes to the journey.

(i) Find the most likely time it will take the driver to get to work each day.

(ii) Find the probability (correct to four decimal places) that on any particular day it will take the driver at least fifteen minutes to get to work.

(iii) Find the probability (correct to four decimal places) that on any particular day it will take the driver at least 15 minutes to get to work given that at most four of the traffic lights are red.

(iv) Find the probability (correct to two decimal places) that in any particular week (five days) it will take the driver at least fifteen minutes to get to work on exactly three days.

Paper 2

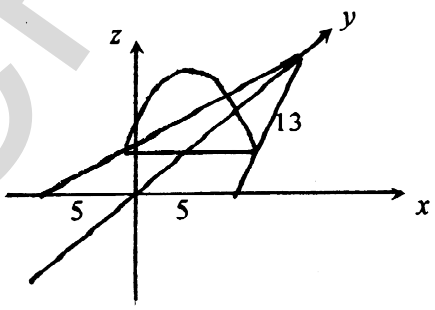
1. (a) (i) Find $\int \frac{dx}{x^2-6x+13}$ **(ii)** $\int \frac{x\ dx}{(x-1)^3(x+1)}$
(b) Leaving your answer in exact form, evaluate **(i)** $\int_{-1}^{2} x\sqrt{2-x}\,dx$ **(ii)** $\int_0^1 x^2e^{-x}\,dx$
(c) Given $I_{2n+1} = \int_0^1 x^{2n+1}e^{x^2}\ dx$, where n is a positive integer, show that $I_{2n+1} = \frac{1}{2}e - nI_{2n-1}$. Hence, or otherwise, evaluate $\int_0^1 x^5e^{x^2}\ dx$.

2. (a) Given the complex number $\omega = \frac{5+3i}{2-i}$, find **(i)** $\overline{\omega}$ **(ii)** $\omega\overline{\omega}$ **(iii)** $|\omega|$
(b) Express $z = \frac{\sqrt{2}}{1-i}$ in modulus-argument form and hence find z^5 in the form $x+iy$
(c) (i) Find the fifth roots of $\sqrt{3}+i$ **(ii)** Show the roots on an Argand diagram
(iii) Find the area of the pentagon formed by the roots.
(d) Show that $(1+\cos 2\theta + i\sin 2\theta)^n = 2^n\cos^n\theta(\cos n\theta + i\sin n\theta)$.

3. Sketch the following curves on separate axes, showing all intercepts and turning points.
(a) $y = \sin x$, and hence $y^2 = \sin x$ (in the domain: $-2\pi \le x \le 2\pi$)
(b) $y = x^3 - x$, and hence $y = |x|^3 - |x|$ (in the domain: $-2 \le x \le 2$)
(c) $y = x\tan x$ (in the domain: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$)
(d) $y = \frac{x^2-4}{x^2+2x-3}$ (in the domain: $-4 \le x \le 3$)

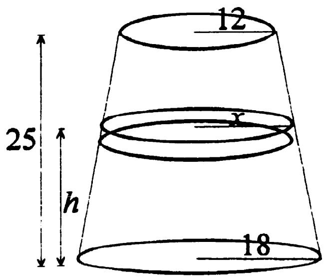
4. (a) Derive the equation of the tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point (x_1, y_1) and hence deduce that the equation of the chord of contact to this hyperbola from an external point $E(x_0, y_0)$ is $\frac{xx_0}{16} - \frac{yy_0}{9} = 1$. If the chord of contact passes through a focus, show that E lies on a directrix.
(b) Let $C_1 \equiv 3x^2 + y^2 - 1$ and $C_2 \equiv 7x^2 + 11y^2 - 3$ and let k be a real number.
(i) Show that $C_1 + kC_2 = 0$ is the equation of a curve passing through the points of intersection of the ellipses $C_1 = 0$ and $C_2 = 0$.
(ii) Determine the values of k for which $C_1 + kC_2 = 0$ is the equation of an ellipse.

5. (a)



A solid shape has a triangular base with sides 13 cm, 13 cm, and 10 cm. Each cross-section (perpendicular to the axis of symmetry of the base) is in the shape of a parabola, with its latus rectum lying in the base. Find the volume of the solid.

(b)



Calculate the volume of the frustrum of a cone, with radii of the top and bottom circles being 12 cm and 18 cm respectively. The height of the frustrum is 25 cm.

6. (a) Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ if it has a root of multiplicity 4
(b) Find the condition that the roots of $x^3 + kx^2 + lx + m = 0$ are in geometric progression.
(c) The equation $x^4 + 3x^3 + 5x^2 - 7x + 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equation with roots $\alpha^2, \beta^2, \gamma^2, \delta^2$.
(d) Solve $x^4 + x^3 - 16x^2 - 4x + 48 = 0$, given that the product of two of the roots is 6.

7. (a) A sequence $\{T_n\}$ is such that $T_1 = 3, T_2 = 5$ and $T_{n+2} = 3T_{n+1} - 3T_n$. Prove by mathematical induction that $T_n = 3^{n-1} + 2$.
(b) How many different sums of money can be made up from eight \$50 notes, four \$20 notes, three \$10 notes, one \$5 note, ten \$2 coins, and fifteen 50¢ pieces?
(c) Find an expansion for $\tan(A + B + C)$.
(d) If $\hat{A}$, $\hat{B}$, and $\hat{C}$ are the angles of a triangle show that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

8. (a) Six letters are chosen from the letters of the word MATHEMATICS. These six letters are then placed alongside each other to form a six-letter arrangement. Find the number of distinct six-letter arrangements which are possible, considering all the choices.
(b) Show that $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$
(c) (i) A particle of mass M is projected downwards under gravity in a medium whose resistance is equal to the velocity of the particle multiplied by $\frac{Mg}{V}$. Show that the velocity tends to the value V.
(ii) A particle is projected vertically upwards in the above medium with velocity U. Show that it reaches a height $\frac{UV}{g} + \frac{V^2}{g} \ln\left(\frac{V}{U+V}\right)$.