Question 1

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• uses the factor theorem to write an equation for k	1
• solves for k	1

Answer

$(x-2)$ is a factor $\therefore P(2)=0$. Hence $8+4+k=0$. $\therefore k=-12$

b. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• writes a numerical expression for $\tan\theta$	1
• finds the angle to the nearest degree.	1

Answer

The lines have gradients 3 and -1. $\therefore \tan\theta = \left|\dfrac{3-(-1)}{1+3\times(-1)}\right| = 2$ $\therefore \theta \approx 63°$

c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• realises that the tallest boy and girl can be positioned in two ways	1
• includes the factor 4! to arrange the remaining students	1

Answer

• • ∗ ∗ • • Position the tallest boy and girl (∗'s) in 2 ways , then the others (•'s) in 4! ways.
Hence number of arrangements is $2\times 4! = 48$.

d. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
• quotes the derivative of the inverse tan function	1
• applies the product rule	1

Answer

$$\frac{d}{dx}\left(e^x \tan^{-1} x\right) = e^x \tan^{-1} x + \frac{e^x}{1+x^2}$$

1e. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • explains why *AECF* is cyclic	1
ii • quotes property of cyclic quad *ABCD* to deduce $\angle BDC = \angle BAC$	1
• quotes property of cyclic quad *AECF* to deduce $\angle EAC = \angle EFC$	1
• identifies equal corresponding angles to deduce $BD \parallel EF$	1

Answer

i. In quadrilateral *AECF*, opposite interior angles at *A* and *C* are supplementary. Hence *AECF* is cyclic.

ii. $\angle BDC = \angle BAC$ ($\angle$'s *subtended at the circumference of circle ABCD by same arc BC are equal*)
$\angle EAC = \angle EFC$ ($\angle$'s *subtended at the circumference of circle AECF by same arc EC are equal*)
$\therefore \angle BDC = \angle EFC$ ($\angle BAC$, $\angle EAC$ *same angle, since B, E, A collinear*)
$\therefore BD \parallel EF$ (*equal corresponding* $\angle$'s *on transversal CF*)

Question 2
a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• rearranges subject of limit into appropriate form	1
• applies known limit	1

Answer

$$\lim_{x\to 0}\frac{\sin 3x}{2x} = \frac{3}{2}\lim_{x\to 0}\frac{\sin 3x}{3x} = \tfrac{3}{2}\times 1 = \tfrac{3}{2}$$

b. Outcomes assessed : P4, PE3

Marking Guidelines

Criteria	Marks
• finds one inequality for *x*	1
• correctly combines this with a second inequality to quote all the solutions	1

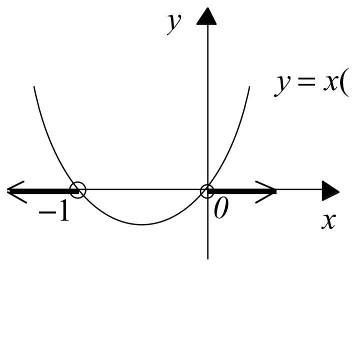
Answer

Multiplying both sides by x^2,

$\frac{x+1}{x} > 0 \Rightarrow x(x+1) > 0$ and $x \neq 0$

$\therefore x < -1$ *or* $x > 0$

(by inspection of the graph)



2c. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• finds the x coordinate	1
• finds the y coordinate	1

Answer

$$A(-2\,,5) \qquad B(7\,,-1)$$

$$2 \quad : \quad 1$$

$$P\left(\frac{1\times(-2)+2\times 7}{2+1}\,,\ \frac{1\times 5+2\times(-1)}{2+1}\right) \qquad \therefore P(4\,,1)$$

d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• uses an appropriate trigonometric identity	1
• finds the primitive function	1

Answer

$$\int \cos^2 2x \;\, dx = \tfrac{1}{2}\int \left(1+\cos 4x\right)\; dx = \tfrac{1}{2}x+\tfrac{1}{8}\sin 4x + c$$

e. Outcomes assessed : PE2, PE4

Marking Guidelines

Criteria	Marks
i • finds gradient of normal at P	1
• finds gradient of QP	1
ii • finds and simplifies x and y coordinates of M in terms of t	1
• eliminates t to find Cartesian equation of locus of M	1

Answer

i. At the general point $(2t\,,t^2)$

$$y=t^2 \;\Rightarrow\; \frac{dy}{dt}=2t \qquad \therefore \frac{dy}{dx}=\frac{2t}{2}=t$$

$$x=2t \;\Rightarrow\; \frac{dx}{dt}=2$$

Hence gradient of normal at P is $\dfrac{-1}{t}$.

Join of $P\left(2t\,,t^2\right)$ and $Q\left(-2(t+\tfrac{2}{t})\,,(t+\tfrac{2}{t})^2\right)$ has gradient

$$\frac{(t+\frac{2}{t})^2-t^2}{-2(t+\frac{2}{t})-2t}=\frac{4+4(\frac{1}{t})^2}{-4t-4(\frac{1}{t})}$$

$$=\frac{4(t^2+1)}{-4t(t^2+1)}$$

$$=\frac{-1}{t}$$

$\therefore QP$ is normal to the parabola at P.

2e ii. At M,

$$x=\tfrac{1}{2}\left\{2t-2\left(t+\tfrac{2}{t}\right)\right\}=-\tfrac{2}{t}$$

$$y=\tfrac{1}{2}\left\{t^2+\left(t+\tfrac{2}{t}\right)^2\right\}=t^2+2+\tfrac{2}{t^2}$$

$\therefore$ locus of M has equation

$$y=\tfrac{4}{x^2}+2+\tfrac{1}{2}x^2$$

Question 3

a. Outcomes assessed : P5, H6

Marking Guidelines

Criteria	Marks
i • formally proves f is odd	1
ii • finds $f'(x)$ and considers sign to deduce f is decreasing	1
iii • shows branches of curve for $\lvert x\rvert>1$ with asymptotes $x=1,\ x=-1,\ y=0$	1
• shows branch of curve for $\lvert x\rvert<1$	1

Answer

i. $f(-x)=\dfrac{3(-x)}{(-x)^2-1}=-\dfrac{3x}{x^2-1}$

$\therefore f(-x)=-f(x)$ for all $x\neq\pm1$

$\therefore f(x)$ is an odd function.

ii.

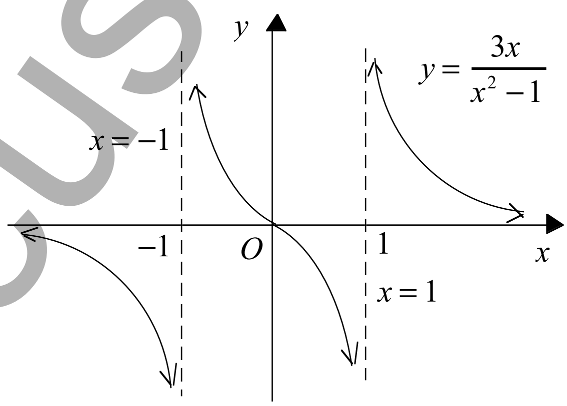
$$f'(x)=3\ \frac{1.(x^2-1)-x.2x}{(x^2-1)^2}$$

$$=\frac{-3(x^2+1)}{(x^2-1)^2}$$

$\therefore f'(x)<0$ for $x\neq\pm1$

$\therefore f(x)$ is decreasing throughout its domain.

iii.



b. Outcomes assessed : H3, H5

Marking Guidelines

Criteria	Marks
i • writes one of cosec x or cot x in terms of t	1
• writes the second reciprocal trig. ratio in terms of t, then simplifies sum to obtain result	1
ii • finds primitive function	1
• evaluates, giving surd values for trigonometric ratios	1
• uses log laws to obtain simplest exact answer.	1

Answer

i. $t=\tan\frac{x}{2}\ \Rightarrow\ \operatorname{cosec}x+\cot x=\dfrac{1+t^2}{2t}+\dfrac{1-t^2}{2t}=\dfrac{1+t^2+1-t^2}{2t}=\dfrac{1}{t}$

$\therefore \operatorname{cosec}x+\cot x=\cot\frac{x}{2}$

3b ii. Let $I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\operatorname{cosec} x + \cot x\right) dx$

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot \tfrac{x}{2}\, dx$$
$$= 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\frac{1}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}\, dx$$
$$= 2\left[\ln\left(\sin\tfrac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$\therefore I = 2\left\{\ln\left(\sin\tfrac{\pi}{4}\right) - \ln\left(\sin\tfrac{\pi}{6}\right)\right\}$$
$$= 2\left(\ln\tfrac{1}{\sqrt{2}} - \ln\tfrac{1}{2}\right)$$
$$= 2\ln\left(2^{-\frac{1}{2}} \div 2^{-1}\right)$$
$$= 2\ln 2^{\frac{1}{2}}$$
$$= \ln 2$$

c. Outcomes assessed : HE5, HE7

Marking Guidelines

Criteria	Marks
• writes the area A as a function of t and differentiates • finds either t or $\sqrt{t+1}$ when $r = 1$ • substitutes for t to find the required rate to stated accuracy, giving units	1 1 1

Answer

$$A = \pi\left\{\sqrt{t+1} - 1\right\}^2$$
$$\frac{dA}{dt} = 2\pi\left\{\sqrt{t+1} - 1\right\}.\tfrac{1}{2}(t+1)^{-\frac{1}{2}}$$
$$= \pi\ \frac{\sqrt{t+1} - 1}{\sqrt{t+1}}$$

$$r = 1 \ \Rightarrow\ \sqrt{t+1} - 1 = 1$$
$$\sqrt{t+1} = 2$$
$$\therefore\ \frac{dA}{dt} = \pi\ \frac{2-1}{2} = \frac{\pi}{2}$$

Ans. $1\cdot 57$ km/h

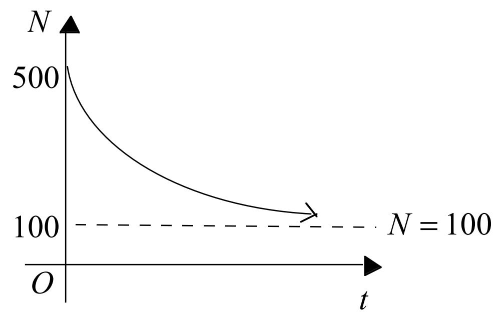
Question 4
a. Outcomes assessed : H3, HE3

Marking Guidelines

Criteria	Marks
i • shows vertical intercept 500 with decreasing function • shows curve approaching asymptote $N = 100$ as $t \to \infty$ ii • finds value of $e^{-0\cdot 1\,t}$ • solves equation for t	1 1 1 1

Answer

i. $N = 100 + 400\, e^{-0\cdot 1\, t}$
$t = 0 \quad \Rightarrow\ N = 500$ and $t \to \infty\ \Rightarrow\ N \to 100$



ii. $N = 250\ \Rightarrow\ 400 e^{-0\cdot 1\, t} = 150$
$$e^{-0\cdot 1\, t} = \tfrac{3}{8}$$
$$-\tfrac{1}{10}\, t = \ln\tfrac{3}{8}$$
$\therefore t = -10\ln\tfrac{3}{8} \approx 9\cdot 8$
Ans. 10 years (to the nearest year)

4b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes integral in terms of u	1
• rearranges integrand into sum of index expressions in u	1
• finds primitive function	1
• evaluates	1

Answer

$u = x+1$

$du = dx$

$x = 0 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 4$

$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_1^4 \frac{u-1}{\sqrt{u}}\,du$$
$$= \int_1^4 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du$$
$$= \left[\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$$
$$= \tfrac{2}{3}(4^{\frac{3}{2}} - 1) - 2(4^{\frac{1}{2}} - 1)$$
$$= 2\tfrac{2}{3}$$

c. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• defines an appropriate sequence of statements to be tested by Mathematical Induction	1
• verifies that the first statement is true	1
• shows that the truth of the k^{th} statement implies the truth of the next statement in the sequence	1
• completes the process of Mathematical Induction	1

Answer

Let $S(n)$, $n = 2, 3, 4, \ldots$ be the sequence of statements defined by $S(n)$: $3^n - 2n - 1 = 4I$ for some integer I.

Consider $S(2)$: $3^2 - 2\times 2 - 1 = 4\times 1$ Hence $S(2)$ is true.

If $S(k)$ is true, $k \ge 2$, $3^k - 2k - 1 = 4I$ for some integer I. *

Consider $S(k+1)$: $3^{k+1} - 2(k+1) - 1 = 3\left\{3^k - 2k - 1\right\} + 4k$

$= 3\times 4I + 4k$ if $S(k)$ is true, using *

$= 4\,(3I + k)$ where $(3I + k)$ is an integer.

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(2)$ is true, hence $S(3)$ is true and then $S(4)$ is true and so on.

$\therefore S(n)$ is true, ie. $3^n - 2n - 1$ is divisible by 4, for all positive integers $n \ge 2$.

Question 5

a. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i • finds domain	1
• finds range	1
ii • sketches curve with correct shape, position and intercepts on the coordinate axes	1
iii • finds the equation of the inverse function	1

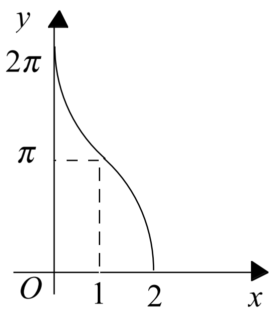
Answer

i. $f(x) = 2\cos^{-1}(x-1)$

$-1 \le x - 1 \le 1$ and $0 \le \cos^{-1}(x-1) \le \pi$

$\therefore$ Domain: $\{x : 0 \le x \le 2\}$ Range : $\{y : 0 \le y \le 2\pi\}$

ii.



iii. $0 \le y \le 2\pi$ and

$$y = 2\cos^{-1}(x-1)$$
$$\tfrac{1}{2}y = \cos^{-1}(x-1)$$
$$\cos(\tfrac{1}{2}y) = x-1$$
$$x = 1 + \cos(\tfrac{1}{2}y)$$

Hence, interchanging x and y,

$$f^{-1}(x) = 1 + \cos(\tfrac{1}{2}x), \quad 0 \le x \le 2\pi$$

b. Outcomes assessed : H5, HE3

Marking Guidelines

Criteria	Marks
i • recognises that there are 4! ways for 4 people to choose different restaurants	1
• divides by 4^4 to find the probability	1
ii • recognises the binomial distribution and writes an expression for the probability	1
• calculates the probability	1

Answer

i. $\dfrac{4!}{4^4} = \dfrac{4\times3\times2\times1}{4\times4\times4\times4} = \dfrac{3}{32}$

ii. Probability that a person chooses restaurant A is $\frac{1}{4}$

$$P(exactly\ 2\ choose\ A) = {}^4C_2(\tfrac{1}{4})^2(\tfrac{3}{4})^2 = \tfrac{27}{128}$$

c. Outcomes assessed : HE5

Marking Guidelines

Criteria	Marks
i • notes that initially v, a have opposite signs to deduce that particle is slowing down	1
ii • writes a as a derivative with respect to x and integrates to find expression for v^2	1
• uses initial conditions to evaluate the constant of integration	1
iii • considers $v = 0$ to find that particle changes direction when it reaches $x = 1$	1

5c. Answer

i. $a = x + \frac{3}{2}$. Initially $x = 5$ and $v = -6$

$\therefore a = \frac{13}{2} > 0$ and $v < 0$. Hence particle is slowing down.

ii. $$\frac{d(\frac{1}{2}v^2)}{dx} = x + \frac{3}{2}$$
$$\tfrac{1}{2}v^2 = \tfrac{1}{2}x^2 + \tfrac{3}{2}x + c_1$$
$$v^2 = x^2 + 3x + c$$
$$\left.\begin{array}{r} x = 5 \\ v = -6 \end{array}\right\} \Rightarrow \begin{array}{l} 36 = 25 + 15 + c \\ \therefore c = -4 \end{array}$$
$$\therefore v^2 = x^2 + 3x - 4$$

iii. $v = 0 \Rightarrow (x+4)(x-1) = 0$

Particle starts at $x = 5$ and moves left towards O.
When particle reaches $x = 1$, $v = 0$ and $a = \frac{5}{2} > 0$.
Hence particle is instantaneously at rest at $x = 1$, then moves off to the right and subsequently continues moving right while speeding up ($v > 0$ and $a > 0$)
Hence only change of direction is at a point 1 m to the right of *O*.

Question 6

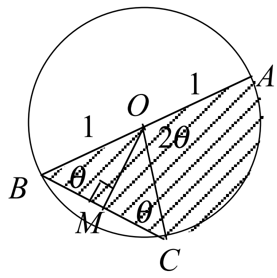
a. Outcomes assessed : H5, PE3, HE1

Marking Guidelines

Criteria	Marks
i • finds BC in terms of θ	1
• uses perimeter in terms of θ to write equation	1
ii • shows that $f(0\cdot 8)$, $f(0\cdot 9)$ have opposite signs	1
• notes that f is continuous and makes required deduction about θ	1
iii • finds $f'(\theta)$ and substitutes into Newton's formula	1
• calculates next approximation.	1

Answer

i.



Let *M* be the foot of the perpendicular form *O* to *BC*.
Then $BC = 2BM = 2\cos\theta$ (*radius* $\perp$ *chord bisects chord*)

In ΔOBC , $\angle OCB = \angle OBC = \theta$ (*equal* $\angle$'*s opp. equal sides*)
$\therefore \angle AOC = 2\theta$ (*ext.* $\angle$ *is sum of int. opp.* $\angle$'*s*)
Hence *Arc* $AC = 2\theta$

Perimeter is 5 cm $\therefore 2\theta + 2\cos\theta + 2 = 5$
$$\theta + \cos\theta - 1\cdot 5 = 0$$

ii. Let $f(\theta) = \theta + \cos\theta - 1\cdot 5$.
Then $f'(\theta) = 1 - \sin\theta$. We want roots of $f(\theta) = 0$.
f is a continuous, increasing function such that $f(0\cdot 8) \approx -0\cdot 003 < 0$ and $f(0\cdot 9) \approx 0\cdot 022 > 0$
Hence there is exactly one root θ and $0\cdot 8 < \theta < 0\cdot 9$.

iii. Applying Newton's method with initial approximation $\theta_0 = 0\cdot 85$, next approximation is

$$\theta \approx 0\cdot 85 - \frac{0\cdot 85 + \cos 0\cdot 85 - 1\cdot 5}{1 - \sin 0\cdot 85} \approx 0\cdot 85 - \frac{0\cdot 009983}{0\cdot 248720} \approx 0\cdot 81$$

6b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • differentiates once to find $\dot{x}$	1
• differentiates a second time and rearranges to obtain required result	1
ii • shows curve with correct position, shape and endpoints	1
• shows intercepts on t axis	1
iii • states time when v increases most rapidly	1
• states maximum rate of increase	1

Answer

i. $x = 4\cos^2 t - 1$

$= 2(1+\cos 2t) - 1$

$= 1 + 2\cos 2t$

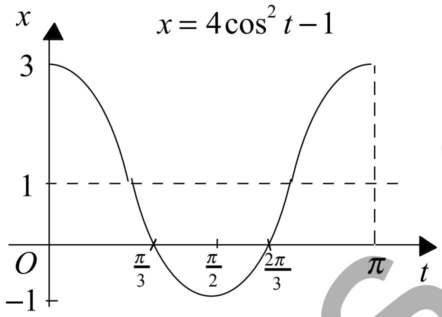
$\dot{x} = -4\sin 2t$

$\ddot{x} = -8\cos 2t$

$= -4(2\cos 2t)$

$= -4(x-1)$

ii.



iii. v is increasing most rapidly when $\ddot{x}$ takes its greatest positive value, ie. when x takes its least value.
This extreme value of $\ddot{x}$ is $-4(-1-1) = 8$ when $t = \frac{\pi}{2}$ and $x = -1$.
Hence v increases most rapidly at time $\frac{\pi}{2}$ seconds and this maximum rate of increase is 8 ms^{-2}.

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • finds expression for x by integration	1
• finds expression for y by integration	1
ii • finds t at greatest height	1
• finds x and y in terms of V, α for this t value	1
• uses the gradient of OP to establish result	1
iii • finds $\tan\alpha$ and hence the exact value of α	1
• finds the exact value of $\sin\alpha$ and writes an equation for V^2	1
• finds the exact value of V	1

Answer

i. $t = 0 \Rightarrow x = y = 0, \quad \dot{x} = V\cos\alpha, \ \dot{y} = V\sin\alpha$

$\ddot{x} = 0 \Rightarrow \dot{x}$ is constant

$\therefore \dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + c_1, \quad c_1$ constant

$t = 0, \ x = 0 \Rightarrow c_1 = 0$

$\therefore x = Vt\cos\alpha$

$\ddot{y} = -10 \Rightarrow \dot{y} = -10t + c_2, \quad c_2$ constant

$t = 0, \dot{y} = V\sin\alpha \Rightarrow c_2 = V\sin\alpha \qquad \therefore \dot{y} = -10t + V\sin\alpha$

$y = -5t^2 + Vt\sin\alpha + c_3, \quad c_3$ constant

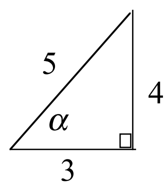
$t = 0, \ y = 0 \Rightarrow c_3 = 0 \qquad \therefore y = Vt\sin\alpha - 5t^2$

7a ii. At greatest height, $\dot{y}=0 \Rightarrow 10t = V\sin\alpha \qquad \therefore t=\frac{1}{10}V\sin\alpha$

$\therefore$ at point P, $\quad x=\frac{1}{10}V^2\sin\alpha\cos\alpha \quad$ and $\quad y=(\frac{1}{10}-\frac{5}{100})V^2\sin^2\alpha=\frac{1}{20}V^2\sin^2\alpha$

Then $\tan\beta = gradient\ OP = \dfrac{\frac{1}{20}V^2\sin^2\alpha}{\frac{1}{10}V^2\sin\alpha\cos\alpha} = \frac{1}{2}\tan\alpha$

iii. $P(120\,,\ 80) \Rightarrow \tan\beta=\frac{80}{120} \Rightarrow \frac{1}{2}\tan\alpha=\frac{2}{3} \qquad \therefore \tan\alpha=\frac{4}{3}$



$$80=\tfrac{1}{20}V^2\sin^2\alpha=\tfrac{1}{20}V^2\times\tfrac{16}{25} \qquad \therefore V^2=2500$$

$$\therefore \alpha=\tan^{-1}\tfrac{4}{3}\ ,\quad V=50$$

b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • writes expansion, with or without using sigma notation	1
ii • substitutes $x=1$	1
• rewrites sum using only binomial coefficients $\binom{2n+1}{r}$, $0\le r\le n$	1
• deduces required result	1

Answer

i. $(1+x)^{2n+1}=\sum_{r=0}^{2n+1}\binom{2n+1}{r}x^r=\binom{2n+1}{0}+\binom{2n+1}{1}x+\ldots+\binom{2n+1}{n}x^n+\binom{2n+1}{n+1}x^{n+1}+\ldots+\binom{2n+1}{2n+1}x^{2n+1}$

ii. Substituting $x=1$, $\quad 2^{2n+1}=\binom{2n+1}{0}+\binom{2n+1}{1}+\ldots+\binom{2n+1}{n}+\binom{2n+1}{n+1}+\ldots+\binom{2n+1}{2n}+\binom{2n+1}{2n+1}$

$$2^{2n+1}=\binom{2n+1}{0}+\binom{2n+1}{1}+\ldots+\binom{2n+1}{n}+\binom{2n+1}{n}+\ldots+\binom{2n+1}{1}+\binom{2n+1}{0}$$

using $\binom{2n+1}{k}=\binom{2n+1}{2n+1-k}$

$$2^{2n+1}=2\sum_{r=0}^{n}\binom{2n+1}{r}$$

$$\therefore \sum_{r=0}^{n}\binom{2n+1}{r}=2^{2n}=4^n$$