3 UNIT EXTENSION 1 TRIAL PAPERS

2003

SYDNEY GRAMMAR
SCOTS COLLEGE
ABBOTSLEIGH
CATHOLIC
BARKER
HILLS GRAMMAR
KING'S
NEWINGTON
St George Girls
Syd – Boys
" " Girls

Sydney Girls High School

2003
TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics

Extension 1

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2003 HSC Examination Paper in this subject.

General Instructions

- Reading Time – 5 mins
- Working Time – 2 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Marks

Question 1 (12 marks)

a) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$ (1)

b) Evaluate i) $\int_0^{\pi} \sin^2 x dx$ (2)

ii) $\int_{-1}^{0} x\sqrt{1+x} dx$ (3)

using the substitution u = 1 + x

c) Find the point which divides the line joining A (1, 3) and B (-2, 6) externally in the ratio 2 : 1 (2)

(d) Write 3 cosθ + 4 sinθ in the form R cos (θ - ∝) and (4)

Hence solve $3\cos\theta + 4\sin\theta = 2,\ 0 \le \theta \le 2\pi$

(Give you answer correct to 2 decimal places)

Marks

Question 2 (12 marks)

(a) Solve $\frac{2x}{x-1} < 1$ (3)

(b) Given that $x = 1.7$ is a first approximation to the positive root of $x = 2\sin x$, use Newton's method once to find a second approximation to this root. (Correct to 1 decimal place) (3)

(c) Find the size of the acute angle between the lines $4x - 3y + 1 = 0$ and $x + 4y + 1 = 0$. (Give answer to the nearest degree). (3)

(d) A polynomial is given by

$P(x) = x^3 + ax^2 + bx - 18$ (3)

Find the values of a and b if $(x+2)$ is a factor of $P(x)$ and if -24 is the Remainder when $P(x)$ is divided by $(x-1)$.

Marks

Question 3 (12 marks)

(a) For the function $y = 2\sin^{-1}x$, find the equation of the tangent to the curve at the point where $x = \frac{1}{\sqrt{2}}$ (3)

(b) The acceleration of a particle moving in a straight line is given by (5)

$$\frac{d^2x}{dt^2} = 2x - 3$$

where x is the position (in metres) from the origin 0 and t is the time in seconds. Initially the particle is at rest at $x = 4$ m.

i. If the velocity is v m/s show that $v^2 = 2x^2 - 6x - 8$

ii. Show that the particle does not pass through the origin.

iii. Find the position when $v = 10$ m/s

(c) The Volume (V) of a sphere of radius r cm is increasing at a constant rate of 200 cm^3 per second. (4)

i. Find $\frac{dr}{dt}$ in terms of r

ii. Hence find the rate of increase of the surface Area (A) when the radius of the sphere is 50 cm.

Marks

Question 4 (12 marks)

(a) Prove the identity $\frac{2\tan\Theta}{1+\tan^2\Theta} = \sin 2\Theta$ (1)

(b) For the function $f(x) = 2\cos^{-1}\frac{x}{3}$ (3)

i. Evaluate f (0)

ii. State the domain and range of $y = f(x)$

iii. Sketch the graph of $y = f(x)$

(c) A particle moves in simple harmonic motion about the origin 0. Its position x metres from 0 at time t seconds is given by (4)

$$x = 3\cos(2t + \frac{\Pi}{3})$$

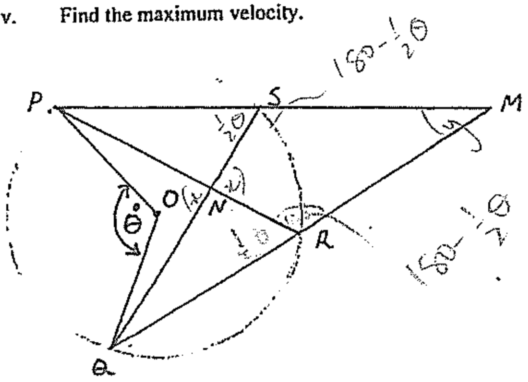
i. Find the acceleration in terms of position.

ii. Find its amplitude

iii. State the position x for maximum velocity.

iv. Find the maximum velocity.

(d) (4)



0 is the centre of a circle and $\angle POQ = \Theta^\circ$ Lines PS and QR produced intersect at M and lines PR and QS intersect at N

i. Copy this diagram into your exam booklet

ii. Prove that $\angle PRM = (180 - \frac{1}{2}\Theta)^\circ$

iii. Prove that $\angle PNQ + \angle PMQ = \Theta$

S 2 quad

$2(180-\frac{1}{2}\theta)+x+y=360$

$360-\theta+x+y=360$

$x+y=\theta$

Question 5 (12 marks)

(a) If α, β, γ are the roots of $x^3 - 3x + 1 = 0$ (3)

Find i $\alpha + \beta + \gamma$

ii $\alpha\beta\gamma$

iii $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(b) i. Factorise the polynomial $f(x) = 3x^3 - 7x^2 + 4$ (3)

ii Hence solve $f(x) = 0$

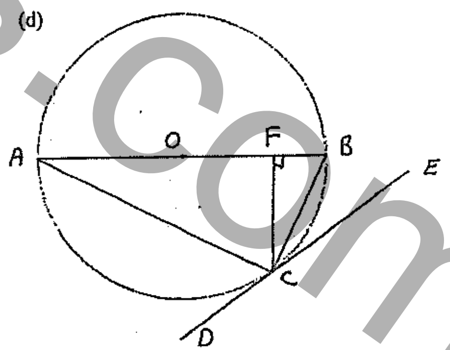
(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ (3)

i. Derive the equation of the tangent at P

ii. Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q

iii. If these tangents intersect at 45^0 show that $p = 1 + q + pq$, if $p > q$

(d) (3)



CF is perpendicular to AB
O is the centre of the circle
DCE is a tangent at C

i Copy this diagram into your workbook

ii Prove that BC bisects angle FCE

Marks

Question 6 **(12 marks)**

(a) In a population study, the population N is given by the equation (3)

$$N = 200 + Ae^{kt}$$

Initially N = 300 and when t = 3 seconds, N = 500

i Find the values of A and k (correct to 4 decimal places)

ii Find the population after 5 seconds

(b) i If $x = 4 \sin\Theta$ show that $\cos\Theta = \dfrac{\sqrt{16-x^2}}{4}$ (5)

ii By using the substitution $x = 4\sin\Theta$

show that $\displaystyle\int \frac{x^2 dx}{\sqrt{16-x^2}} = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + c$

(c) The chord of contact of the tangents to the parabola $x^2 = 4ay$ from an external point $P(x_1, y_1)$ cuts the directrix at Q. Prove that PQ subtends a right angle at the focus of the parabola. (4)

Marks

Question 7 **(12 marks)**

(a) Prove, using the Principle of Mathematical Induction, that (3)

$1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$

for all positive integers $n \geq 1$

(b) An projectile at the highest point of its trajectory has a velocity 8 metres per second and its position is 8 metres above the ground. (5)

Find i. the angle of projection (to nearest degree)

ii the initial velocity (correct to 1 decimal place)

(take $g = 9.8\ ms^{-2}$)

(c) (4)

i. Restrict the domain of $y = x^2 - 4x$ so that it will have an inverse function of the largest possible domain, and will include the point $x = 3$.

ii. Determine the equation of the inverse, writing y as the subject.

iii. Write down the co-ordinates of any points shared by the original curve and its inverse.

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2003
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections. Section A (Questions 1,2 and 3), Section B (Questions 4 and 5) and Section C (Questions 6 and 7).
- Start each NEW section in a separate answer booklet.

Total Marks - 84 Marks
- Attempt Sections A - C
- All questions are of equal value.

Examiner: *B. Opferkuch*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks-84.
Attempt Questions 1-7.
All questions are of equal value.

Answer each Section in a SEPARATE writing booklet. Extra writing booklets are available

Section A Use a SEPARATE writing booklet

Question 1 (12 marks) Marks

(a) Differentiate

(i) $x\sin 3x$ 1

(ii) e^{1-x^2}. 1

(b) Find the acute angle between the lines $3y = 2x + 8$ and $5x - y - 9 = 0$. 2

(c) Evaluate

(i) $\int_0^2 \frac{dx}{4+x^2}$ 2

(ii) $\int_0^1 \frac{x^2}{2+x^3}\,dx$ 2

(d) The letters of the word INTEGRAL are arranged in a row. 2

If one of these arrangements is selected at random, what is the probability that the vowels are in the same position?

(e) Solve the inequality $\frac{\theta - 4}{\theta} > 0$. 2

Section A continued.

Question 2. (12 marks) **Marks**

(a) If α, β and γ are the roots of the equation $2x^3 - 5x^2 - 3x + 1 = 0$, evaluate

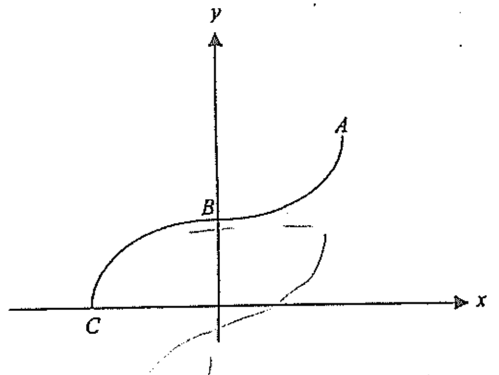
(i) $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$. 1

(ii) $\alpha^2 + \beta^2 + \gamma^2$. 2

(b) Use the substitution $u = x^2 + 4$ to find the exact value of $\int_0^{2\sqrt{3}} \frac{x}{\sqrt{x^2+4}}\,dx$. 3

(c) Determine the exact value of $\cos\left(\tan^{-1}\left(\frac{8}{15}\right)\right)$. 2

(d)



The diagram shows the graph of $y = \pi + 2\sin^{-1} 3x$.

(i) Find the coordinates of A and C. 2

(ii) Find the gradient of the tangent at B. 2

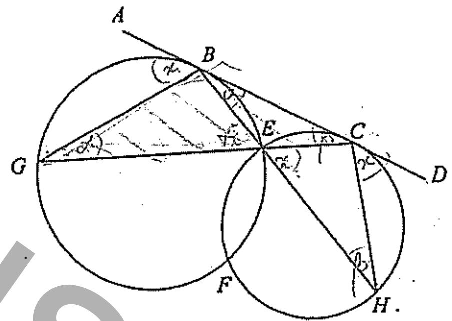
Section A continued.

Question 3. (12 marks) **Marks**

(a) A function is defined as $f(x) = 1 + e^{2x}$. 2
Find the inverse function $f^{-1}(x)$ and state the domain and range.

(b) Consider the quadratic expression $Q(x) = (5k-4)x^2 - 6x + (6k+3)$, 3
where k is a constant.
Find the values of k for which $Q(x) = 0$ has rational roots.

(c)



In ΔBCG and ΔBCH
$B\hat{C}G = B\hat{H}C$ (β)
$B\hat{E}G = C\hat{E}H$ (vert.)
∴ ~~ean~~ equiangular
∴ ||| Δs

$ABCD$ is a common tangent to the two circles.

(i) Prove that $\angle ABG = \angle DCH$! 2

(ii) Prove that $\Delta BCG \,|||\, \Delta BCH$. 2

(d) Consider the series $2^N + 2^{N-1} + 2^{N-2} + \ldots + 2^{1-N} + 2^{-N}$,
where N is a positive integer.

$-3, -2, -1, 0, 1, 2, 3$

$N=3$ $\sum$ $-N$

$x, N=3$

$S_T \to N=3$

(i) Find an expression in terms of N for the number of terms in the series. 2

(ii) Find an expression in terms of N for the sum of the series. 1

∴ terms
$n = 2N+1$

Section B Use a SEPARATE writing booklet.

Question 4. (12 marks)

Marks

(a) Consider the function $f(\theta)=\dfrac{\sin\theta+\sin\frac{\theta}{2}}{1+\cos\theta+\cos\frac{\theta}{2}}$

(i) Show that $f(\theta)=t$ where $t=\tan\frac{\theta}{2}$. 3

(ii) Write down the general solution of $f(\theta)=1$. 1

(a) A certain particle moves along the straight line in accordance with the law: $t=2x^2-5x+3$, where x is measured in centimetres and t in seconds.

Initially, the particle is $1\cdot5$ centimetres to the right of the origin O, and moving away from O.

(i) Show that the velocity, v cms^{-1}, is given by 1

$$v=\frac{1}{4x-5}$$

(ii) Find an expression for the acceleration, a cms^{-2}, of the particle, in terms of x. 2

(iii) Find the velocity and acceleration of the particle when: 3

(α) $x=2$ cm
(β) $t=6$ seconds

(iv) Describe carefully in words the motion of the particle. 2

Section B continued.

Question 5.

Marks

a) (i) Prove the identity $\dfrac{\cos y-\cos(y+2\alpha)}{2\sin\alpha}=\sin(y+\alpha)$ 2

(ii) Hence prove by mathematical induction that for positive integers n, 4

$\sin\alpha+\sin3\alpha+\sin5\alpha+\ldots+\sin(2n-1)\alpha=\dfrac{1-\cos2n\alpha}{2\sin\alpha}$.

(b) (i) Show that the curve $y=\dfrac{x^3+4}{x^2}$ has one stationary point and no points of inflexion. 2

(ii) Write down the equation(s) of any asymptotes. 1

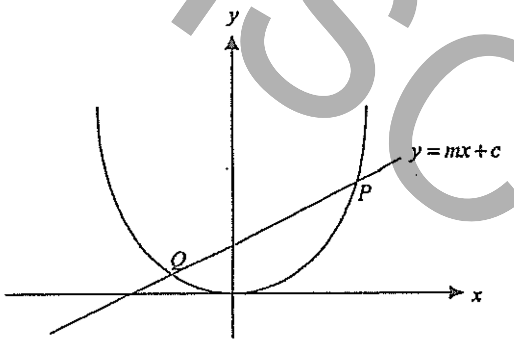
(iii) Sketch the curve. 1

(iv) Hence, use the graph to find the values of k for which the equation $x^3-kx^2+4=0$ has 3 real roots. 2

Section C Use a SEPARATE writing booklet.

Question 6. (12 marks) Marks

The straight line $y = mx + c$ meets the parabola $x = 2t,\ y = t^2$ in real distinct points P and Q which correspond respectively to the values $t = p$ and $t = q$.



(i) Prove that $pq = -c$. 2

(ii) Prove that $p^2 + q^2 = 4m + 2c$. 2

(iii) Show that the equation of the normal to the parabola at P is $x + py = 2p + p^3$. 2

(iv) The point N is the point of intersection of the normals to the parabola at P and Q.
Show that the coordinates at N are $\left(-pq(p+q),\ (2 + p^2 + pq + q^2)\right)$ 2

(v) If the chord PQ is free to move while maintaining a fixed gradient.

- (α) Show that the locus of N is a straight line. 2
- (β) Hence, or otherwise, show that this straight line is a normal to the parabola. 2

Section C continued.

Question 7. (12 marks) Marks

(a) When the polynomial $P(x)$ is divided by $(x+4)$ the remainder is 5 and when $P(x)$ is divided by $(x-1)$ the remainder is 9. Find the remainder when $P(x)$ is divided by $(x-1)(x+4)$. 3

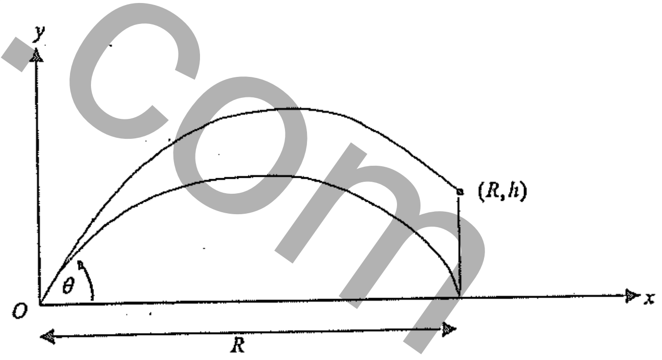
(b) A projectile is fired from a point on horizontal ground with initial speed V ms^{-1} and angle of projection θ.
The cartesian equation of the path is given by

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

where x and y are the horizontal and vertical displacements of the particle from O, the point of projection.
The acceleration due to gravity is g and air resistance has been neglected.

(i) Use the given equation to show that the maximum range R on the horizontal plane is given by $R = \dfrac{V^2}{g}$. 2

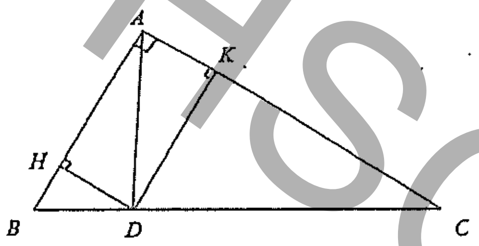
(ii) Show that to hit a target h metres above the ground at the same horizontal distance R using the same angle of projection θ, the speed of projection must be increased to $\dfrac{V^2}{\sqrt{V^2 - gh}}$. 4



Question 7.

Marks

(c)



In the triangle ABC, $< BAC = 90°$. AD bisects $< BAC$.
$DH \perp AB$ and $DK \perp AC$.

Copy the diagram.

(i) Show that $\dfrac{AD}{DH} = \sqrt{2}$. 1

(ii) By considering the areas of the triangles or otherwise, show that $\dfrac{\sqrt{2}}{AD} = \dfrac{1}{AB} + \dfrac{1}{AC}$. 2

THIS IS THE END OF THE PAPER

St George Girls' High School

Trial Higher School Certificate Examination

2003

Mathematics Extension 1

Total Marks – 84

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Attempt ALL questions.
- Begin each question on a new page
- Write your student number on each page
- All necessary working must be shown.
- Diagrams are not to scale.
- Board-approved calculators may be used.
- The mark allocated for each question is listed at the side of the question.

Question	Mark
Q1	/12
Q2	/12
Q3	/12
Q4	/12
Q5	/12
Q6	/12
Q7	/12
Total	/84

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (12 marks) – Start a new page Mark

a) Find the exact value of $\int_2^3 \frac{x^2}{x^3-7}\,dx$

b) Solve for x: $\frac{2}{x-1} \le 1$ 3

c) $P(19, -15)$ is the point which divides the line interval 'AB' externally in the ratio 3:2. Find the coordinates of $B(x, y)$ given $A(-2, 3)$. 3

d) (i) Find $\frac{d}{dx}(\tan^{-1}x + x)$

(ii) Hence, evaluate $\int_0^1 \frac{x^2+2}{x^2+1}\,dx$ 4

(leave in exact form).

Question 2 – (12 marks) – Start a new page

Marks

a) The equation $x^3 - mx + 2 = 0$ has two of its roots equal. 4

(i) Write down expressions for the sum of the roots and for the product of the roots.

(ii) Hence, find the value of m.

b) The polynomial equation $8x^3 - 36x^2 + 22x + 21 = 0$ has roots which form an arithmetic progression. 3

Find the roots of the polynomial.

c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola with the equation $x^2 = 4ay$.

It is given that the chord PQ has equation $y = \left(\frac{p+q}{2}\right)x - apq$ 3

(i) Show that the gradient of the tangent at P is p.

(ii) Prove that if PQ passes through the focus, then the tangent at P is parallel to the normal at Q.

d) (i) Write down the equation for the inverse function of $y = 2^x$, write your response with y as the subject. 2

(ii) Write down the domain of the inverse function from part (i).

Question 3 – (12 marks) – Start a new page

Marks

a) Find the term independent of x in the expansion of $\left(x - \frac{2}{x^3}\right)^{12}$ 3

b) Find the greatest coefficient in the expansion of $(2+3x)^{14}$ 4

c) (i) Show that $\sqrt{12}\sin x + 2\cos x \equiv 4\cos\left(x - \frac{\pi}{3}\right)$ 5

(ii) Hence, solve the equation $\sqrt{12}\sin x + 2\cos x = -2\sqrt{2}$ for $0 \le x \le 2\pi$

[Give all answers correct to two decimal places]

SYDNEY GRAMMAR Ext 1 Trial, 2003.

QUESTION ONE (Start a new answer booklet)

Marks

(a) Solve the inequation $\dfrac{1}{x-3} < 3$. $\boxed{2}$

(b) Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$, giving your answer in exact form. $\boxed{2}$

(c) Differentiate with respect to x:

(i) $y = \tan^{-1} 2x$ $\boxed{1}$

(ii) $y = \log_e \cos x$ $\boxed{2}$

(d) Find, correct to the nearest degree, the acute angle between the straight lines $y = 3$ and $y = -\frac{5}{3}x + 2$. $\boxed{2}$

(e) Let α, β and γ be the roots of $2x^3 - x^2 + 3x - 2 = 0$. Find the value of $\boxed{3}$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}.$$

QUESTION TWO (Start a new answer booklet)

Marks

(a) Use the substitution $u = 1 + \tan x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$. $\boxed{3}$

(b) Find the term independent of x in the expansion of $\left(x^2 - \dfrac{3}{x^2}\right)^6$. $\boxed{3}$

(c) Using the t-substitutions, or otherwise, prove the identity $\boxed{3}$

$$\frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} = \tan^2\theta.$$

(d) An object, always spherical in shape, is increasing in volume at a constant rate of $8\,\text{m}^3/\text{min}$.

(i) Find the rate at which the radius is increasing when the radius is 4 metres. (Note: You may assume the volume formula $V = \frac{4}{3}\pi r^3$). $\boxed{2}$

(ii) Find the rate at which the surface area is increasing when the radius is 4 metres. (Note: You may assume the surface area formula $S = 4\pi r^2$). $\boxed{1}$

Exam continues next page ...

QUESTION THREE (Start a new answer booklet)

Marks

(a) Consider the function $f(x) = 3\sin^{-1}(x+1)$.

(i) Write down the domain and the range of $f(x)$. $\boxed{2}$

(ii) Sketch $y = f(x)$, giving the coordinates of its endpoints and any intercepts with the coordinate axes. $\boxed{2}$

(b) A particle moves according to the equation $v^2 = 2x(6-x)$.

(i) Show that the particle moves in the interval $0 \le x \le 6$. $\boxed{1}$

(ii) Write down the centre of the motion. $\boxed{1}$

(iii) Find the maximum speed of the particle. $\boxed{1}$

(iv) Find the acceleration function. $\boxed{1}$

(c) The expression $\left(2 + \dfrac{x}{3}\right)^n$ is expanded. The ratio of the coefficients of the terms in x^6 and x^7 is 7 : 8. Find the value of n. $\boxed{4}$

QUESTION FOUR (Start a new answer booklet)

Marks

(a) The polynomial $2x^3 + ax^2 + bx + 6$ has $x - 1$ as a factor and leaves a remainder of -12 when divided by $x + 2$. Find the values of a and b. $\boxed{4}$

(b) Given that the equation $x^3 + px^2 + qx + r = 0$ has a triple root, use the sums and products of roots to show that $pq = 9r$. (Hint: Let the roots be α, α and α). $\boxed{4}$

(c) (i) Show that the coefficient of x^5 in the expansion of $(1+x)^4(1+x)^4$ is given by $\boxed{3}$

$${}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4.$$

(ii) Hence, by equating the coefficients of x^5 on both sides of the identity $\boxed{1}$

$$(1+x)^4(1+x)^4 = (1+x)^8,$$

prove that ${}^4C_0 \times {}^4C_1 + {}^4C_1 \times {}^4C_2 + {}^4C_2 \times {}^4C_3 + {}^4C_3 \times {}^4C_4 = \dfrac{8!}{3! \times 5!}$.

${}^4C_0 = {}^4C_4$

${}^4C_1 = {}^4C_3$

Exam continues overleaf ...

QUESTION FIVE (Start a new answer booklet)

(a) The temperature of a body is changing at the rate $\frac{dT}{dt} = -k(T-20)$, where T is the temperature at time t minutes and k is a positive constant.

The temperature of the surrounding environment is 20°C. The initial temperature of the body is 36°C and it falls to 35°C in 5 minutes.

Marks

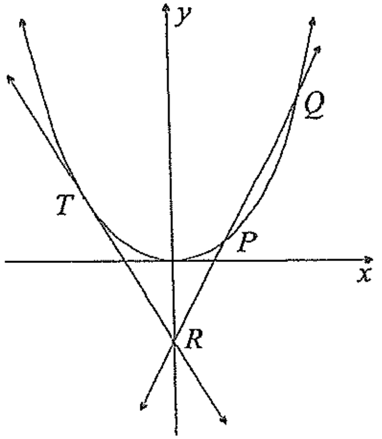
(i) Show that $T = 20 + Ae^{-kt}$ is a solution of $\frac{dT}{dt} = -k(T-20)$, where A is a constant. $\boxed{1}$

(ii) Prove that $A = 16$ and $k = -\frac{1}{5}\log_e \frac{15}{16}$. $\boxed{3}$

(iii) Find how long, correct to the nearest minute, it will take the temperature to fall to 27° C. $\boxed{2}$

(iv) Explain why the body will never reach a temperature that is one half of its initial temperature. $\boxed{1}$

(b)



The diagram above shows the parabola $x^2 = 4ay$. The points $T(2at, at^2)$, $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola.

You may assume that the chord PQ has equation $y - \frac{1}{2}(p+q)x + apq = 0$.

(i) Prove that the equation of the tangent to the parabola at the point $T(2at, at^2)$ is $y - tx + at^2 = 0$. $\boxed{2}$

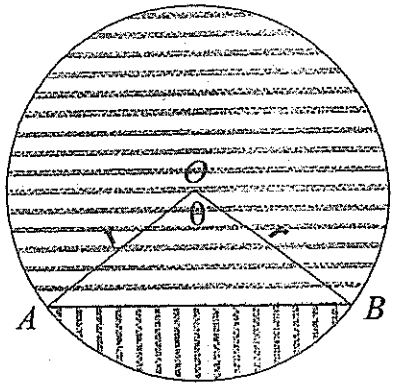
(ii) Let the tangent at T intersect the axis of the parabola at the point R. Find the coordinates of R. $\boxed{1}$

(iii) Given that the chord PQ also passes through R, show that the parameters p, t and q form a geometric sequence. $\boxed{2}$

Exam continues next page ...

QUESTION SIX (Start a new answer booklet)

(a)



In the diagram above, the chord AB subtends an angle of θ radians at the centre O of the circle with radius r.

(i) Show that the ratio of the areas of the two segments is $\boxed{2}$

$$\frac{\text{area of major segment}}{\text{area of minor segment}} = \frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta}.$$

(ii) Now suppose that

$$\frac{\text{area of major segment}}{\text{area of minor segment}} = \frac{\pi - 1}{1}.$$

(α) Prove that $\theta - 2 - \sin\theta = 0$. $\boxed{1}$

(β) Show that the equation $\theta - 2 - \sin\theta = 0$ has a root between $\theta = 2$ and $\theta = 3$.

(γ) Taking $\theta = 2{\cdot}5$ as the first approximation, use Newton's method to find a second approximation to the root. Give your answer correct to two decimal places.

(δ) Determine whether the second approximation of θ yields a smaller value of $|\theta - 2 - \sin\theta|$ than the first approximation.

Exam continues overleaf ...

(b) $\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$

$u = \sqrt{x}$

$= x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$du = \frac{dx}{2\sqrt{x}}$

$= \int_1^{\sqrt{3}} \frac{2\,du}{1+u^2}$

$= 2\left[\tan^{-1}u\right]_1^{\sqrt{3}}$

$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1}1\right]$

$= 2\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$

$= \frac{\pi}{6}$

(c) $T = \frac{2\pi}{n} = 2$

$\therefore n = \pi$

$\left.\begin{array}{l} t=0 \\ v=0 \\ x=10 \end{array}\right\}$

(i) The motion can be described by

$x = 10\cos\pi t$

$\dot{x} = -10\pi\sin\pi t$

$(\dot{x})^2 = 100\pi^2\sin^2\pi t$

$= \pi^2(100\sin^2\pi t)$

ie $v^2 = \pi^2(100 - x^2)$

when $x=6$

$v^2 = \pi^2 \times 64$

$v = \pm 8\pi$

$\therefore$ Speed is 8π m/s

(ii) when $x = 6$

$6 = 10\cos\pi t$

$0.6 = \cos\pi t$

$(\pi t)_{acute} = 0.927\ldots$

First time $\Rightarrow \therefore \pi t = 0.92729\ldots$

$\therefore t = \frac{0.92729\ldots}{\pi}$

$= 0.2951\ldots$

$= 0.30$ (correct to 2 dec pl)

$\therefore$ Time taken is 0.30 s

— 5 MARKS

QUESTION 7 :

(i) P: $\ddot{x}_p = 0$ ——①

$\dot{x} = C$

at $t=0$, $\dot{x}_p = V_p\cos\alpha$

$\therefore C = V_p\cos\alpha$

$\therefore \dot{x}_p = V_p\cos\alpha$ ——②

$\Rightarrow x = (V_p\cos\alpha)t + C$

at $t=0$, $x=0$ $\therefore C=0$

$\therefore x_p = (V_p\cos\alpha)t$ ——③

$\ddot{y}_p = -g$ ——④

$\dot{y} = -gt + C$

at $t=0$, $\dot{y} = V_p\sin\alpha$

$\therefore C = V_p\sin\alpha$

$\therefore \dot{y}_p = -gt + V_p\sin\alpha$ ——⑤

$\Rightarrow y = -\frac{1}{2}gt^2 + (V_p\sin\alpha)t + C$

at $t=0$, $y = h$ $\therefore C = h$

$\therefore y_p = -\frac{1}{2}gt^2 + (V_p\sin\alpha)t + h$ ——⑥

(ii) $x_a = (V_a\cos\beta)t$

$y_a = -\frac{1}{2}gt^2 + (V_a\sin\beta)t$

(iii) at collision, $t = T$, $x_p + x_a = 2a$ and $y_p = y_a$

now $y_p = y_a$

$\Rightarrow -\frac{1}{2}gT^2 + (V_a\sin\beta)T = -\frac{1}{2}gT^2 + (V_p\sin\alpha)T + h$

$\therefore T = \frac{h}{V_a\sin\beta - V_p\sin\alpha}$

Then $x_p + x_a = 2a$

$\Rightarrow (V_p\cos\alpha)\cdot\frac{h}{V_a\sin\beta - V_p\sin\alpha} + (V_a\cos\beta)\cdot\frac{h}{V_a\sin\beta - V_p\sin\alpha} = 2a$

$\therefore V_p\cos\alpha + V_a\cos\beta = 2V_a\sin\beta - 2V_p\sin\alpha$

$V_p(\cos\alpha + 2\sin\alpha) = V_a(2\sin\beta - \cos\beta)$

$\therefore \frac{V_p}{V_a} = \frac{2\sin\beta - \cos\beta}{2\sin\alpha + \cos\alpha}$

— 6 MARKS

(b) (i) $\sin(x-y) = \sin x\cos y - \cos x\sin y$

(ii) $c^2 + cd + d^2$

$= \sin^2\alpha + \sin\alpha\left[\sin(60-\alpha)\right] + \left[\sin(60-\alpha)\right]^2$

$= \sin^2\alpha + \sin\alpha\left[\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right] + \left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right)^2$

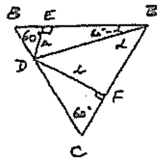
$= \sin^2\alpha + \frac{\sqrt{3}}{2}\sin\alpha\cos\alpha - \frac{1}{2}\sin^2\alpha + \frac{3}{4}\cos^2\alpha - \frac{\sqrt{3}}{2}\sin\alpha\cos\alpha$

$+ \frac{1}{4}\sin^2\alpha$

$= \frac{3}{4}\sin^2\alpha + \frac{3}{4}\cos^2\alpha$

$= \frac{3}{4}$

(iii)



In $\triangle AED$ $\sin(60°-\alpha) = \frac{a}{AD} = d$

In $\triangle AFD$ $\sin\alpha = \frac{b}{AD} = c$

From (ii) above,

$c^2 + cd + d^2 = \frac{3}{4}$

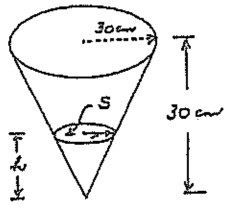
$\Rightarrow \frac{a^2}{(AD)^2} + \left(\frac{a}{AD}\right)\cdot\left(\frac{b}{AD}\right) + \frac{b^2}{(AD)^2} = \frac{3}{4}$

$a^2 + ab + b^2 = \frac{3}{4}(AD)^2$

$\therefore AD = \frac{2}{\sqrt{3}}(a^2 + ab + b^2)$

But true for $n=1 \Rightarrow$ true for $n=2$ and hence by the Principle of mathematical Induction it is true for all integer $n \geq 1$.

(c) $\frac{dV}{dt} = 24 \text{ cm}^3/s$



By similar triangles
$\frac{r}{h} = \frac{30}{30}$
$\therefore r = h$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = \pi h^2$

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \frac{1}{\pi h^2} \cdot 24$ cm/s

when $h = 16$

$\frac{dh}{dt} = \frac{1}{\pi . 16^2} \cdot 24$ cm/s

$= \frac{3}{32\pi}$ cm/s

$\frac{dS}{dt} = \frac{dS}{dh} \cdot \frac{dh}{dt}$ where $S = \pi r^2$
$= \pi h^2$
$\frac{dS}{dh} = 2\pi h$

$= 2\pi h \times \frac{3}{32\pi}$ cm²/s —— 4 MARKS

$h=16 \Rightarrow \frac{dS}{dt} = 3$ cm²/s

OR $\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt}$

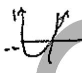
$= 2\pi h \times \frac{1}{\pi h^2} \times 24$ cm²/s

$h=16 \Rightarrow \frac{dS}{dt} = 32\pi \times \frac{1}{256\pi} \times 24$ cm²/s

$= 3$ cm²/s

(b) $y = \log\left(\frac{2x}{2+x}\right)$

(i) must have $\frac{2x}{2+x} > 0$



$2x(2+x) > 0$

$\therefore x < -2$ or $x > 0$ —— 2 MARKS

$\therefore$ Domain is $x < -2$ or $x > 0$

(ii) $y = 0 \Rightarrow \log\left(\frac{2x}{2+x}\right) = 0$

$\frac{2x}{2+x} = 1$

$2x = 2+x$ —— 1 MARK

$\therefore x = 2$

(iii) $y = \log\left(\frac{2x}{2+x}\right)$

$= \log 2x - \log(2+x)$

$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2+x}$

$= \frac{2+x-x}{x(2+x)}$

$= \frac{2}{x(2+x)}$ —— 2 MARKS

now $2x(2+x) > 0$,
$\therefore x(2+x) > 0$
$\therefore \frac{dy}{dx} > 0$ for all x in D

$\therefore$ fn is increasing

(iv) $\frac{dy}{dx} = \frac{2}{2x+x^2}$

$\frac{d^2y}{dx^2} = \frac{(2x+x^2).0 - 2(2+2x)}{(2x+x^2)^2}$

$= \frac{-4-4x}{(2x+x^2)^2}$

Possible pt of inflexion when $y'' = 0$
$-4-4x = 0$
$x = -1$

But this is outside the domain

$\therefore$ There are no points of inflexion —— 1 MARK

QUESTION 5: (12 MARKS)

(a) $\frac{dT}{dt} = k(T-R)$ —— ①

(i) $T = R + Ae^{kt}$ —— ②

$\Rightarrow \frac{dT}{dt} = kAe^{kt}$

$= k(T-R)$ from ② —— 1 MARK

(ii) $\frac{dT}{dt} = k(T - -40)$

$= k(T+40)$

$\Rightarrow T = -40 + Ae^{kt}$

$\left.\begin{matrix} t=0 \\ T=24 \end{matrix}\right\} \Rightarrow 24 = -40 + A$

$\therefore A = 64$

$\therefore T = -40 + 64e^{kt}$

$\left.\begin{matrix} t=5 \\ T=19 \end{matrix}\right\}$ $19 = -40 + 64e^{5k}$

$59 = 64e^{5k}$

$\frac{59}{64} = e^{5k}$

$5k = \ln\left(\frac{59}{64}\right)$

$\therefore k = \frac{1}{5}\ln\left(\frac{59}{64}\right)$

$= -0.016269...$

$T=0 \Rightarrow 0 = -40 + 64e^{kt}$

$40 = 64e^{kt}$

$e^{kt} = \frac{5}{8}$

$kt = \ln\left(\frac{5}{8}\right)$

$6 = \frac{\ln\left(\frac{5}{8}\right)}{k}$

$= 28.889...$

$= 29$ (to nearest whole)

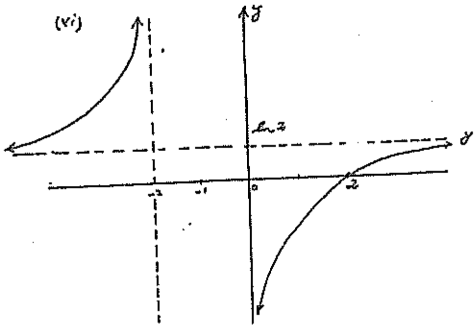
$\therefore$ It will take 29 s.

(v) $\lim_{x\to\infty}\left[\log\left(\frac{2x}{2+x}\right)\right]$

$= \lim_{x\to\infty}\left[\log\left(\frac{2}{\frac{2}{x}+1}\right)\right]$ —— 1 MARK

$= \log 2$

(vi) —— 1 MARK



QUESTION 6: (12 MARKS)

(a) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$ —— ①

(i) sub $x=1$ in ①

$\Rightarrow 2^n = \sum_{r=0}^{n} {}^nC_r$ —— 1 MARK

(ii) now $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_k x^k + \dots + {}^nC_n$

Differentiating with respect to x:

$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2 x + \dots + k.{}^nC_k x^{k-1} + \dots + n.{}^nC_n$

$= \sum_{r=1}^{n} {}^nC_r . r . x^{r-1}$

let $x = 1$

$\Rightarrow n . 2^{n-1} = \sum_{r=1}^{n} r . {}^nC_r$ —— 2 MARKS

Question 7 (cont'd) Marks

b) (i) Write down an expression for $\sin(x-y)$ 6

(ii) If $\sin\alpha = c$ and $\sin(60^\circ - \alpha) = d$, prove that $c^2 + cd + d^2 = \frac{3}{4}$

(iii) If ΔABC is equilateral and D is any point on the side BC and if a and b are the lengths of the perpendiculars from D to AB and AC respectively, prove

$$AD = \frac{2}{\sqrt{3}}\sqrt{a^2 + ab + b^2}$$

EXTENSION 1 - SOLUTIONS

QUESTION 1 : (12 MARKS)

(a) $\int_2^3 \frac{x^2}{x^3-7}\,dx = \frac{1}{3}\Big[\ln(x^3-7)\Big]_2^3$

$= \frac{1}{3}(\ln 20 - \ln 1)$

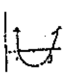
$= \frac{1}{3}\ln 20$ —

(b) $\frac{2}{x-1} \leq 1 \qquad x \neq 1$

$2(x-1) \leq (x-1)^2$

$0 \leq (x-1)^2 - 2(x-1)$

$0 \leq (x-1)(x-3)$ –



$\Longrightarrow x \leq 1$ OR $x \geq 3$

but $x \neq 1$

$\therefore x < 1$ OR $x \geq 3$ — 3

(c) $\underline{A(-2,3) \qquad B(x,y)}$

$3 : -2$

Hence $(19,-15) = \left(\frac{3x+4}{1}, \frac{3y-6}{1}\right)$

$\Longrightarrow \left.\begin{array}{l} x = 5 \\ y = -3 \end{array}\right\}$ — 3

(d) (i) $\frac{d}{dx}(\tan^{-1}x + x) = \frac{1}{1+x^2} + 1$

(ii) $\int_0^1 \left(\frac{x^2+2}{x^2+1}\right)dx = \int_0^1\left(\frac{x^2+1}{x^2+1} + \frac{1}{x^2+1}\right)dx$

$= \int_0^1\left(1 + \frac{1}{x^2+1}\right)dx$

$= \Big[x + \tan^{-1}x\Big]_0^1$

$= (1 + \tan^{-1}1) - (0+0)$

$= 1 + \frac{\pi}{4}$

— 4

QUESTION 2 : (12 MARKS)

(a) Let roots be α, α, β — 4 MARKS.

(i) $2\alpha + \beta = 0$ —— ①

$\alpha^2\beta = -2$ —— ②

(ii) from ①, $\beta = -2\alpha$ and ∴②

$\alpha^2 \times (-2\alpha) = -2$

$\alpha^3 = 1$

$\left.\begin{array}{l} \alpha = 1 \\ \therefore \beta = -2 \end{array}\right\}$

ie Roots are, 1, 1, −2

now $\Sigma\alpha\beta \Rightarrow \alpha^2 + 2\alpha\beta = -m$

ie $1 - 4 = -m$

$\therefore \underline{m = 3}$

(b) Let roots be $\alpha - d, \alpha, \alpha + d$ — 3 MARKS

$\Sigma\alpha \Longrightarrow (\alpha - d) + \alpha + (\alpha + d) = \frac{36}{8}$

$3\alpha = \frac{36}{8}$

$\alpha = \frac{3}{2}$

$\angle \alpha\beta \rightarrow \frac{3}{2}(\frac{3}{2}-\alpha) + (\frac{3}{2}-d)(\frac{3}{2}+d) + \frac{3}{2}(\frac{3}{2}+d) = \frac{11}{4}$

$\frac{9}{4} - \frac{3d}{2} + \frac{9}{4} - d^2 + \frac{9}{4} + \frac{3}{2}d = \frac{11}{4}$

ie $\frac{27}{4} - d^2 = \frac{11}{4}$

$27 - 4d^2 = 11$

$4d^2 = 16$

$d^2 = 4$

$d = 2, -2$

$d = 2 \Rightarrow$ roots are $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}$

$d = -2 \Rightarrow$ roots are $\frac{7}{2}, \frac{3}{2}, -\frac{1}{2}$

} ie Roots are $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}$

——— 3 MARKS.

(c) (i) $x^2 = 4ay$

$\Rightarrow y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

at $P(2ap, ap^2)$: $\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$

(ii) PQ through focus $(0, a)$

$\Longrightarrow a = 0 - apq$

ie $pq = -1$

normal at Q has gradient $-\frac{1}{q}$

tangent at Q has gradient q

$\therefore pq = -1 \Longrightarrow$ tangent at P $\perp$ tangent at Q

ie tangent at P // normal at Q

(d) (i) $f: y = 2^x$ — D: all real x, R: $y > 0$

$f^{-1}: x = 2^y$

$\therefore \log_2 x = y$

Then $\frac{P_{k+1}}{P_k} > 1 \Longrightarrow P_{k+1} > P_k$

Then... $\frac{45-3k}{2k} > 1 \Longrightarrow 45 > 5k$

$k < 9 \Longrightarrow P_{k+1} > P_k$

ie $P_9 > P_8 \ldots\ldots P_3 > P_2 > P_1$

$\therefore P_9$ is largest co-efficient

and $P_9 = {}^{14}C_8 \cdot 2^6 \cdot 3^8$

$= 1\,260\,971\,712$ ——— 4 MARKS.

(c) (i) $4\cos(x - \frac{\pi}{3}) = 4[\cos x \cos\frac{\pi}{3} + \sin x \sin\frac{\pi}{3}]$

$= 4[\cos x . \frac{1}{2} + \sin x . \frac{\sqrt{3}}{2}]$

$= 2\cos x + \sqrt{12}\sin x$

(ii) $\sqrt{12}\sin x + 2\cos x = -2\sqrt{2}$

$\Longrightarrow 4\cos(x - \frac{\pi}{3}) = -2\sqrt{2}$

$\cos(x - \frac{\pi}{3}) = \frac{-\sqrt{2}}{2}$

$0 \le x \le 2\pi$

$\rightarrow -\frac{\pi}{3} \le x - \frac{\pi}{3} \le \frac{5\pi}{3}$

$(x - \frac{\pi}{3})_{\text{base}} = \frac{\pi}{4}$

$\therefore x - \frac{\pi}{3} = \frac{3\pi}{4}, \frac{5\pi}{4}$

$\therefore x = \frac{13\pi}{12}, \frac{19\pi}{12}$

$= 3.403\ldots, 4.974\ldots$

$= 3.40, 4.97$ (correct to 2 dec. places)

$^{-1}: y \quad \log_2$

(ii) D: $x > 0$

D: $x > 0$
R: all real y

QUESTION 3: (12 MARKS)

(a) $(x - \frac{2}{x^3})^{12}$: $T_{k+1} = {}^{12}C_k\, x^{12-k} \cdot (-\frac{2}{x^3})^k$

$= {}^{12}C_k \cdot x^{12-k} \cdot (-2)^k \cdot x^{-3k}$

$= {}^{12}C_k\, (-2)^k \cdot x^{12-4k}$

Independent of $x \Rightarrow 12 - 4k = 0$

$k = 3$

$\therefore$ Term is $T_4 = {}^{12}C_3 (-2)^3$

$= -1760$ ——— 3 MARK

(b) $(2 + 3x)^{14}$: $T_{k+1} = {}^{14}C_k \cdot 2^{14-k} \cdot (3x)^k$

$\therefore T_k = {}^{14}C_{k-1} \cdot 2^{14-(k-1)} \cdot (3x)^{k-1}$

$= {}^{14}C_{k-1} \cdot 2^{15-k} \cdot (3x)^{k-1}$

Then co-efficients P_k, P_{k+1}

$\Longrightarrow \frac{P_{k+1}}{P_k} = \frac{{}^{14}C_k\, 2^{14-k}\, 3^k}{{}^{14}C_{k-1}\, 2^{15-k}\, 3^{k-1}}$

$= \frac{\cancel{14!}}{(14-k)!\,k!} \times \frac{(15-k)!\,(k-1)!}{\cancel{14!}} \times \frac{3}{2}$

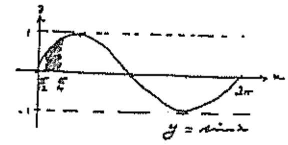
$= \frac{(15-k) \times 3}{2k}$

$= \frac{45-3k}{2k}$

QUESTION 4: (12 MARKS)

——— 4 MARKS

(a)



$V = \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} y^2\,dx$

$= \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2 x\,dx$

$= \frac{\pi}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1 - \cos 2x)\,dx$

$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left[(\frac{\pi}{4} - \frac{1}{2}) - (\frac{\pi}{12} - \frac{1}{2} \cdot \frac{1}{2})\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{6} - \frac{1}{4}\right]$

$\therefore$ Volume is $\frac{\pi}{2}(\frac{\pi}{6} - \frac{1}{4})$ units^3

(b) Proposition : $7^n + 5$ is divisible by 6 for all integers $n \ge 1$

(i) Test for $n = 1$: $7^1 + 5 = 12$

$\therefore$ True for $n = 1$

(ii) Assume proposition is true for some integer $n = k$

ie $7^k + 5 = 6M$, M integer ——— ①

Then $7^{k+1} + 5 = 7(7^k) + 5$

$= 7(6M - 5) + 5$ from ①

$= 42M - 30$

$= 6(7M - 5)$

$= 6N$ N integer

$\therefore$ If proposition is true for $n = k$ then it is also true for $n = k+1$

Question 6 – (12 marks) – Start a new page　　Marks

a) By noting that $(1+x)^n = \sum_{r=0}^{n} {}^nC_r\ x^r$,

prove that

(i) $\sum_{r=0}^{n} {}^nC_r = 2^n$　　1

(ii) $\sum_{r=1}^{n} r.\ {}^nC_r = n.2^{n-1}$　　2

b) Evaluate $\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$ using the substitution $u=\sqrt{x}$, give the EXACT value.　　4

c) A particle moving in Simple Harmonic Motion starts from rest at a distance 10 metres to the right of its centre of oscillation O. The period of the motion is 2 seconds.

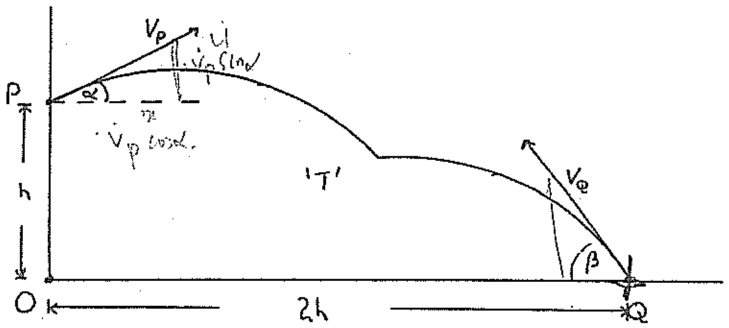
(i) Find the speed of the particle when it is 4 metres from its starting point.　　5

(ii) Find the time taken by the particle to first reach the point 4 metres from its starting point, in seconds correct to two decimal points.

Question 7 – (12 marks) – Start a new page　　Marks

a) O and Q are two points 2h metres apart on horizontal ground. P is a point h metres directly above O.　　6

A particle is projected from P towards Q with speed $V_P\ ms^{-1}$ at an angle 'α' above the horizontal. At the same time another particle is projected from Q towards P with speed $V_Q\ ms^{-1}$ at an angle of 'β' above the horizontal. The two particles collide 'T' seconds after projection.



(i) For the projectile travelling from P towards Q the equations of motion are $\ddot{x}=0$ and $\ddot{y}=-g$.

Use calculus to show that at time t seconds, its horizontal distance x_P from O and its vertical height y_P from O are given by $x_P=(V_P\cos\alpha)t$ and $y_P=(V_P\sin\alpha)t-\frac{1}{2}gt^2+h$

(ii) For the particle going from Q towards P, write down expressions for the horizontal distance x_Q from Q and its vertical height y_Q from Q at time t seconds.

(iii) Hence, show that $\frac{V_P}{V_Q}=\frac{2\sin\beta-\cos\beta}{2\sin\alpha+\cos\alpha}$

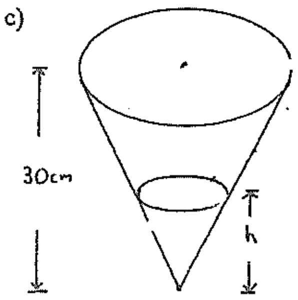
Question 4 – (12 marks) – Start a new page **Marks**

a) The region bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{12}$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution about the x-axis. Find the volume of the solid so formed. 4

[Give your answer in terms of π].

b) Use Mathematical induction to show that the expression $7^n + 5$ is divisible by 6 for all positive integers n. 4

c)



[NOT TO SCALE]

Water is poured into a conical vessel at a constant rate of 24cm^3 per second. The depth of water is h cm at any time t seconds. 4

What is the rate of increase of the area of the surface 'S' of the water when the depth is 16cm?

Question 5 – (12 marks) – Start a new page **Marks**

a) Newton's Law of Cooling states that when an object at temperature $T°$ is placed in an environment at a temperature of $R°$, then the rate of temperature loss is given by the equation

$$\frac{dT}{dt} = k(T - R)$$

where t is the time in seconds and k is a constant.

(i) Show that $T = R + Ae^{kt}$ is a solution to the equation. 1

(ii) A packet of peas, initially at 24°C is placed in a snap-freeze refrigerator in which the internal temperature is maintained at –40°C.

After 5 seconds the temperature of the packet is 19°C. How long will it take for the packet's temperature to reduce to 0°C? 3

b) Consider the function $y = \log_e\left(\frac{2x}{2+x}\right)$

(i) Show that the domain of the function is: $x < -2,\ x > 0$ 2

(ii) Find the value of x for which $y = 0$ 1

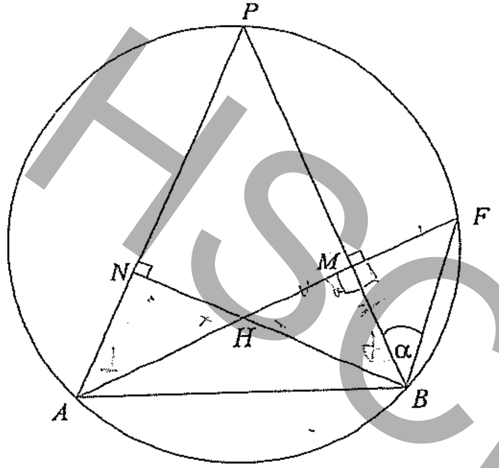
(iii) Show that $\frac{dy}{dx} = \frac{2}{x(2+x)}$ and hence show that the function is increasing for all x in the domain. 2

(iv) Find ~~any possible points of inflexion.~~ 1

(v) Find $\lim_{x \to \infty} \left[\log_e\left(\frac{2x}{2+x}\right)\right]$ 1

(vi) Sketch the graph of the function. 1

(b)



In the diagram above, ABP is a triangle inscribed in a circle.

The altitudes BN and AM of the triangle intersect at H.

The altitude AM is produced to meet the circumference of the circle at F.

Copy the diagram into your examination booklet.

Let $\angle PBF = \alpha$.

(i) Why is $\angle PAF = \alpha$? $\boxed{1}$

(ii) Why are the points A, N, M, and B concyclic? $\boxed{1}$

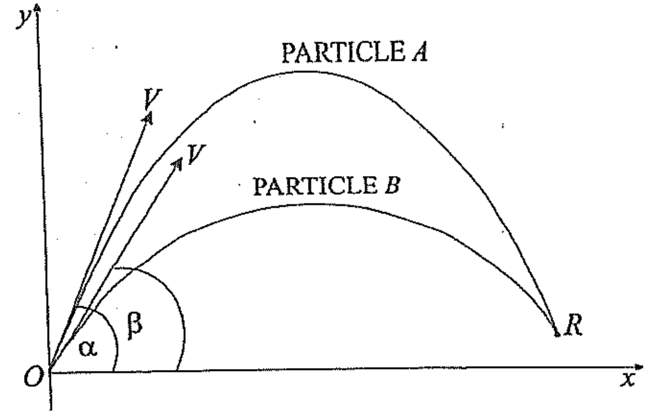
(iii) Why is $\angle NBM = \alpha$? $\boxed{1}$

(iv) Show that M bisects HF. $\boxed{2}$

(v) If AB is a fixed chord of the circle and P moves on the major arc AB, show that α is independent of the position of P. $\boxed{1}$

QUESTION SEVEN (Start a new answer booklet)

(a)



The diagram above shows two particles A and B projected from the origin.

Particle A is projected with initial velocity V m/s at an angle α.

Particle B is projected T seconds later with the same initial velocity V m/s but at an angle of β.

The particles collide at the point R.

Marl

(i) You may assume that the equations of the paths of A and B are: $\boxed{3}$

For A: $y = -\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha$

For B: $y = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta$

Show that the x-coordinate of the point R of collision is

$$x = \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}.$$

5/6

(ii) You may assume that the equation of the horizontal displacement of A is

$$x = Vt\cos\alpha.$$

(α) Write down the equation for the horizontal displacement of B. (Remember that B is projected T seconds after A).

(β) Show that the difference T in the times of projection is

$$T = \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)}.$$

(b) (i) Prove by mathematical induction that for all positive integers n, $\boxed{4}$

$$\sin(n\pi + x) = (-1)^n \sin x.$$

(ii) Let $S = \sin(\pi + x) + \sin(2\pi + x) + \sin(3\pi + x) + \cdots + \sin(n\pi + x)$, for $0 < x < \frac{\pi}{2}$ and for all positive integers n. Show that $\boxed{2}$

$$-1 < S \leq 0.$$

GJ

Trial 2003 3/4 UNIT MATHEMATICS FORM VI Solutions

STION ONE

)

$$\frac{1}{x-3} < 3,\ x \neq 3$$

$$\frac{1}{x-3} \times (x-3)^2 < 3(x-3)^2$$

$$x-3 < 3(x-3)^2 \quad \boxed{\checkmark}$$

$$3(x-3)^2 - (x-3) > 0$$

$$(x-3)(3(x-3)-1) > 0$$

$$(x-3)(3x-10) > 0$$

$$x < 3 \text{ or } x > \tfrac{10}{3}. \quad \boxed{\checkmark}$$

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\tfrac{x}{3}\right]_0^3 \quad \boxed{\checkmark}$$

$$= \sin^{-1}1 - \sin^{-1}0$$

$$= \tfrac{\pi}{2}. \quad \boxed{\checkmark}$$

) (i) $y = \tan^{-1}2x$

$$\frac{dy}{dx} = \frac{2}{1+4x^2}. \quad \boxed{\checkmark}$$

(ii) $y = \log_e \cos x$

$$\frac{dy}{dx} = -\frac{\sin x}{\cos x} \quad \boxed{\checkmark \text{ for } -\sin x\ \checkmark \text{ for quotient}}$$

(d) $\tan\theta = |-\tfrac{5}{3}|$ $\boxed{\checkmark}$

$$\theta \doteqdot 59^\circ \quad \boxed{\checkmark}$$

(e) $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} \quad \boxed{\checkmark}$$

$$= \frac{3}{2} \div 1$$

$$= \frac{3}{2} \quad \boxed{\checkmark\checkmark \text{ any correct method}}$$

QUESTION TWO

(a) $$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx = \int_1^2 \frac{du}{u^{\frac{1}{2}}} \quad \boxed{\checkmark}$$

$$= \int_1^2 u^{-\frac{1}{2}}\,du$$

$$= \left[2u^{\frac{1}{2}}\right]_1^2 \quad \boxed{\checkmark}$$

$$= 2\sqrt{2} - 2 \quad \boxed{\checkmark}$$

Let $u = 1 + \tan x$

$du = \sec^2 x\,dx$

When $x = 0,\ u = 1,$

When $x = \frac{\pi}{4},\ u = 2.$

(b) General term $= {}^6\mathrm{C}_r\,(x^2)^{6-r}\,(-1)^r\,(3x^{-2})^r$

$$= {}^6\mathrm{C}_r\,(x)^{12-2r}\,(-1)^r\,(3)^r\,(x)^{-2r}$$

$$= {}^6\mathrm{C}_r\,(-1)^r\,(3)^r\,(x)^{12-4r} \quad \boxed{\checkmark}$$

Let $12 - 4r = 0$

$r = 3$ $\boxed{\checkmark}$

Term independent of x $= {}^6\mathrm{C}_3\,(-1)^3\,(3)^3$

$$= -540. \quad \boxed{\checkmark}$$

(c) $$LHS = \frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta}$$

Let $t = \tan\theta$

$$LHS = \left(\frac{2t}{1-t^2} - t\right) \div \left(\frac{2t}{1-t^2} + \frac{1}{t}\right) \quad \boxed{\checkmark}$$

$$= \frac{2t - t + t^3}{1-t^2} \times \frac{t(1-t^2)}{2t^2 + 1 - t^2}$$

$$= \frac{t(1+t^2)}{1-t^2} \times \frac{t(1-t^2)}{t^2+1}$$

$\boxed{\checkmark \text{ correct method of simplification of the algebraic fractions}}$

$$= t^2 \quad \boxed{\checkmark}$$

$$= \tan^2\theta$$

$$= RHS$$

(d) (i) $V = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \boxed{\checkmark}$$

$$8 = 64\pi \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{8\pi}\ \text{m/min} \quad \boxed{\checkmark}$$

(ii) $S = 4\pi r^2$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$= 8\pi r \times \frac{1}{8\pi}$$

$$= 4\,\text{m}^2/\text{min}. \quad \boxed{\checkmark}$$

UESTION THREE

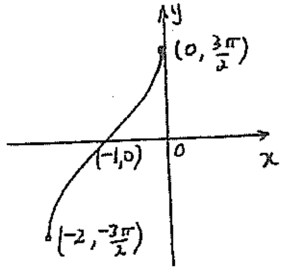
a) (i) $f(x) = 3\sin^{-1}(x+1)$

Domain: $-1 \le x+1 \le 1$

$-2 \le x \le 0$ ✓

Range: $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$. ✓

(ii)



✓ Shape
✓ Labels

(b) (i) $v^2 = 2x(6-x)$

$2x(6-x) \ge 0$

$0 \le x \le 6$ ✓

(ii) $x = 3$ ✓

(iii) Maximum speed when $x = 3$.

$v^2 = 6 \times 3$

$|v| = 3\sqrt{2}$ ✓

(iv) $v^2 = 2x(6-x)$

$\frac{1}{2}v^2 = 6x - x^2$

$\frac{d}{dx}(\frac{1}{2}v^2) = 6 - 2x$

$\ddot{x} = 6 - 2x$ ✓

(c) Given $\left(2 + \frac{x}{3}\right)^n$:

term in $x^6 = {}^nC_6 \times 2^{n-6} \times (\frac{x}{3})^6$

$$\text{Ratio of coefficients} = \frac{\frac{n!}{6!(n-6)!} \times 2^{n-6} \times (\frac{1}{3})^6}{\frac{n!}{7!(n-7)!} \times 2^{n-7} \times (\frac{1}{3})^7}$$

$$= \frac{n!}{6!(n-6)!} \times \frac{7!(n-7)!}{n!} \times 3 \times 2 \quad ✓$$

$$= \frac{42}{n-6} \quad ✓$$

$$\text{so } \frac{7}{8} = \frac{42}{n-6}$$

$$n - 6 = 48$$

$$n = 54. \quad ✓$$

QUESTION FOUR

(a) Let $P(x) = 2x^3 + ax^2 + bx + 6$

$$P(1) = 2 + a + b + 6$$
$$0 = a + b + 8$$
$$a + b = -8 \quad \cdots (1) \quad \boxed{\surd \text{ for any correct form}}$$
$$P(-2) = -16 + 4a - 2b + 6$$
$$-12 = 4a - 2b - 10$$
$$4a - 2b = -2$$
$$2a - b = -1 \quad \cdots (2) \quad \boxed{\surd \text{ for any correct form}}$$
$$(1) + (2) \quad 3a = -9$$
$$a = -3 \quad \boxed{\surd}$$
$$b = -5 \quad \boxed{\surd}$$

(b) $x^3 + px^2 + qx + r = 0$

$$3\alpha = -p \quad \cdots (1) \quad \boxed{\surd}$$
$$3\alpha^2 = q \quad \cdots (2) \quad \boxed{\surd}$$
$$\alpha^3 = -r \quad \cdots (3) \quad \boxed{\surd}$$
$$(1) \times (2) \quad 9\alpha^3 = -pq$$
$$-9r = -pq$$
$$pq = 9r \quad \boxed{\surd}$$

(c) (i) $(1+x)^4(1+x)^4 = \left({}^4\mathrm{C}_0 + {}^4\mathrm{C}_1 x + {}^4\mathrm{C}_2 x^2 + {}^4\mathrm{C}_3 x^3 + {}^4\mathrm{C}_4 x^4\right)$
$\times \left({}^4\mathrm{C}_0 + {}^4\mathrm{C}_1 x + {}^4\mathrm{C}_2 x^2 + {}^4\mathrm{C}_3 x^3 + {}^4\mathrm{C}_4 x^4\right) \quad \boxed{\surd}$

Term in $x^5 = {}^4\mathrm{C}_1 x \times {}^4\mathrm{C}_4 x^4 + {}^4\mathrm{C}_2 x^2 \times {}^4\mathrm{C}_3 x^3 + {}^4\mathrm{C}_3 x^3 \times {}^4\mathrm{C}_2 x^2 + {}^4\mathrm{C}_4 x^4 \times {}^4\mathrm{C}_1 x \quad \boxed{\surd}$

Coefficient $= {}^4\mathrm{C}_1 \times {}^4\mathrm{C}_4 + {}^4\mathrm{C}_2 \times {}^4\mathrm{C}_3 + {}^4\mathrm{C}_3 \times {}^4\mathrm{C}_2 + {}^4\mathrm{C}_4 \times {}^4\mathrm{C}_1$
$= {}^4\mathrm{C}_0 \times {}^4\mathrm{C}_1 + {}^4\mathrm{C}_1 \times {}^4\mathrm{C}_2 + {}^4\mathrm{C}_2 \times {}^4\mathrm{C}_3 + {}^4\mathrm{C}_3 \times {}^4\mathrm{C}_4$, by symmetry. $\boxed{\surd}$

(ii) Coefficient of x^5 in $(1+x)^8 = {}^8\mathrm{C}_5$
$$= \frac{8!}{3! \times 5!} \quad \boxed{\surd}$$

Now $(1+x)^4(1+x)^4 = (1+x)^8$,

so ${}^4\mathrm{C}_0 \times {}^4\mathrm{C}_1 + {}^4\mathrm{C}_1 \times {}^4\mathrm{C}_2 + {}^4\mathrm{C}_2 \times {}^4\mathrm{C}_3 + {}^4\mathrm{C}_3 \times {}^4\mathrm{C}_4 = \dfrac{8!}{3! \times 5!}$.

QUESTION FIVE

(a) (i) Given $T = 20 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$
$$= -k(T-20). \quad \boxed{\checkmark}$$

So $T = 20 + Ae^{-kt}$ is a solution.

(ii) When $t = 0,\ T = 36$

so $36 = 20 + Ae^0$

$A = 16. \quad \boxed{\checkmark}$

When $t = 5,\ T = 35$

so $35 = 20 + 16e^{-5k}$

$$15 = 16e^{-5k}$$
$$e^{-5k} = \frac{15}{16} \quad \boxed{\checkmark}$$
$$-5k = \log_e \frac{15}{16}$$
$$k = -\frac{1}{5}\log_e \frac{15}{16}. \quad \boxed{\checkmark}$$

(iii) When $T = 27$,

$$27 = 20 + 16e^{-kt}$$
$$e^{-kt} = \frac{7}{16} \quad \boxed{\checkmark}$$
$$t = \frac{\log_e \frac{7}{16}}{-k}$$
$$= 64{\cdot}045.....$$

It will take 64 minutes. $\boxed{\checkmark}$

(iv) As $t \to \infty,\ T \to 20$ from above.
The temperature does not drop below 20°C and so will never reach 18°C. $\boxed{\checkmark}$

(b) (i)

$$y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \frac{x}{2a}$$

At T,

$$\frac{dy}{dx} = \frac{2at}{2a}$$
$$= t. \quad \boxed{\checkmark}$$

Now

$$y - at^2 = t(x - 2at)$$
$$y - at^2 = tx - 2at^2$$

(ii) Let $x = 0$

so $y = -at^2$

R is the point $(0, -at^2)$. $\boxed{\checkmark}$

(iii) R lies on PQ.

$$y - \tfrac{1}{2}(p+q)x + apq = 0$$
$$-at^2 + apq = 0 \quad \boxed{\checkmark}$$
$$t^2 = pq,\ a \neq 0$$
$$\frac{t}{p} = \frac{q}{t}$$

So p, t, and q form a geometric sequence. $\boxed{\checkmark}$

UESTION SIX

1 (i) Area of minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

Area of major segment $= \pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)$

$$\text{Ratio of areas} = \frac{\pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)}{\frac{1}{2}r^2(\theta - \sin\theta)} \quad ☑$$

$$= \frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta} \quad ☑$$

(ii) (α)

$$\frac{2\pi - \theta + \sin\theta}{\theta - \sin\theta} = \frac{\pi - 1}{1}$$

$$\pi\theta - \pi\sin\theta - \theta + \sin\theta = 2\pi - \theta + \sin\theta$$

$$\theta - 2 - \sin\theta = 0 \quad ☑$$

(β) Let $f(\theta) = \theta - 2 - \sin\theta$

$$\begin{aligned} f(2) &= -\sin 2 \\ &\doteqdot -0{\cdot}909 \\ &< 0 \\ f(3) &= 1 - \sin 3 \\ &\doteqdot 0{\cdot}859 \\ &> 0. \end{aligned}$$

So the root lies between $\theta = 2$ and $\theta = 3$, ☑

(γ) $f(\theta) = \theta - 2 - \sin\theta$

$f'(\theta) = 1 - \cos\theta$.

Let θ_0 be the first approximation.

$$\theta_1 = \theta_0 - \frac{\theta_0 - 2 - \sin\theta_0}{1 - \cos\theta_0}$$

$$\theta_1 = 2{\cdot}5 - \frac{2{\cdot}5 - 2 - \sin 2{\cdot}5}{1 - \cos 2{\cdot}5} \quad ☑$$

$$\doteqdot 2{\cdot}55$$

(δ) When $\theta = 2{\cdot}5$,
$|\theta - 2 - \sin\theta| \doteqdot 0{\cdot}09847$.

(ε) When $\theta = 2{\cdot}55$,
$|\theta - 2 - \sin\theta| \doteqdot 0{\cdot}00768$. So $\theta = 2{\cdot}55$ yields a smaller value. ☑

(b) (i) $\angle PAF = \angle PBF$ angles at circumference standing on the same arc ☑

$\angle PAF = \alpha$.

(ii) $\angle ANB = \angle AMB$ (both given as rightangles).
These lie on the same interval AB and so A,N,M and B are concyclic. ☑

(iii) $\angle NBM = \angle MAN$ (angles standing on the same arc of circle $ANMB$) ☑

$\angle NBM = \alpha$.

(iv) $\triangle BHM \equiv \triangle BFM$ (AAS test) ☑

$HM = MF$ (matching sides of congruent triangles) ☑

(v) $\angle APB$ stands on fixed chord AB and its size is independent of the position of P (angles at circumference standing on the same chord). So α is independent of the position of P. ☑

QUESTION SEVEN

(a) (i) For A: $y = -\dfrac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \cdots (1)$

For B: $y = -\dfrac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \cdots (2)$

At R the coordinates are identical, so substitute (1) in (2).

$$-\frac{gx^2}{2V^2}\sec^2\alpha + x\tan\alpha = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta \quad \boxed{\checkmark}$$

$$\frac{gx^2}{2V^2}(\sec^2\alpha - \sec^2\beta) = x(\tan\alpha - \tan\beta)$$

$$\frac{gx}{2V^2}(\tan^2\alpha - \tan^2\beta) = (\tan\alpha - \tan\beta),\ x \neq 0 \quad \boxed{\checkmark}$$

$$\begin{aligned}\frac{gx}{2V^2} &= \frac{(\tan\alpha - \tan\beta)}{(\tan^2\alpha - \tan^2\beta)}\\ x &= \frac{2V^2}{g} \times \frac{1}{\tan\alpha + \tan\beta},\ \tan\alpha \neq \tan\beta\\ &= \frac{2V^2}{g} \times \frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}\\ &= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)} \quad \boxed{\checkmark}\end{aligned}$$

(ii) (α) $x = V(t-T)\cos\beta$. $\boxed{\checkmark}$

(β) When A is at R:

$$\begin{aligned}Vt\cos\alpha &= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}\\ t &= \frac{2V\cos\beta}{g\sin(\alpha+\beta)} \quad \cdots (3) \quad \boxed{\checkmark}\end{aligned}$$

When B is at R:

$$\begin{aligned}V(t-T)\cos\beta &= \frac{2V^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)}\\ t - T &= \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}\\ T &= t - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)}\\ &= \frac{2V\cos\beta}{g\sin(\alpha+\beta)} - \frac{2V\cos\alpha}{g\sin(\alpha+\beta)},\ \text{from (3)}\\ &= \frac{2V(\cos\beta - \cos\alpha)}{g\sin(\alpha+\beta)} \quad \boxed{\checkmark}\end{aligned}$$

(b) (i) Prove by mathematical induction the proposition that for all positive integer:

$$\sin(n\pi + x) = (-1)^n\sin x,\ \text{for } 0 < x < \tfrac{\pi}{2}.$$

A. When $n = 1$,

$$\begin{aligned}LHS &= \sin(\pi + x)\\ &= -\sin x\\ &= RHS.\end{aligned}$$

The proposition is true for $n = 1$. $\boxed{\checkmark}$

B. Assume the proposition is true for some positive integer k so that $\sin(k\pi + x) = (-1)^k\sin x \cdots (*)$

We are required to prove the proposition true for $n = k + 1$.

That is, $\sin[(k+1)\pi + x] = (-1)^{k+1}\sin x$. $\boxed{\checkmark}$

Now

$$\begin{aligned}LHS &= \sin[(k+1)\pi + x]\\ &= \sin[\pi + (k\pi + x)] \quad \boxed{\checkmark}\\ &= \sin\pi\cos(k\pi + x) + \cos\pi\sin(k\pi + x)\\ &= -1 \times \sin(k\pi + x)\\ &= -1 \times (-1)^k\sin x,\ \text{from } (*) \quad \boxed{\checkmark}\\ &= (-1)^{k+1}\sin x\\ &= RHS\end{aligned}$$

It follows from A and B by mathematical induction that for all positive integ
n, $\sin(n\pi + x) = (-1)^n\sin x$, for $0 < x < \frac{\pi}{2}$.

(ii)

$$\begin{aligned}S &= \sin(\pi + x) + \sin(2\pi + x) + \sin(3\pi + x) + \cdots + \sin(n\pi\\ &= -\sin x + \sin x - \sin x + \cdots + \sin(n\pi + x)\end{aligned}$$

When n is odd $\quad S = -\sin x$

so $\quad -1 < S < 0$, for $0 < x < \frac{\pi}{2}$. $\boxed{\checkmark}$

When n is even $\quad S = 0$.

So $\quad -1 < S \leq 0$. $\boxed{\checkmark}$

SCOTS COLLEGE, EXT 1. 2003

QUESTION 1

(a) Find the acute angle between the lines $2x - y = 0$ and $x + 3y = 0$, giving the answer correct to the nearest minute. [2]

(b) Solve the inequality $\frac{x}{x-3} \le 3$ [3]

(c) If u, v and w are the roots of $x^3 - 4x + 1 = 0$, find the value of $\frac{1}{u} + \frac{1}{v} + \frac{1}{w}$. [3]

(d) Solve the equation $\sin 2x = \tan x$ for $0 \le x \le \pi$. [4]

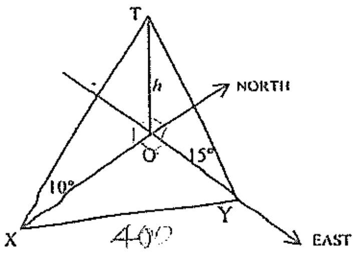
QUESTION 2 [START A NEW BOOKLET]

(a) A is the point $(-2, 1)$ and B is the point (x, y). The point $P(13, -9)$ divides AB externally in the ratio 5 : 3. Find the values of x and y. [3]

(b) (i) Show that the equation of the normal to the parabola $x^2 = 4ay$ at the point $T\left(2at, at^2\right)$ is $x + ty = 2at + at^3$. [2]

(ii) Hence show that there is only one normal to the parabola which passes through its focus. [1]

(c) A surveyor at X observes a tower due north. The angle of elevation to the top of the tower is 10°. He then walks 400m to a position Y which is due east of the tower. The angle of elevation from Y to the top of the tower is 15°.



(i) Write an expression for OY in terms of h. [1]

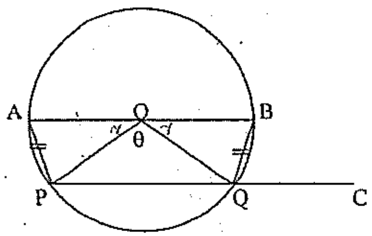
(ii) Calculate h to the nearest metre. [4]

(iii) Find the bearing of Y from X. [1]

QUESTION 3 [START A NEW BOOKLET]

(a) Evaluate $\int_0^{2\pi} \cos^2 2x \, dx$. [3]

(b)



The points A, B, P and Q lie on the circle with centre at O.

AB is a diameter and PC passes through Q.

AP is equal to BQ and ∠POQ = θ

$2x + \theta = 180$

$2x = \frac{180-\theta}{2}$

(i) Express ∠AOP in terms of θ. [1]

(ii) Prove that AB is parallel to PC. [2]

(c) By graphing or some other justification, simplify [3]

(i) $\sin^{-1} x + \sin^{-1}(-x)$

(ii) $\tan^{-1} x + \tan^{-1}(-x)$

(iii) $\sin^{-1} x - \cos^{-1}(-x)$

(d) Find $\int_0^2 2x\sqrt{1-\frac{x}{2}}\, dx$ using the substitution $u = 1 - \frac{x}{2}$ [3]

QUESTION 4 [START A NEW BOOKLET]

(a) The surface area of a cube is increasing at a rate of 10cm^2 per second. Find the rate of increase of the volume of the cube when the edge of the cube has length 12cm. [4]

(b) N is the number of animals in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = -k(N - 1000)$ for some constant k.

(i) Verify where A is constant, that $N = 1000 + Ae^{-kt}$ is a solution of the equation. [2]

(ii) Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of A and k, to 2 decimal places. [2]

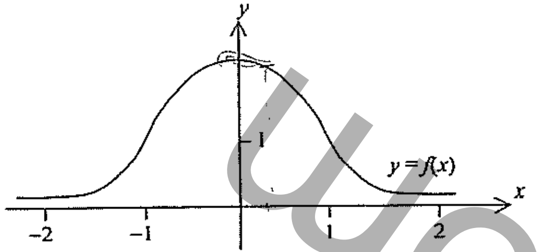
(iii) Find when the number of animals has fallen to 1300. [2]

(iv) Sketch the graph of the population size against time. [2]

SCOTS COLLEGE, EXT 1, 2003

QUESTION 5 [START A NEW BOOKLET]

(a) The graph below shows the derivative of $y = 2\tan^{-1} x$.



(i) Where does $y = 2\tan^{-1} x$ have its greatest slope and what is this slope? [2]

(ii) Calculate the x values correspond with $\frac{dy}{dx} = \frac{1}{3}$? [1]

(iii) Write an integral that represents the area in the first quadrant bounded by this curve, the x axis and $x = k$, where $k > 0$. [1]

(iv) By considering the limit as $k \to \infty$ determine the total area bounded by this curve and the x axis. [1]

(b) (i) Sketch the graph of function $f(x) = e^x - 4$. [1]

(ii) On the same diagram sketch the graph of the inverse function f^{-1}. [2]

(iii) Explain why the x coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$. [1]

(iv) Show that the equation $e^x - x - 4 = 0$ has a root between $x = 1$ and $x = 2$. Use one application of Newton's method to approximate the root, to 2 decimal places. [3]

QUESTION 6 [START A NEW BOOKLET]

(a) Prove by Mathematical Induction that $1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$ for all positive integers n. [5]

(b) A particle moves in a straight line so that its displacement x from a fixed point 0 at time t is given by $x = 3\sin 2t + 4\cos 2t$.

(i) If the motion is expressed in the form of $x = R\sin(2t + \alpha)$ where α is in radians, evaluate the constants R and α, to 2 decimal places. [3]

(ii) Show that the motion is Simple Harmonic. [1]

(iii) What is the period of oscillation? [1]

(iv) Determine the maximum displacement from the centre of motion. [2]

QUESTION 7 [START A NEW BOOKLET]

(a) A projectile has an initial velocity V and an angle of projection θ.

(i) Assuming $\frac{d^2y}{dt^2} = -10$, $\frac{d^2x}{dt^2} = 0$ and initially $x = 0$, $y = 10$, find expressions for x and y. [3]

(ii) If $V = 13ms^{-1}$ and $\theta = \tan^{-1}\left(\frac{5}{12}\right)$ find the range of the projectile. [2]

(b) (i) Use the Chain Rule to show that

$$\frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$ [1]

(ii) The acceleration due to gravity is inversely proportional to the square of the distance x from the centre of the earth.

This can be written as $\frac{dv}{dt} = \frac{-k}{x^2}$. Find k if $\frac{dv}{dt} = -g$ when $x = R$. [1]

(iii) Hence show that $v^2 = \frac{2R^2g}{x} + u^2 - 2gR$ where the initial velocity of a rocket is u ms^{-1}, g is the acceleration due to gravity and R is the radius of the earth. [2]

(iv) Find the maximum distance that the rocket will travel from the centre of the earth. (Answer in terms of g, R and u). [2]

(v) Taking g = 9.8ms^{-2}, R = 6400km find the value of u in ms^{-1} for which the rocket will escape the gravity of the earth. [1]

Standard Integrals

$\int x^n dx = \frac{1}{n+1}x^{n+1}+C, \quad n \neq -1; x \neq 0, \text{if } n<0$

$\int \frac{1}{x} dx = \ln x + C, \quad x>0$

$\int e^{ax} dx = \frac{1}{a}e^{ax}+C, \quad a \neq 0$

$\int \cos ax\, dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$

$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$

$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$

$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + C, \quad a>0, \ -a<x<a$

$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right)+C, \quad x>a>0$

$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)+C$

NOTE : $\ln x \equiv \log_e x, \quad x>0$

2003 Ext 1 Trial Paper.

Question 1

a) $m_1 = 2, \quad m_2 = -\frac{1}{3}$

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

$= \left|\frac{2+\frac{1}{3}}{1-2(\frac{1}{3})}\right|$

$= |7|$

$\therefore \theta = 81°52'$

b) $\frac{x}{x-3} \le 3$

$x(x-3) \le 3(x-3)^2$

$x^2 - 3x \le 3x^2 - 18x + 27$

$2x^2 - 15x + 27 \ge 0$

$(2x-9)(x-3) \ge 0$

$\therefore x < 3$ or $x \ge \frac{9}{2}$

c) $x^3 - 4x + 1 = 0.$

$u+v+w = 0$

$uv + uw + vw = -4$

$uvw = -1$

$\frac{1}{u}+\frac{1}{v}+\frac{1}{w} = \frac{vw+uw+uv}{uvw}$

$= \frac{-4}{-1}$

$= 4.$

d) $\sin 2x = \tan x \qquad 0 \le x \le \pi$

$2\sin x\cos x = \frac{\sin x}{\cos x}$

$2\sin x\cos^2 x - \sin x = 0$

$\sin x(2\cos^2 x - 1) = 0$

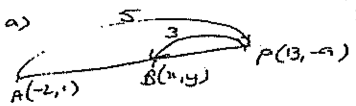
$\therefore \sin x = 0$ or $\cos x = \pm\frac{1}{\sqrt{2}}$

$x = 0, \pi \qquad x = \frac{\pi}{4}, \pi - \frac{\pi}{4}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}.$

Question 2

a)



$m:n$
$5:-3$

$x_4 = \frac{nx_1+mx_2}{m+n} \qquad y = \frac{ny_1+my_2}{m+n}$

$13 = \frac{-3(-2)+5x}{5-3} \qquad -9 = \frac{-3(1)+5y}{5-3}$

$26 = 6+5x \qquad -18 = -3+5y$

$5x = 20 \qquad 5y = -15$

$x = 4 \qquad y = -3$

b) $x^2 = 4ay$

(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

At $x = 2at,$

$\frac{dy}{dx} = t.$

$\therefore$ grad of normal $= -\frac{1}{t}$

$\therefore$ eqn of normal

$y - at^2 = -\frac{1}{t}(x-2at$

$ty - at^3 = -x + 2at$

$x + ty = 2at + at^3$

(ii) $S(0,a)$

$at = 2at + at^3$

$at + at^3 = 0$

$at(1+t^2) = 0$

$\therefore t = 0$ or $t^2 = -1$ no solution.

c) (i) $\tan 15° = \frac{h}{OY}$

$OY = \frac{h}{\tan 15°}$

(ii) $\tan 10° = \frac{h}{OX}$

$OX = \frac{h}{\tan 10°}$

$\therefore OX^2 + OY^2 = 400^2$

$$\frac{h^2}{\tan^2 10^\circ} + \frac{h^2}{\tan^2 15^\circ} = 160000$$

$$h^2\left(\frac{1}{\tan^2 10^\circ} + \frac{1}{\tan^2 15^\circ}\right) = 160000$$

$$h^2 = 3471.345..$$

$$h = 58.918..$$

$$h = 59m$$

(iii) $\cancel{\tan}\theta = \frac{04}{400}$

$$\sin\theta = \frac{59}{400\tan 15^\circ}$$

$$\theta = 033^\circ T.$$

Question 3.

a) $\cos^2 x = \frac{1}{2}(\cancel{2\cos 2x} - 1)$

$\cos^2 2x = \frac{1}{2}(\cancel{2}\cos 4x + 1)$

$\therefore \int_0^{2\pi} \cos^2 2x \, dx$

$= \frac{1}{2}\int_0^{2\pi} \cancel{2}\cos 4x + 1 \; dx$

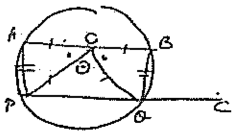
$= \frac{1}{2}\left[\frac{\sin 4x}{\cancel{4}} + x\right]_0^{2\pi}$

$= \frac{1}{2}\left(\frac{\sin 8\pi}{4} + 2\pi - \left(\frac{\sin 0}{2} - 0\right)\right)$

$= \pi.$

(b)

(i)



$\triangle AOP \equiv \triangle BOQ$ (SSS)

$\therefore \angle AOP = \angle BOQ$

$\therefore \angle AOP = \frac{180 - \theta}{2}$

$= 90^\circ - \frac{\theta}{2}$

(ii) $\angle OPQ = \frac{180 - \theta}{2}$ (base ∠s of isosceles △)

$= 90 - \frac{\theta}{2}$

$\therefore \angle AOP = \angle OPQ$ & they are alternate

$\therefore AB \parallel PQ.$

c) (i) $\sin^{-1}(x) + \sin^{-1}(-x)$

$= \sin^{-1}(x) - \sin^{-1}(x)$

$= 0.$

(ii) $\tan^{-1}(x) + \tan^{-1}(-x)$

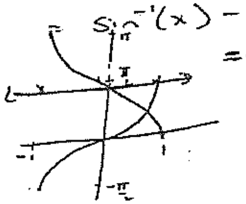
$= \tan^{-1}(x) - \tan^{-1}(x)$

$= 0$

(iii) $\sin^{-1}(x) - \cos^{-1}(-x)$

$\sin^{-1}(x) - \cos^{-1}(x)$

$= -\frac{\pi}{2}$



d) $u = 1 - \frac{x}{2}$ $\quad x = 0 \quad u = 1$

$\frac{du}{dx} = -\frac{1}{2}$ $\quad x = 2, \quad u = 0$

$x = 2(1-u)$

$\int_0^2 2x\sqrt{1 - \frac{x}{2}} \, dx$

$= 2.-2\int_1^0 2(1-u)\sqrt{u} \, . du$

$= -8\int_0^1 u^{\frac{1}{2}}(1-u) \, du$

$= 8\int_1^0 u^{\frac{1}{2}} - u^{3/2} \, du$

$= 8\left[\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^1$

$= 8\left(\frac{2}{3} - \frac{2}{5} - 0\right)$

$= \frac{32}{15}$

Question 4

a) $\frac{dA}{dt} = 10$ $\quad x = 12$

$\frac{dV}{dt} = ?$

$A = 6x^2$ $\quad V = x^3$

$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$

$10 = 12x \cdot \frac{dx}{dt}$

$x = 12, \quad 10 = 144 \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{10}{144}$

$\frac{dx}{dt} = \frac{5}{72}$

$\therefore \frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$

$= 3x^2 \cdot \frac{dx}{dt}$

$= 3(12)^2 \cdot \frac{5}{72}$

$= 30 \, cm^3/s$

(b)

(i) $N = 1000 + Ae^{-kt}$

$\frac{dN}{dt} = Ae^{-kt} \cdot -k$

$= -k(N - 1000)$

(ii) $t = 0, \quad N = 2500$

$t = 2, \quad N = 2200$

$2500 = 1000 + A$

$A = 1500$

$2200 = 1000 + 1500e^{-2k}$

$1500e^{-2k} = 1100$

$e^{-2k} = \frac{11}{15}$

$-2k = \ln\left(\frac{11}{15}\right)$

$k = \frac{\ln\left(\frac{11}{15}\right)}{-2}$

$k = 0.16$ (2dp)

(iii) $1300 = 1000 + 1500e^{-0.16t}$

$1500e^{-0.16t} = 300$

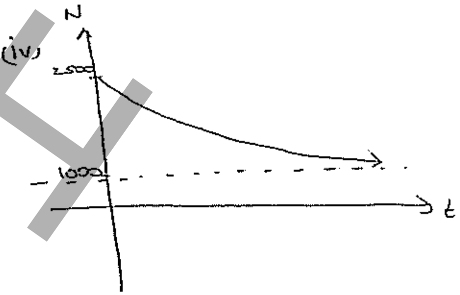
$e^{-0.16t} = \frac{1}{5}$

$-0.16t = \ln\left(\frac{1}{5}\right)$

$t = \frac{\ln\left(\frac{1}{5}\right)}{-0.16}$

$t = 10.06$ years

(iv)



Question 5

a)(i) $y = 2\tan^{-1}x$

$\frac{dy}{dx} = \frac{2}{1+x^2}$

greatest slope occurs at
$x = 0, \quad \frac{dy}{dx} = 2$

(ii) $\frac{2}{1+x^2} = \frac{1}{3}$

$6 = x^2+1$

$x^2 = 5$

$x = \pm\sqrt{5}$

(iii) $A = \int_0^k f(x)\,dx$

(iv) As $k \to \infty$, ~~Area under~~
Area ~~Area~~ under curve

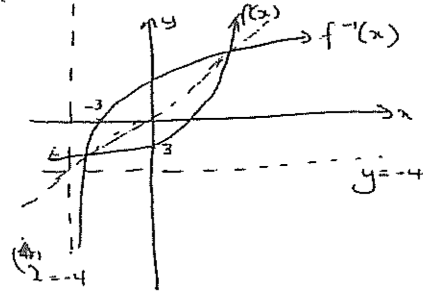
$A = 2\int_0^\infty f(x)\,dx$

$= 2.\left[2\tan^{-1}x\right]_0^\infty$

$= 2.\,2.\frac{\pi}{2}$

$= 2\pi$ sq units.

(b)(i) $f(x) = e^x - 4$



(iii) $f(x)$ & $f^{-1}(x)$ are reflections along the line $y = x$.

Points of intersection are $y = e^x - 4$ & $y = x$ hold true

ie $e^x - 4 = x$

$e^x - x - 4 = 0$.

(iv) let $f(x) = e^x - x - 4$

$f(1) = e - 1 - 4 < 0$

$f(2) = e^2 - 2 - 4 > 0$

∴ root lies between $x = 1$ & $x = 2$

$f'(x) = e^x - 1 \qquad x_1 = 1.5$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.5 - \frac{f(1.5)}{f'(1.5)}$

$= 1.79$ (2 dp).

Question 6

a) Step 1 : Need to prove $n=1$ is true

LHS $= 1$, RHS $= \frac{1(3(1)-1)}{2}$
$= 1$
$=$ LHS

∴ $n=1$ is true

Step 2. Assume that $n=k$ is true

ie $1+4+7+\dots+(3k-2) = \frac{k(3k-1)}{2}$

Need to prove that $n=k+1$ is true

ie $1+4+7\dots+(3k-2)+(3k+1) = \frac{(k+1)(3k+2)}{2}$

LHS $= 1+4+7+\dots+(3k-2)+(3k+1)$

$= \frac{k(3k-1)}{2} + (3k-1)$

$= \frac{1}{2}(3k^2 - k + 6k - 2)$

$= \frac{1}{2}(3k^2 + 5k - 2)$

$= \frac{1}{2}(3k+1)(k-2)$

$=$ RHS

∴ $n=k+1$ is also true

Step 3 : Since $n=1$, $n=k$ & $n=k+1$ are all true
Then $n=2, n=3\dots$ are true

∴ $1+4+7+\dots+(3n-2) = \frac{n(3n-1)}{2}$.

(b) $x = 3\sin 2t + 4\cos 2t$

(i) $3\sin 2t + 4\cos 2t$
$= R\sin(2t+\alpha)$
$= R\sin 2t\cos\alpha + R\cos 2t\sin\alpha$

∴ $R\cos\alpha = 3$ $\qquad R^2 = 3^2+4^2$

$R\sin\alpha = 4$ $\qquad R = \sqrt{25}$

$\tan\alpha = \frac{4}{3}$ $\qquad R = 5$

$\alpha = 0.93$

(ii) $x = 5\sin(2t+0.93)$

$\dot{x} = 10\cos(2t+0.93)$

$\ddot{x} = -20\sin(2t+0.93)$

$= -4x$

∴ motion is S.H.

(iii) period $= \frac{2\pi}{2}$
$= \pi$

(iv) max disp when $\dot{x} = 0$

$10\cos(2t+0.93) = 0$

$\cos(2t+0.93) = 0$

$2t + 0.93 = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

$2t = \frac{\pi}{2} - 0.93, \ \frac{3\pi}{2} - 0.93$

$t = 0.32, \ 1.89\dots$

At $t = 0.32$,

$x = 5\sin(2(0.32)+0.93)$

$x = 5$.

Question 7

a) $\frac{d^2x}{dt^2} = 0$ $\qquad$ Initially, $\dot{x} = V\cos\theta$, $\dot{y} = V\sin\theta$

$\frac{dx}{dt} = c_1$

$t=0$, $\frac{dx}{dt} = V\cos\theta$, ∴ $\frac{dx}{dt} = V\cos\theta$

$x = \int V\cos\theta\,dt$

$x = Vt\cos\theta + c_2$

$t=0$, $x=0$, $c_2 = 0$. ∴ $x = Vt\cos\theta$

$\frac{d^2y}{dt^2} = -10$

$\frac{dy}{dt} = \int -10\,dt$

$= -10t + c_3$

$t=0$, $\frac{dy}{dt} = V\sin\theta$, ∴ $c_3 = V\sin\theta$

∴ $\frac{dy}{dt} = -10t + V\sin\theta$

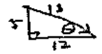
$y = \int -10t + V\sin\theta\,dt$

$y = -5t^2 + Vt\sin\theta + c_4$

$t=0$, $y=10$, ∴ $c_4 = 10$

∴ $y = -5t^2 + Vt\sin\theta + 10$

(ii) $V = 13$



Sub $y=0$, $-5t^2 + t.13.\frac{5}{13} + 10 = 0$

$-5t^2 + 5t + 10 = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

∴ $t = 2$ or $t = -1$ but $t \geq$

∴ $t = 2$

When $t = 2$, $x = 13.2.\frac{12}{13} = 2$

b) (i) $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

$= v \cdot \frac{dv}{dx}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \cdot 2v \cdot \frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$

$\therefore \frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(ii) $\frac{dv}{dt} = \frac{-k}{x^2}$

sub $\frac{dv}{dt} = -g$, $x = R$,

$-g = -\frac{k}{R^2}$

$R^2 g = k$.

(iii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{R^2 g}{x^2}$

$\therefore \frac{1}{2}v^2 = \int -\frac{R^2 g}{x^2}\, dx$

$\frac{1}{2}v^2 = \frac{R^2 g}{x} + c$

when $x = R$, $v = u$

$\frac{1}{2}u^2 = \frac{R^2 g}{R} + c$

$\therefore c = \frac{1}{2}u^2 - Rg$

$\therefore \frac{1}{2}v^2 = \frac{R^2 g}{x} + \frac{1}{2}u^2 - Rg$

$\therefore v^2 = \frac{2R^2 g}{x} + u^2 - 2gR$

iv) max distance, $v = 0$

$\frac{2R^2 g}{x} + u^2 - 2gR = 0$

$\frac{2R^2 g}{x} = 2gR - u^2$

$\therefore x = \frac{2R^2 g}{2gR - u^2}$

(v) $g = 9.8$, $R = 6400$

as $x \to \infty$, $u^2 = 2gR$

$\underline{u^2 = 2(9.8)(6400)}$

$u = \pm 11200$

but $u > 0$ $\therefore u = 11200 \text{ ms}^{-1}$.

$2gR - u^2 = \frac{2R^2 g}{x}$

$u^2 = 2gR - \frac{2R^2 g}{x}$

$x \to \infty$,

ABBOTSLEIGH

AUGUST 2003
YEAR 12
ASSESSMENT 4
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used
- A table of standard integrals is provided with this paper
- All necessary working should be shown in every question

Total marks - 84
- Attempt Questions 1-7
- All questions are of equal value

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $\frac{4}{x-1} \geq 1$ 3

(b) A is the point $(-2,-1)$ and B is the point $(1,5)$. Find the coordinates of the point Q which divides AB externally in the ratio $5:2$. 2

(c) Given $f(x) = \tan^{-1}(\sin x)$ find $f'(\pi)$ 2

(d) Prove $\frac{1+\sin x-\cos x}{1+\sin x+\cos x} = \tan\frac{x}{2}$ 2

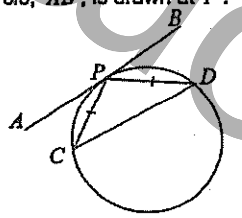
(e) Find the exact value of $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{(3-4x^2)}}$ 3

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the equation $2\sin^2\theta = \sin 2\theta$ for $0 \le \theta \le 2\pi$ 2

(b) PC and PD are equal chords of a circle.
A tangent to the circle, AB, is drawn at P.



Copy the diagram into your answer booklet and prove that AB is parallel to CD. 2

(c) (i) Find $\int \frac{x}{x^2+9}\,dx$ 2

(ii) Evaluate $\int_0^4 x\sqrt{x^2+9}\,dx$ using the substitution $u = x^2+9$ 3

(d) (i) Sketch $y = |x+1|$ 1

(ii) Using your graph, or otherwise, solve $|x+1| > -2x$ for x 2

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the polynomial $P(x) = x^3 - kx^2 - x + 2$

(i) Find the value of k if $x-1$ is a factor of $P(x)$ 1

(ii) Hence factorise $P(x)$ completely. 2

(b) Find the term which is independent of x in the expansion of $\left(x^2 + \frac{2}{x}\right)^9$ 2

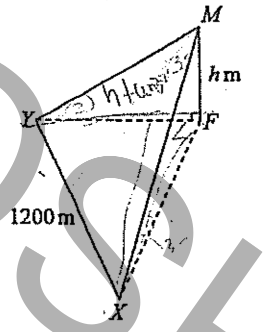
(c) For the function $f(x) = 4\sin^{-1}(x-2)$

(i) Evaluate $f\left(1\frac{1}{2}\right)$ 1

(ii) Sketch $y = f(x)$ clearly indicating the domain and range. 2

(iii) Find $\int_1^3 4\sin^{-1}(x-2)\,dx$ 1

(d) In the diagram, Point X is due south and point Y is due west of the foot, F, of a mountain. From X and Y, the angles of elevation of the top of the mountain M are 35° and 43° respectively.



Not to scale

If X and Y are 1200 metres apart, show that the height, h metres, of the mountain is given by $h = 1200\left(\tan^2 55° + \tan^2 47°\right)^{-\frac{1}{2}}$ and evaluate h. 3

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the graph of $y = \cos x, -\pi \le x \le \pi$ and use this graph to show that $\cos x + x = 0$ has only one solution. 2

(ii) Use Newton's method with a first approximation of $x = -1$ to find a second approximation to the root of $\cos x + x = 0$. 2

(b) The inside of a vessel used for water has the shape of a solid of revolution obtained by the rotation of the parabola $9y = 8x^2$ about the y-axis. The depth of the vessel is 8cm.

(i) Prove that the volume of water $h\text{cm}$ from its base is $\frac{9}{16}\pi h^2$ 1

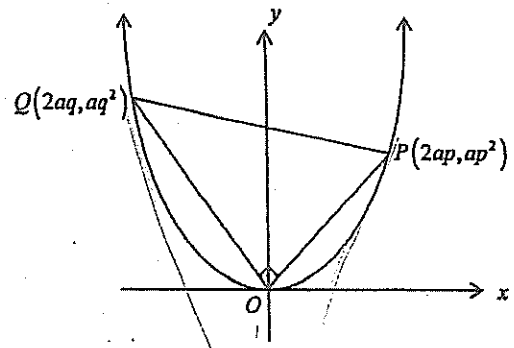
(ii) If water is poured into the vessel at a rate of $20\text{cm}^3/\text{sec}$, find the rate at which the level of water is rising when the vessel is half full. 3

(c) Use the Principle of Mathematical Induction to prove that $2^{3n} - 3^n$ is divisible by 5 for all positive integers n. 4

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram, PQ is a variable chord of the parabola $x^2 = 4ay$. It subtends a right angle at the vertex O. Let p and q be the parameters corresponding to the points P and Q respectively.



(i) Show that the equation of the tangent to $x^2 = 4ay$ at P is $y - px + ap^2 = 0$ 1

(ii) Hence, write down the equation of the tangent at Q, and then find R, the point of intersection of the two tangents drawn at P and Q. 2

(iii) Find the gradients of OP and OQ and hence prove $pq = -4$ 2

(iv) Show that the locus of R, the point of intersection of the two tangents drawn at P and Q is $y = -4a$ 1

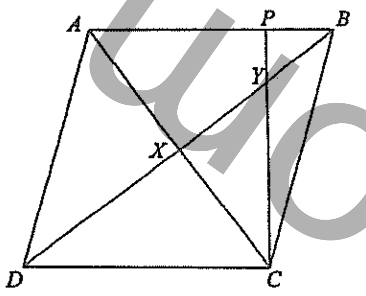
(b) By considering $f(x) = (1+x)^n$ in $\int_0^1 f(x)\,dx$, prove that

$$\sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{2^{n+1}-1}{n+1}$$ 3

Question 5 continues on page 7

Question 5 (continued) **Marks**

(c) $ABCD$ is a rhombus whose diagonals intersect at X. The perpendicular CP from C to AB cuts BD at Y.



Not to scale

Copy the diagram into your writing booklet and prove that B, P, X, C are concyclic. 3

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \sin^2 x \cos^2 x \, dx$ 2

(b) A particle moves in a straight line so that its velocity after t seconds is v ms^{-1} and its displacement is x.

(i) Given that $\frac{d^2x}{dt^2} = 2x^3 - 10x$ and that initially $v = 0$ when $x = -1$, find v in terms of x. 3

(ii) Explain why this motion can only exist between $x = -1$ and $x = 1$. 2

(iii) Describe briefly what would have happened if the initial conditions were $v = 0$ when $x = 0$. 1

(c) In a colony of 400 ants the number, N, diseased at time, t, is given by $N = \frac{400}{1 + ke^{-400t}}$ where k is a constant and t is time in years. (Assume one year is 365 days.)

(i) If at time $t = 0$ one ant was infected, after how many days will half the colony be infected? 3

(ii) Show that eventually all the ants will be infected. 1

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle is projected from a point on level ground with a speed of $V\,\text{ms}^{-1}$ and angle of projection, α. Assume that acceleration due to gravity is $g\,\text{ms}^{-2}$ and that there is no air resistance.

(i) Show that the horizontal and vertical displacements, x and y, of the particle in metres from the point of projection at time t seconds after projection are given by **2**

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$

(ii) Show that the greatest height of the particle is $\dfrac{V^2\sin^2\alpha}{2g}$ **1**

(iii) Show that the range of the particle is $\dfrac{V^2\sin 2\alpha}{g}$ **1**

(iv) Two particles are projected from the same point on level ground with the same speed $V\,\text{ms}^{-1}$ and with angles of projection α and $90° - \alpha$ respectively.

The greatest heights the two particles reach are h_1 and h_2 respectively.

Show that, for any α, $h_1 + h_2 = \frac{1}{2}R$ where R is the maximum range. **3**

(b) A_n and B_n are two series given by

$A_n = 1^2 + 5^2 + 9^2 + 13^2 + \ldots + (4n-3)^2$

$B_n = 3^2 + 7^2 + 11^2 + 15^2 + \ldots$ for $n = 1, 2, 3, \ldots$

(i) Find the nth term of B_n. **1**

(ii) If $S_{2n} = A_n - B_n$, prove that $S_{2n} = -8n^2$. **2**

(iii) Hence, or otherwise, evaluate **2**

$101^2 - 103^2 + 105^2 - 107^2 + \ldots + 2001^2 - 2003^2$

End of Paper

MATHEMATICS EXTENSION 1 - ASSESSMENT 4 TRIAL EXAMINATION AUGUST

Question 1

(a) $\frac{4}{x-1} \geqslant 1$ $\boxed{x \neq 1}$

$4(x-1) \geqslant (x-1)^2$

$(x-1)^2 - 4(x-1) \leqslant 0$

$(x-1)[x-1-4] \leqslant 0$

$(x-1)(x-5) \leqslant 0$ 1 5

$1 < x \leqslant 5$

(b) $A(-2,-1)$ $B(1,5)$ $5:-2$

$x = \frac{5\times1-2\times-2}{5-2}$ $y = \frac{5\times5-2\times-1}{5-2}$

$= 3$ $= 9$

$Q(3,9)$

(c) $f(x) = \tan^{-1}(\sin x)$

$f'(x) = \frac{\cos x}{1+\sin^2 x}$

$f'(\pi) = \frac{\cos\pi}{1+\sin^2\pi} = -1$

(d) $\text{LHS} = \frac{1+\sin x-\cos x}{1+\sin x+\cos x}$

$= \frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$

$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2}$

$= \frac{2t^2+2t}{2+2t}$

$= 2t(t+1)$ ...

(e) $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{3-4x^2}} = \int_0^{\sqrt{3}/2} \frac{dx}{2\sqrt{\frac{3}{4}-x^2}}$

$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{\sqrt{3}}\right]_0^{\sqrt{3}/2}$

$= \frac{1}{2}(\sin^{-1}1 - \sin^{-1}0)$

$= \frac{\pi}{4}$

Question 2

(a) $2\sin^2\theta = \sin 2\theta$, $0 \leqslant \theta \leqslant 2\pi$

$2\sin^2\theta = 2\sin\theta\cos\theta$

$\sin^2\theta - \sin\theta\cos\theta = 0$

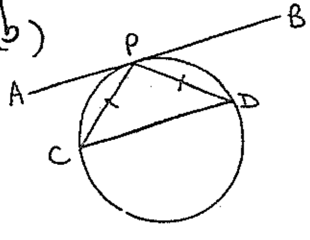
$\sin\theta(\sin\theta - \cos\theta) = 0$

$\sin\theta = 0$, $\sin\theta = \cos\theta$

$\tan\theta = 1$

$\theta = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{5\pi}{4}$

(b)



$\angle PCD = \angle PDC$ given equal chords $\triangle PDC$ isosceles with equal base angle

$\angle BPC = \angle PCD$ angle between tangent + chord = $\angle$ in alternate segment

$\therefore \angle BPC = \angle PDC$

The alternate angles are equal

(c) (i) $\int \frac{x}{x+9}\,dx = \int \frac{x+9-9}{x+9}\,dx$

$= \int\left(\frac{x+9}{x+9} - \frac{9}{x+9}\right)dx$

$= \int\left(1 - \frac{9}{x+9}\right)dx$

$= x - 9\ln(x+9) + C$

(ii) $\int_0^4 x\sqrt{x^2+9}\,dx$

$u = x^2+9$

$\frac{du}{dx} = 2x$

$x=0$ $u=9$

$x=4$ $u=25$

$= \frac{1}{2}\int_0^4 \sqrt{x^2+9} \times 2x\,dx$

$= \frac{1}{2}\int_9^{25} \sqrt{u}\,du$

$= \frac{1}{2}\int_9^{25} u^{\frac{1}{2}}\,du$

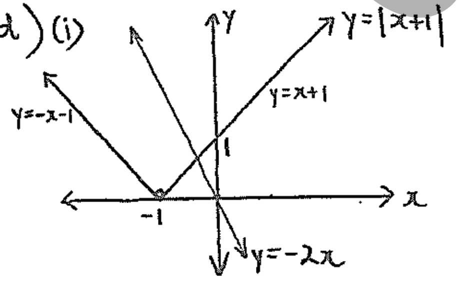
$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_9^{25}$

$= \frac{1}{3}\left((\sqrt{25})^3 - (\sqrt{9})^3\right)$

$= \frac{1}{3}(125 - 27)$

$= \frac{98}{3}$

(d) (i)



(ii) One pt of intersection

Solve $x+1 = -2x$

$3x = -1$

$x = -\frac{1}{3}$

$\therefore |x+1| > -2x$ when $x > -\frac{1}{3}$

Question 3

(a) $P(x) = x^3 - kx^2 - x + 2$

(i) $x-1$ is a factor $\therefore P(1) = 0$

$0 = 1 - k - 1 + 2$

$k = 2$

(ii) $P(x) = x^3 - 2x^2 - x + 2$

$= x^2(x-2) - (x-2)$

$= (x-2)(x^2-1)$

$= (x-2)(x-1)(x+1)$

(b) $T_{k+1} = {}^9C_k \left(\frac{2}{x}\right)^k (x^2)^{9-k}$

$= {}^9C_k 2^k x^{-k} x^{18-2k}$

$= {}^9C_k 2^k x^{18-3k}$

term indept. of x ie x^0

$\therefore 0 = 18 - 3k$

$k = 6$

7th term is independent of x

$T_7 = {}^9C_6 2^6$

$= 2^8 \times 3 \times 7$

$= 5376$

) $f(x) = 4\sin^{-1}(x-2)$

(i) $f(1\frac{1}{2}) = 4\sin^{-1}(-\frac{1}{2})$

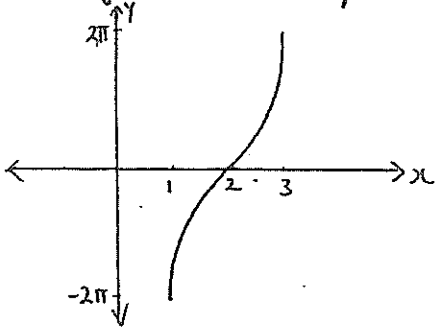
$= 4 \times -\frac{\pi}{6}$

$= -\frac{2\pi}{3}$

(ii) Domain $-1 \leq x-2 \leq 1$

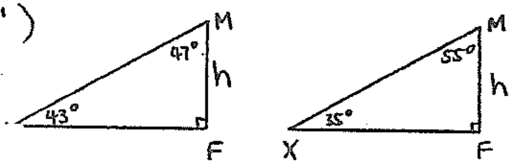
$1 \leq x \leq 3$

Range $-2\pi \leq y \leq 2\pi$



(iii) $\int_1^3 4\sin^{-1}(x-2)\,dx = 0$

)



In ΔMYF $\tan 47° = \frac{YF}{h}$

$YF = h\tan 47°$

In ΔMXF $\tan 55° = \frac{XF}{h}$

$XF = h\tan 55°$

In ΔYFX

$1200^2 = YF^2 + XF^2$

$1200^2 = h^2\tan^2 47° + h^2\tan^2 55°$

$h^2(\tan^2 47° + \tan^2 55°) = 1200^2$

$h^2 = \frac{1200^2}{(\tan^2 47° + \tan^2 55°)}$

$h = \frac{1200}{\sqrt{\tan^2 47° + \tan^2 55°}}$

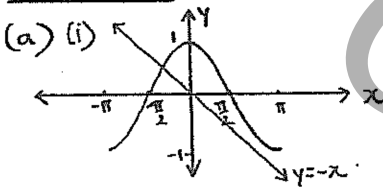
$h = 1200(\tan^2 47° + \tan^2 55°)^{-\frac{1}{2}}$

$h = 671.915\ldots$ m

$h \doteqdot 671.9$ m (to 1 dp)

Question 4

(a) (i)



$\cos x + x = 0 \Rightarrow \cos x = -x$

Consider $y = \cos x$ & $y = -x$

Graphs intersect at one pt. only

$\therefore \cos x + x = 0$ has only one soln.

(ii) $f(x) = \cos x + x$

$f'(x) = -\sin x + 1$

$x_1 = -1$ (in radians)

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= -1 - \frac{\cos(-1) - 1}{1 - \sin(-1)}$

$= -0.75036\ldots$

2nd approximation is

b) (i) $V = \pi\int_0^h \frac{9y}{8}\,dy$

$= \frac{9\pi}{8}\left[\frac{1}{2}y^2\right]_0^h$

$= \frac{9\pi}{8} \times \frac{1}{2}h^2 - 0$

$= \frac{9\pi h^2}{16}$

(ii) $\frac{dV}{dt} = 20$ $\frac{dV}{dh} = \frac{18\pi h}{16} = \frac{9\pi h}{8}$

Find h when $\frac{1}{2}$ full.

Full $\Rightarrow h = 8$cm $V = 36\pi$

$\frac{1}{2}$ full $\Rightarrow V = 18\pi = \frac{9\pi h^2}{16}$

$h^2 = 32$

$h = 4\sqrt{2}$.

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$20 = \frac{9\pi h}{8} \times \frac{dh}{dt}$ $h = 4\sqrt{2}$

$\frac{20 \times 8}{9\pi \times 4\sqrt{2}} = \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{40}{9\pi\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{20\sqrt{2}}{9\pi}$ cm/sec

(c) $2^{3n} - 3^n$ divisible by 5.

Show true for n=1

$2^3 - 3^1 = 8 - 3$

$= 5$ which is $\div$ by 5

Assume true for n=k

ie assume $2^{3k} - 3^k = 5M$ where M is a positive integer

Show true for n=k+1

$2^{3(k+1)} - 3^{k+1} = 2^{3k+3} - 3^{k+1}$

$= 2^3 \times 2^{3k} - 3 \times 3^k$

$= 2^3(5M + 3^k) - 3 \times 3^k$ using assumption

$= 40M + 8 \times 3^k - 3 \times 3^k$

$= 40M + 5 \times 3^k$

$= 5(8M + 3^k)$ which is divisible by 5

$\therefore$ true for n=k+1 if true for n=k

Since true for n=1 it is also tru for n=1+1=2 & thus true for n=2+1=3 and so on for all positive integers n.

Question 5

(a) (i) $y = \frac{x^2}{4a} \Rightarrow y' = \frac{x}{2a}$

at $P(2ap, ap^2)$ $m_T = \frac{2ap}{2a} = p$

eq. tangent $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

) at Q tangent eq. $y - qx + aq^2 = 0$

$y = px - ap^2$, $y = qx - aq^2$

$px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p-q)(p+q)$

$x = a(p+q)$ $p \neq q$

$y = ap(p+q) - ap^2$

$y = apq$

$R\,(a(p+q), apq)$

(iii) $m_{OP} = \frac{ap^2 - 0}{2ap - 0} = \frac{p}{2}$

Similarly $m_{OQ} = \frac{q}{2}$

$OP \perp OQ$ $\therefore m_{OP} \times m_{OQ} = -1$

ie $\frac{pq}{4} = -1 \Rightarrow pq = -4$

(iv) at R $x = a(p+q)$

$y = apq$

from (iii) $pq = -4$ $\Big\rangle$ $y = -4a$ indept. of $p+q$

$\therefore$ locus of R is $y = -4a$

(b) $f(x) = (1+x)^n$

$$\int_0^1 (1+x)^n\,dx = \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1$$

$$= \frac{2^{n+1}}{n+1} - \frac{1^{n+1}}{n+1}$$

$$= \frac{2^{n+1} - 1}{n+1}$$

Also

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots$

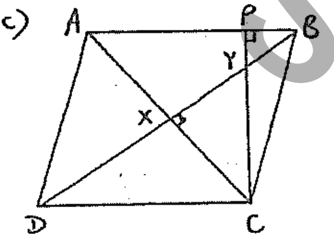
$$\int_0^1 (1+x)^n\,dx = \left[\binom{n}{0}x + \tfrac{1}{2}\binom{n}{1}x^2 + \tfrac{1}{3}\binom{n}{2}x^3 + \tfrac{1}{r+1}\binom{n}{r}x^{r+1} + \dots \tfrac{1}{n+1}\binom{n}{n}x^{n+1}\right]_0^1$$

$$= \binom{n}{0} + \tfrac{1}{2}\binom{n}{1} + \tfrac{1}{3}\binom{n}{2} + \dots + \tfrac{1}{r+1}\binom{n}{r} + \dots + \tfrac{1}{n+1}\binom{n}{n} - 0$$

$$= \sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r}$$

$$\therefore \sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{2^{n+1} - 1}{n+1}$$

(c)



$\angle BPC = 90°$ given

$\angle BXC = 90°$ given rhombus; diagonals bisect each other at right angles.

Interval BC subtends equal angles on the same side of it at points P and X

$\therefore$ B, C, X, P are concyclic

Question 6

(a) $\int \sin^2 x \cos^2 x \, dx$

$= \int (\sin x \cos x)^2 dx$

$= \int (\tfrac{1}{2}\sin 2x)^2 dx$

$\int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int \sin^2 2x \, dx$

$= \frac{1}{4}\int \frac{1}{2}(1 - \cos 4x)\,dx$

$= \frac{1}{8}(x - \frac{1}{4}\sin 4x) + C$

$= \frac{x}{8} - \frac{1}{32}\sin 4x + C$

(b) (i) $\frac{d}{dx}(\frac{1}{2}v^2) = 2x^3 - 10x$

$\frac{1}{2}v^2 = \frac{x^4}{2} - 5x^2 + C$

$v = 0$ when $x = -1$

$0 = \frac{1}{2} - 5 + C$

$C = \frac{9}{2}$

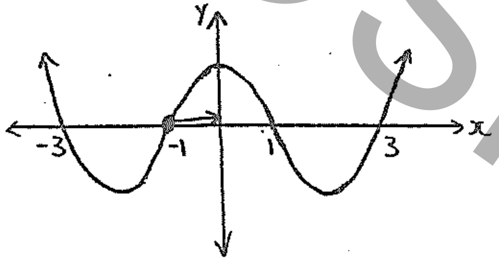
$\frac{1}{2}v^2 = \frac{x^4}{2} - 5x^2 + \frac{9}{2}$

$v^2 = x^4 - 10x^2 + 9$

$v = \pm\sqrt{x^4 - 10x^2 + 9}$

(ii) $v^2 = (x^2 - 9)(x^2 - 1)$

$v^2 = (x-3)(x+3)(x-1)(x+1)$



Starts at $x = -1$ and $v^2 \geq 0$

$\therefore$ only movement from $x = -1$ is to $x = 1$ + back. Exists between

(iii) If $x = 0$ then $a = 0$

With $v = 0$ and $a = 0$ there is no acceleration + no velocity

$\therefore$ particle would not move.

(c) (i) $t = 0$ $N = 1$

$1 = \frac{400}{1 + ke^0}$

$1 + k = 400$

$k = 399$

find t when $N = 200$

$200 = \frac{400}{1 + 399e^{-400t}}$

$1 + 399e^{-400t} = 2$

$399e^{-400t} = 1$

$e^{-400t} = \frac{1}{399}$

$-400t = \ln\frac{1}{399}$

$t = -\frac{1}{400}\ln\frac{1}{399}$

$= 0.0149\ldots$ y

$\doteqdot 5.5$ days

(ii) as $t \to \infty$

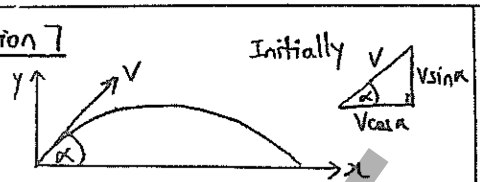
$e^{-400t} \to 0$

$\therefore N \to \frac{400}{1+0} = 400$

ie all the ants will be infected.

Question 7

(a) (i)



Horizontal

$\ddot{x} = 0$

$\dot{x} = c_1$

when $t=0$ $\dot{x} = V\cos\alpha$

$\therefore c_1 = V\cos\alpha$

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + C_2$

when $t=0$ $x=0$

$\therefore C_2 = 0$

$x = Vt\cos\alpha$

Vertical

$\ddot{y} = -g$

$\dot{y} = -gt + C_3$

when $t=0$ $\dot{y} = V\sin\alpha$

$\therefore C_3 = V\sin\alpha$

$\dot{y} = V\sin\alpha - gt$

$y = Vt\sin\alpha - \frac{1}{2}gt^2 + C_4$

when $t=0$ $y=0$

$\therefore C_4 = 0$

$y = Vt\sin\alpha - \frac{1}{2}gt^2$

(ii) greatest ht when $\dot{y} = 0$

$V\sin\alpha = gt$

$t = \frac{V\sin\alpha}{g}$

$y = V \times \frac{V\sin\alpha}{g}\sin\alpha - \frac{1}{2}g\frac{V^2\sin^2\alpha}{g^2}$

$= \frac{V^2\sin^2\alpha}{2g}$

(iii) range $\Rightarrow$ $y=0$

$0 = t\left(V\sin\alpha - \frac{1}{2}gt\right)$

$t = 0$, $t = \frac{2V\sin\alpha}{g}$

$x = V \times \frac{2V\sin\alpha}{g}\cos\alpha$

$= \frac{V^2 \times 2\sin\alpha\cos\alpha}{g} = \frac{V^2\sin 2\alpha}{g}$

(iv) max. range when $\sin 2\alpha = 1$

$\therefore R = \frac{V^2}{g}$

$h_1 = \frac{V^2\sin^2\alpha}{2g}$ $h_2 = \frac{V^2\sin^2(90-\alpha)}{2g}$

$h_1 + h_2 = \frac{V^2}{2g}\left(\sin^2\alpha + \cos^2\alpha\right)$

$= \frac{V^2}{2g}$

$= \frac{1}{2}R$

(b) (i) $B_n = 3^2 + 7^2 + \ldots + (4n-1)^2$

n-th term $(4n-1)^2$

(ii) $S_{2n} = A_n - B_n$

$= 1^2 + 5^2 + \ldots + (4n-3)^2$

$- 3^2 - 7^2 - \ldots - (4n-1)^2$

$S_{2n} = (1^2-3^2) + (5^2-7^2) + (9^2-11^2)$

$+ \ldots + \left[(4n-3)^2 - (4n-1)^2\right]$

$= (1-3)(1+3) + (5-7)(5+7) + \ldots + (9-11)(9+11)$

$+ \ldots + (4n-3-4n+1)(4n-3+4n-1)$

$= -2(4) - 2(12) - 2(20) - \ldots - 2(8n-4)$

$= -2\left(4 + 12 + 20 + \ldots + (8n-4)\right)$

$= -2\left(\frac{n}{2}\left(8 + (n-1)8\right)\right)$

$= -n(8 + 8n - 8)$

$= -8n^2$

(iii) $101^2 - 103^2 + 105^2 - \ldots + 2001^2 - 2003^2$

starts from $n=26$ to $n=501$

$S_{501} - S_{25} = -8(501)^2 + 8(25)^2$

$= -2003008$

Centre Number ☐☐☐☐☐

Student Number ☐☐☐☐☐☐☐☐☐

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

Pymble Ladies' College
Mathematics Department

2003
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 1

Afternoon Session
Tuesday 12 August 2003

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks (84)
- Attempt Questions 1 – 7
- All questions are of equal value

Disclaimer

Every effort has been made to prepare these 'Trial' Higher School Certificate Examinations in accordance with the Board of Studies documents, *Principles for Setting HSC Examinations in a Standards-Referenced Framework* (BOS Bulletin, Vol 8, No 9, Nov/Dec 1999), and *Principles for Developing Marking Guidelines Examinations in a Standards Referenced Framework* (BOS Bulletin, Vol 9, No 3, May 2000). No guarantee or warranty is made or implied that the 'Trial' Examination papers mirror in every respect the actual HSC Examination question paper in any or all courses to be examined. These papers do not constitute 'advice' nor can they be construed as authoritative interpretations of Board of Studies intentions. The CSSA accepts no liability for any reliance use or purpose related to these 'Trial' question papers. Advice on HSC examination issues is only to be obtained from the NSW Board of Studies.

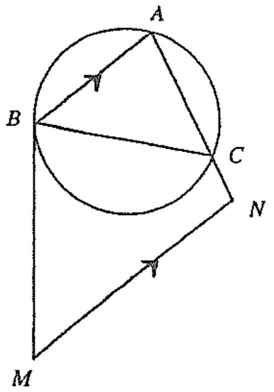
Question 1 **Begin a new page** Mar

(a) Find the value of $\lim_{n\to\infty}\frac{5(10^n)+3}{2(10^n)+1}$. 2

(b) $A(-2,5)$ and $B(7,-1)$ are two points. Find the coordinates of the point $M(x,y)$ which divides the line AB internally in the ratio $2:1$. 2

(c) Solve the inequality $\frac{2}{x} > x-1$. 4

(d)



ABC is a triangle inscribed in a circle. M is a point on the tangent to the circle at B and N is a point on AC produced so that MN is parallel to BA.

(i) Copy the diagram.

(ii) Give a reason why $\angle MBC = \angle BAC$. 1

(iii) Show that $MNCB$ is a cyclic quadrilateral. 3

Question 2 **Begin a new page** **Marks**

(a) If $y=\left(x^2+1\right)^5$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **2**

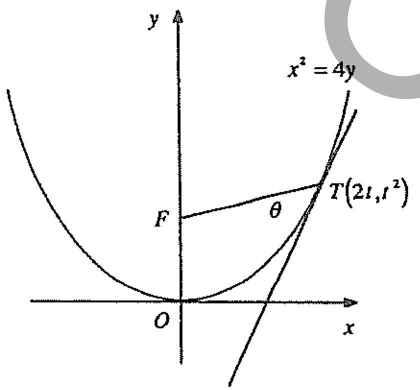
(b) Find the value of $\sum_{n=2}^{5} {}^nC_2$. **2**

(c)

(i) Show that $(\sin A-\cos A)^2=1-\sin 2A$. **2**

(ii) Hence find the exact value of $\sin 15°-\cos 15°$. **2**

(d)



$T\left(2t,t^2\right)$ is a point on the parabola $x^2=4y$ with focus F. The tangent to the parabola at T makes an acute angle θ with the line FT.

(i) Show that the tangent to the parabola at T has gradient t. **1**

(ii) Find $\tan\theta$ in simplest form in terms of t. **3**

Question 3 **Begin a new page.** **Marks**

(a) Consider the function $f(x)=x-e^{-x}$.

(i) Show that the function is increasing and its graph is concave down for all values of x in its domain. **2**

(ii) Use one application of Newton's Method with an initial approximation of $x=0\cdot5$ to find the value of the x intercept on the graph of $y=f(x)$, giving the answer correct to one decimal place. **2**

(iii) Sketch the graph of $y=f(x)$ showing clearly the intercepts on the axes and the equations of any asymptotes. **3**
On the same diagram sketch the graph of the inverse function $y=f^{-1}(x)$.

(b) A particle is moving in a straight line. At time t seconds it has displacement x metres (where $0\le x<\frac{\pi}{2}$) from a fixed point O on the line, velocity v ms^{-1} given by $v=\cos^2 x$ and acceleration a ms^{-2}. The particle starts at O.

(i) Find expressions for a in terms of x and for x in terms of t. **2**

(ii) Sketch the graph of x against t. **1**

(iii) Describe the motion of the particle from its initial position to its limiting position. **2**

Marks

Question 4

Begin a new page.

(a) Consider the function $f(x)=\cos^{-1}\sqrt{x}$.

(i) Find the domain and the range of the function and sketch the graph of $y=f(x)$. 3

(ii) Use Simpson's Rule with three function values to find an approximation to the area bounded by the curve $y=f(x)$ and the coordinate axes. 2

(iii) Use integration to find the exact area bounded by the curve $y=f(x)$ and the coordinate axes. 2

(b) A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point O on the line, velocity $v\,\text{ms}^{-1}$ and acceleration $\ddot{x}\ \text{ms}^{-2}$ given by $\ddot{x}=-4(x-2)$. The particle is at rest at the fixed point O.

(i) Show that $v^2=-4x^2+16x$. 2

(ii) Find the period and amplitude of the motion. 2

(iii) Find the distance travelled by the particle in the first minute of its motion, giving the answer correct to the nearest metre. 1

Question 5

Begin a new page.

(a) In the expansion of $(1+ax)^9$ the coefficient of x^5 is twice the coefficient of x^6. Find the value of the constant a. 4

(b) Evaluate $\int_1^{49}\frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}}\,dx$ using the substitution $u=1+\sqrt{x}$, expressing the answer in the form $\sqrt{n}$ for some positive integer n. 4

(c) A container with capacity A litres is being filled with water. After t minutes the volume V litres of water in the container is given by $V=A\left(1-e^{-kt}\right)$ for some constant $k>0$.

(i) Show that $\frac{dV}{dt}=k(A-V)$. 2

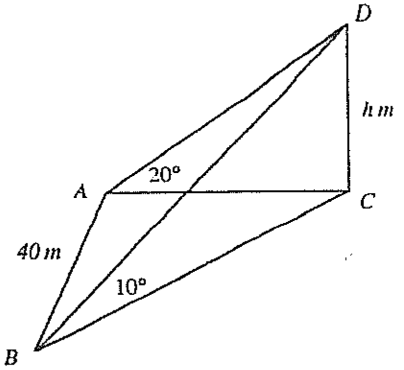
(ii) If one quarter of the container is filled in the first two minutes find what fraction of the container is filled in the next two minutes. 2

Marks

Question 6

Begin a new page.

(a)



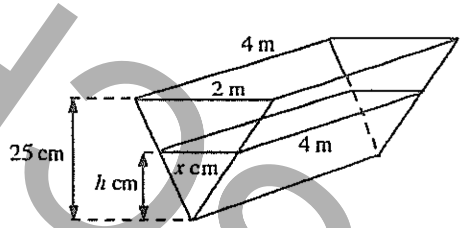
A vertical flagpole CD of height h metres stands with its base C on horizontal ground. A is a point on the ground due West of C and B is a point on the ground 40 metres due South of A. From A and B the angles of elevation of the top D of the flagpole are 20° and 10° respectively. Find the height of the flagpole correct to the nearest metre. 4

(b) A die is biased so that in any single throw the probability of an odd score is p where p is a constant such that $0<p<1,\ p\neq 0\cdot5$.

(i) Show that in six throws of the die the probability of at most one even score is $6p^5-5p^6$. 2

(ii) Find the probability that in six throws of the die the product of the scores is even. 2

(c)



An open flat topped water trough in the shape of a triangular prism is being emptied through a hole in its base at a constant rate of 16 litres per second. Its top measures 2 metres by 4 metres and its triangular end has a vertical height of 25 centimetres. When the water depth is h centimetres the water surface measures x centimetres by 4 metres.

(i) Show that when the water depth is h centimetres the volume $V\,\text{cm}^3$ of water in the trough is given by $V=1600h^2$. 2

(ii) Find the rate at which the depth of water is changing when $h=10\text{cm}$. 2

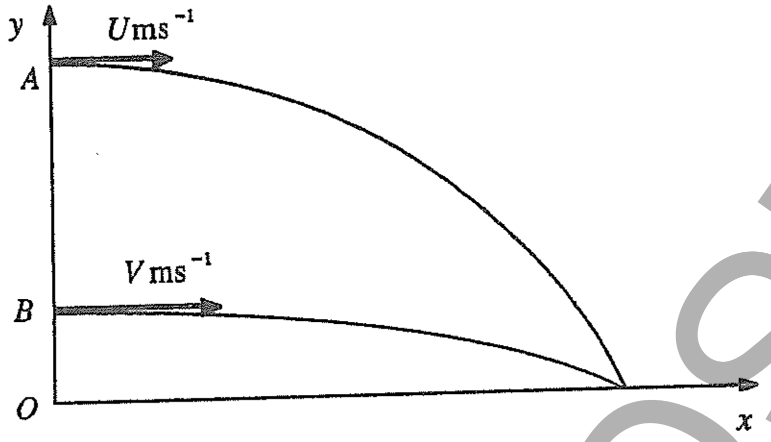
Marks

Question 7 **Begin a new page.**

(a)

$OB = h$ metres

$OA = 4h$ metres



A vertical building stands with its base O on horizontal ground. A and B are two points on the building vertically above each other such that A is $4h$ metres above O and B is h metres above O. A particle is projected horizontally with speed $U\text{ms}^{-1}$ from A and 10 seconds later a second particle is projected horizontally with speed $V\text{ms}^{-1}$ from B. The two particles hit the ground at the same point and at the same time.

(i) Write down expressions for the horizontal and the vertical displacements relative to O of each particle t seconds after the first particle is projected. **2**

(ii) Find the time of flight of each particle. **3**

(iii) Show that $V = 2U$. **1**

(b)

(i) Use Mathematical Induction to show that $\ln(n!) > n$ for all positive integers $n \geq 6$. **3**

(ii) Hence show that $\frac{1}{n!} < \frac{1}{e^n}$ for all positive integers $n \geq 6$. **1**

(iii) Hence show that $\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!}+\ldots\ldots < \frac{103}{60} + \frac{1}{e^5(e-1)}$. **2**

Mathematics Extension I CSSA HSC Trial Examination 2003
Marking Guidelines

Question 1

1(a) Outcomes Assessed : H3

Marking Guidelines

Criteria	Marks
• rearranging limit	1
• finding answer	1

Answer

$$\lim_{n \to \infty} \frac{5(10^n)+3}{2(10^n)+1} = \lim_{n \to \infty} \frac{5+3(10^{-n})}{2+1(10^{-n})} = \frac{5+0}{2+0} = {}^5/_2$$

1(b) Outcomes Assessed : H5

Marking Guidelines

Criteria	Marks
• finding x coordinate	1
• finding y coordinate	1

Answer

$$x = \frac{2(7)+1(-2)}{2+1} = 4 \quad , \quad y = \frac{2(-1)+1(5)}{2+1} = 1$$

1(c) Outcomes Assessed : PE3

Marking Guidelines

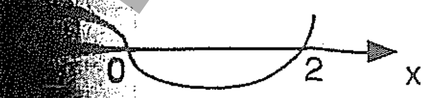
Criteria	Marks
• finding cubic inequality	1
• factoring cubic expression	1
• using diagram	1
• finding answer	1

Answer

[illegible] 1 , $\frac{2}{x} \times x^2 > (x-1) \times x^2$, $2x > x^3 - x^2$, $x^3 - x^2 - 2x < 0$,

[illegible] $-2) < 0$, $x(x+1)(x-2) < 0$

$y = x(x+1)(x-2)$



$x < -1$ or $0 < x < 2$

1(d) Outcomes Assessed : (i) / (ii) PE3 (iii) PE2

Marking Guidelines

Criteria	Marks
(i) copying diagram	0
• (ii) stating alternate segment theorem	1
• (iii) showing that $B\hat{A}C + M\hat{N}C = 180^\circ$	1
• showing that $M\hat{B}C + M\hat{N}C = 180^\circ$	1
• giving a reason why MNCB is cyclic	1

Answer

(i) /

(ii) The angle between the tangent BM and the chord BC is equal to the angle in the alternate segment.

(iii) $B\hat{A}C + M\hat{N}C = 180^\circ$ (cointerior angles supplementary BA parallel to MN)

$M\hat{B}C + M\hat{N}C = 180^\circ$ ($M\hat{B}C = B\hat{A}C$)

MNCB is cyclic (a pair of opposite interior angles is supplementary)

Question 2

2(a) Outcomes Assessed : PE5

Marking Guidelines

Criteria	Marks
• finding first derivative	1
• finding second derivative	1

Answer

$dy/dx = 5(x^2+1)^4 \times 2x = 10x(x^2+1)^4$

$d^2y/dx^2 = 10x \times 4(x^2+1)^3 \times 2x + (x^2+1)^4 \times 10 = 10(x^2+1)^3(9x^2+1)$

2(b) Outcomes Assessed : PE3 , PE6

Marking Guidelines

Criteria	Marks
• writing as a sum of four binomial coefficients	1
• finding answer	1

Answer

$\sum_{n=2}^{n=5} {}^nC_2 = {}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 = 1 + 3 + 6 + 10 = 20$

2(c) Outcomes Assessed : (i) P4 , H5 (ii) P4 , H5

Marking Guidelines

Criteria	Marks
• (i) expanding LHS	1
• showing answer	1
• (ii) using $A = 15^\circ$ to find value of RHS	1
• finding answer	1

Answer

(i) $(\sin A - \cos A)^2 = \sin^2 A + \cos^2 A - 2\sin A\cos A = 1 - \sin 2A$

(ii) $(\sin 15^\circ - \cos 15^\circ)^2 = 1 - \sin 30^\circ = 1 - 1/2 = 1/2$

$\sin 15^\circ - \cos 15^\circ = -1/\sqrt{2}$ (since $\sin 15^\circ < \cos 15^\circ$)

2(d) Outcomes Assessed : (i) PE4 (ii) H5 , PE3

Marking Guidelines

Criteria	Marks
• (i) finding gradient of tangent	1
• (ii) finding gradient of FT	1
• finding initial expression for $\tan\theta$	1
• finding final expression for $\tan\theta$	1

Answer

(i) $y = x^2/4$, $dy/dx = 2x/4 = x/2$

When $x = 2t$, $dy/dx = 2t/2 = t$. The tangent at T has gradient t

(ii) F is the point (0, 1) . FT has gradient $(t^2-1)/2t$

$$\tan\theta = \left|\frac{t-(t^2-1)/2t}{1+t(t^2-1)/2t}\right| = \left|\frac{2t^2-t^2+1}{2t+t^3-t}\right| = \left|\frac{t^2+1}{t^3+t}\right| = \frac{1}{|t|}$$

Question 3

3(a) Outcomes Assessed : (i) H6 (ii) PE3 (iii) HE4

Marking Guidelines

Criteria	Marks
• (i) showing function increasing	1
• showing graph concave down	1
• (ii) finding numerical expression for x intercept	1
• finding answer	1
• (iii) showing intercepts and asymptotes on graph	1
• showing shape of graph	1
• sketching graph of inverse function	1

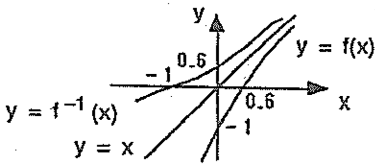
Answer

(i) $f'(x) = 1 + e^{-x} > 0$ for all x , function increasing

$f''(x) = -e^{-x} < 0$ for all x , graph concave down

(ii) $x = 0.5 - \dfrac{f(0.5)}{f'(0.5)} = 0.5 - \dfrac{0.5 - e^{-0.5}}{1 + e^{-0.5}} = 0.6$

(iii)



3(b) Outcomes Assessed : (i) HE5 (ii) HE4 (iii) HE7

Marking Guidelines

Criteria	Marks
• (i) finding a in terms of x	1
• finding x in terms of t	1
• (ii) sketching graph	1
• (iii) describing motion	1
• finding limiting position	1

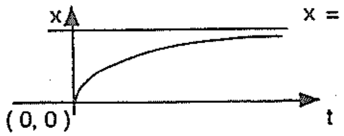
Answer

(i) $a = {}^{d}/_{dx}\,({}^{1}/_{2}\,v^2) = {}^{d}/_{dx}\,({}^{1}/_{2}\cos^4 x) = -2\cos^3 x\sin x$

$^{dx}/_{dt} = \cos^2 x$, $^{dt}/_{dx} = \sec^2 x$, $t = \int \sec^2 x\,dx$

$t = \tan x + c$, $t = 0, x = 0, c = 0$, $t = \tan x$, $x = \tan^{-1} t$

(ii)



(iii) The particle starts at O moving to the right ($v > 0$ for $0 \le x < {}^{\pi}/_{2}$) and slowing down ($v > 0$ and $a < 0$ for $0 < x < {}^{\pi}/_{2}$). It approaches its limiting position of $^{\pi}/_{2}$ metres to the right of O.

Question 4

4(a) Outcomes Assessed : (i) HE4 (ii) H8 (iii) H8

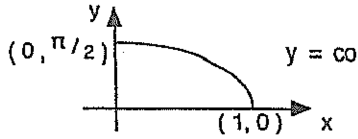
Marking Guidelines

Criteria	Marks
• (i) finding domain	1
• finding range	1
• sketching graph	1
• (ii) finding numerical expression for area	1
• finding answer	1
• (iii) finding primitive	1
• finding answer	1

Answer

(i) domain : $-1 \le \sqrt{x} \le 1$, $0 \le \sqrt{x} \le 1$, $0 \le x \le 1$

range : $\cos^{-1} 1 \le y \le \cos^{-1} 0$, $0 \le y \le {}^{\pi}/_{2}$



(ii)

x	0	$^1/_2$	1
y	$^{\pi}/_{2}$	$^{\pi}/_{4}$	0

Area $= \int_0^1 y\,dx$

Area $= \frac{^1/_2}{3}\{{}^{\pi}/_{2} + 4({}^{\pi}/_{4}) + 0\} = {}^{\pi}/_{4}$ units^2

(iii) Area $= \int_0^{\pi/2} x\,dy = \int_0^{\pi/2} \cos^2 y\,dy = \int_0^{\pi/2} {}^1/_2\,(1 + \cos 2y)\,dy$

$= \left[\,{}^1/_2\,y + {}^1/_4\sin 2y\,\right]_0^{\pi/2} = {}^{\pi}/_{4}$ units^2

4(b) Outcomes Assessed : (i) HE3 (ii) HE3 (iii) HE3

Marking Guidelines

Criteria	Marks
• (i) finding primitive	1
• finding answer	1
• (ii) finding period	1
• finding amplitude	1
• (iii) finding answer	1

Answer

(i) $v^2 = \int 2\ddot{x}\,dx = \int(-8x + 16)\,dx = -4x^2 + 16x + c$

$x = 0, v = 0, c = 0$, $v^2 = -4x^2 + 16x$

(ii) $x = -(2)^2(x - 2)$, $n = 2$, period $= {}^{2\pi}/_{2} = \pi$ seconds.

When $v = 0$, $-4x^2 + 16x = 0$, $-4x(x - 4) = 0$, $x = 0$ or $x = 4$

Centre at $x = 2$. Amplitude $= 2$ metres.

(iii) In π seconds the particle travels 8 metres.

In 1 minute the particle travels $60 \times {}^{8}/_{\pi} = 153$ metres.

Question 5

5(a) Outcomes Assessed : PE3

Marking Guidelines

Criteria	Marks
• finding numerical coefficient of x^5	1
• finding numerical coefficient of x^6	1
• finding equation for a	1
• finding answer	1

Answer

(i) $(1 + ax)^9 = 1 + \ldots + {}^9C_5\,(ax)^5 + {}^9C_6\,(ax)^6 + \ldots + (ax)^9$

$= 1 + \ldots + 126\,a^5 x^5 + 84\,a^6 x^6 + \ldots + a^9 x^9$

$126\,a^5 = 2 \times 84\,a^6$, $a = {}^3/_4$

5(b) Outcomes Assessed : HE6

Marking Guidelines

Criteria	Marks
• finding new integrand	1
• finding new limits	1
• finding primitive	1
• finding answer	1

Answer

$$\int_1^{49} \frac{1}{\sqrt{1+\sqrt{x}}} \frac{1}{\sqrt{x}}\, dx = \int_2^8 \frac{1}{\sqrt{u}} 2\, du = \Big[4\sqrt{u}\Big]_2^8 = 8\sqrt{2} - 4\sqrt{2} = 4\sqrt{2} = \sqrt{32}$$

5(c) Outcomes Assessed : (i) HE3 (ii) HE3

Marking Guidelines

Criteria	Marks
• (i) differentiating	1
• showing answer	1
• (ii) finding expression for k	1
• finding answer	1

Answer

(i) $V = A - Ae^{-kt}$, $Ae^{-kt} = A - V$

$dV/_{dt} = 0 - A(-ke^{-kt}) = Ake^{-kt} = k(Ae^{-kt}) = k(A - V)$

(ii) When $t = 2$, $V = A/_4 \;(= 4A/_{16})$

$A/_4 = A(1 - e^{-2k})$, $1 - e^{-2k} = 1/_4$, $e^{-2k} = 3/_4$

When $t = 4$, $V = A(1 - e^{-4k}) = A(1 - (e^{-2k})^2) = A(1 - (3/_4)^2)$

$= A(1 - 9/_{16}) = 7A/_{16}$. $3/_{16}$ of the container is filled in the next 2 minutes.

Question 6

6(a) Outcomes Assessed : H5

Marking Guidelines

Criteria	Marks
• finding AC and BC in terms of h	1
• using Pythagoras Theorem in Δ ABC	1
• finding expression for h	1
• finding answer	1

Answer

(i) In ΔACD , $\tan 20^\circ = h/_{AC}$, $AC = h \cot 20^\circ$

In ΔBCD , $\tan 10^\circ = h/_{BC}$, $BC = h \cot 10^\circ$

In ΔCAB , $BC^2 = AB^2 + AC^2$, $(h \cot 10^\circ)^2 = 40^2 + (h \cot 20^\circ)^2$

$h^2(\cot^2 10^\circ - \cot^2 20^\circ) = 40^2$, $h = \dfrac{40}{\sqrt{(\cot^2 10^\circ - \cot^2 20^\circ)}} = 8$

6(b) Outcomes Assessed : (i) HE3 (ii) HE3

Marking Guidelines

Criteria	Marks
• (i) using binomial probabilities	1
• showing answer	1
• (ii) using complementary probability	1
• finding answer	1

Answer

(i) P(at most one even score) = P(5 or 6 odd scores) = ${}^6C_5\, p^5(1 - p) + p^6$

$= 6p^5(1 - p) + p^6 = 6p^5 - 6p^6 + p^6 = 6p^5 - 5p^6$

(ii) P(product of scores even) = 1 − P(product of scores odd)

$= 1 -$ P(6 odd scores) $= 1 - p^6$

6(c) Outcomes Assessed : (i) H5 (ii) HE5

Marking Guidelines

Criteria	Marks
• (i) using similar triangles to find expression for x	1
• showing answer	1
• (ii) using chain rule to find expression for $dh/_{dt}$	1
• finding answer	1

Answer

(i) Using similar triangles $x/_{200} = h/_{25}$, $x = 8h$

$V = 1/_2 \times h \times 400 = 1600h^2$

(ii) $dh/_{dt} = dV/_{dt} \div dV/_{dh} = -16000 \div 3200h = -5/_h$

When $h = 10$, $dh/_{dt} = -5/_{10} = -0.5$.

The water level is falling at 0.5 cm s^{-1} .

Question 7

7(a) Outcomes Assessed : (i) HE3 (ii) HE3 (iii) HE3

Marking Guidelines

Criteria	Marks
• (i) writing answers for particle from A	1
• writing answers for particle from B	1
• (ii) eliminating h	1
• finding time of flight for particle from A	1
• finding time of flight for particle from B	1
• (iii) showing answer	1

Answer

(i) $x(A) = Ut$, $y(A) = 4h - 1/_2\, g t^2$

$x(B) = V(t - 10)$, $y(B) = h - 1/_2\, g(t - 10)^2$

(ii) At impact $y(A) = y(B) = 0$

$4h = {}^1/_2\, g t^2$ and $h = {}^1/_2\, g(t-10)^2$, $4(t-10)^2 = t^2$

$2(t-10) = t$ or $2(t-10) = -t$, $t = 20$ or $t = {}^{20}/_3$ (but $t > 10$)

Time of flight for particle from A = 20 seconds

Time of flight for particle from B = 10 seconds

(iii) At impact $x(A) = x(B)$

$20U = 10V$, $V = 2U$

7(b) Outcomes Assessed : (i) HE2 (ii) H3 (iii) H5

Marking Guidelines

Criteria	Marks
• (i) showing S(1) is true	1
• using logarithm law	1
• showing S(k) true implies S(k + 1) true	1
• (ii) showing answer	1
• (iii) using limiting sum	1
• showing answer	1

Answer

(i) $S(n) : \ln n! > n$ for all $n \geq 6$

When $n = 6$, $\ln 6! = 6.58 > 6$. S(1) is true

If S(k) is true for some $k \geq 6$, i.e. if $\ln k! > k$ for some $k \geq 6$

then $\ln(k+1)! = \ln\{(k+1) \times k!\} = \ln(k+1) + \ln k!$

$> \ln e + \ln k!$ (since $k+1 > e$) $> 1 + k$

and so S(k + 1) is also true

It follows that S(n) is true for all $n \geq 6$

(ii) $\ln n! > n$ for all $n \geq 6$, $n! > e^n$ for all $n \geq 6$, $\dfrac{1}{n!} < \dfrac{1}{e^n}$ for all $n \geq 6$

(iii) $\dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!} + \dfrac{1}{8!} + \dfrac{1}{9!} + \ldots$

$< \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{e^6} + \dfrac{1}{e^7} + \dfrac{1}{e^8} + \dfrac{1}{e^9} + \ldots$

$< \dfrac{206}{120} + \dfrac{1}{e^6}\left(1 + \dfrac{1}{e} + \dfrac{1}{e^2} + \dfrac{1}{e^3} + \ldots\right)$

(limiting sum of geometric series in brackets exists since $-1 < r = {}^1/e < 1$)

$< \dfrac{103}{60} + \dfrac{1}{e^6}\,\dfrac{1}{1 - {}^1/e}$

$< \dfrac{103}{60} + \dfrac{1}{e^5(e-1)}$

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Barker College

2003
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1

PM THURSDAY 14 AUGUST

Staff Involved:

- CFR*
- HG*
- DOK
- RMH
- MRB
- BJR
- VAB

90 copies

General Instructions

- **Reading time – 5 minutes**
- **Working time – 2 hours**
- **Write using blue or black pen**
- **Make sure your Barker Student Number is on ALL pages**
- **Board-approved calculators may be used**
- **A table of standard integrals is provided on page 9**
- **ALL necessary working should be shown in every question**
- **Marks may be deducted for careless or badly arranged working**

Total marks – 84

- **Attempt Questions 1 – 7**
- **All questions are of equal value**

Total marks – 84
Attempt Questions 1 – 7
ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (12 marks) **[BEGIN A NEW PAGE]**

(a) If $f(x) = x^2$ and $g(x) = -\sqrt{x}$, what is the value of $f(g(9)) - g(f(9))$? **2**

(b) $y = f(x)$ is a linear function with slope $\frac{1}{2}$

(i) Find an expression for the inverse function of $y = f(x)$ **2**

(ii) Hence find the slope of $y = f^{-1}(x)$ **1**

(c) Find $\int \frac{2}{3\sqrt{16 - x^2}}\,dx$ **1**

(d) Find the coordinates of the point that divides the interval AB, where A is $(-1, 3)$ and B is $(2, 8)$, externally in the ratio of $3 : 2$. **2**

(e) If $\sin 2A = \frac{1}{2}$, what is the value of $\frac{1}{\sin A \cos A}$? **2**

(f) If $0 \leq t \leq 1$, find the Cartesian equation of the curve whose parametric equations are $y = t^2$ and $x = \sqrt{t}$ **2**

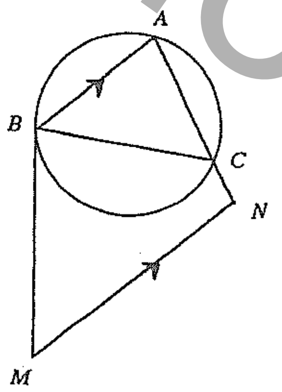
Question 1 **Begin a new page** Marks

(a) Find the value of $\lim_{n \to \infty} \frac{5(10^n)+3}{2(10^n)+1}$. 2

(b) $A(-2,5)$ and $B(7,-1)$ are two points. Find the coordinates of the point $M(x,y)$ which divides the line AB internally in the ratio $2:1$. 2

(c) Solve the inequality $\frac{2}{x} > x-1$. 4

(d)



ABC is a triangle inscribed in a circle. M is a point on the tangent to the circle at B and N is a point on AC produced so that MN is parallel to BA.

(i) Copy the diagram.

(ii) Give a reason why $\angle MBC = \angle BAC$. 1

(iii) Show that $MNCB$ is a cyclic quadrilateral. 3

Marks

Question 3 (12 marks) **[BEGIN A NEW PAGE]**

(a) Differentiate $x\cos^{-1}x$ 2

(b) Find $\int_0^{\pi} \sin^3 x\, dx$ using the result $\sin 3x = 3\sin x - 4\sin^3 x$ 3

(c) A boat is attached by a rope to a jetty 2 m **above** the bow of the boat. The rope is being pulled in at the rate of 1 m s^{-1}.
At what rate is the boat approaching the jetty when 3 m of rope still remains to be pulled in? (Answer correct to 1 decimal place) 4

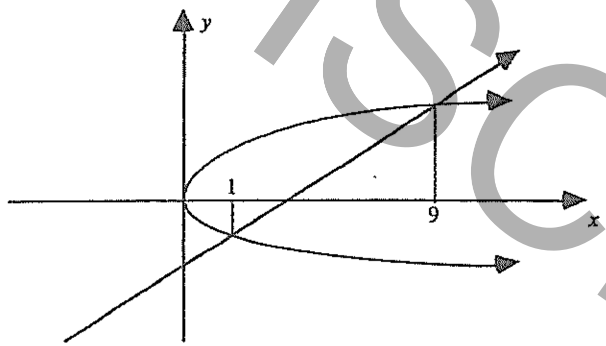
(d) (i) Express $x^2 + x + 1$ in the form $(x - A)^2 + B$ where A, B are constants. 1

(ii) Hence find $\int \frac{dx}{x^2 + x + 1}$ 2

Marks

Question 4 (12 marks) **[BEGIN A NEW PAGE]**

(a) The curves $y^2 = 16x$ and $y = 2x - 6$ intersect at the points where $x = 1$ and $x = 9$.



Find the acute angle between the two curves at the point where $x = 1$ 3

(b) If $\tan\frac{\theta}{2} = t$ and θ is acute, express $\sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}}$ in terms of t 3

(c) Evaluate in exact form $\cos 105°$ 2

(d) Solve $\sqrt{2}\cos x - \sin x = \frac{3}{2}$ for $0° \le x \le 360°$ 4

Marks

Question 5 (12 marks) **[BEGIN A NEW PAGE]**

(a) A body is cooling in a room of constant temperature $15°$C.
At time t minutes its temperature, T, decreases according to the equation

$$\frac{dT}{dt} = -k(T - 15)$$

where k is a positive constant.
The initial temperature of the body is $75°$C, and it cools to $55°$C after 10 minutes.
What is the temperature of the body after a further 5 minutes?
(Answer correct to 1 decimal place) 4

(b) (i) Show that the relation $v^2 = -kx^2 + c$, where k and c are constants, is satisfied by the equation $\frac{d}{dx}\left(\frac{v^2}{2}\right) = -kx$ 1

(ii) A pendulum, P, swings so that it oscillates about its centre of motion according to the equation $\frac{d^2x}{dt^2} = \frac{-x}{9}$, where x is the distance of P from its centre of oscillation at any time t seconds.
Show that $v^2 = \frac{1}{9}(4 - x^2)$, given that its maximum displacement is 2 cm.
Hence find the maximum speed of P. 4

(c) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx$ using $u = \cos x$ 3

Marks

Question 6 (12 marks) **[BEGIN A NEW PAGE]**

(a) Write down the value of ${}^nC_j - {}^nC_{n-j}$ 1

(b) Find the term independent of x in the expansion of $\left(x^3 + \frac{5}{x}\right)^8$ 3

(c) By considering the identity $(1 + x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$, show that

$$\sum_{k=1}^{2n} k\binom{2n}{k} = n4^n$$ 4

(d) What is the greatest coefficient in the expansion of $(2 + 3x)^{30}$? 4

Mark

Question 7 (12 marks) **[BEGIN A NEW PAGE]**

(a) Given that $y = \sin x$, and using the result $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, it can be shown that :

$$\frac{dy}{dx} = \cos x$$

$$= \sin\left(x + \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2} = \cos\left(x + \frac{\pi}{2}\right)$$

$$= \sin\left[\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= \sin\left[x + \frac{2\pi}{2}\right]$$

Similarly :

$$\frac{d^3y}{dx^3} = \sin\left(x + \frac{3\pi}{2}\right)$$

Therefore :

$$\frac{d^ny}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$$

Prove, by induction, that the generalisation given above,
i.e. $\frac{d^ny}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$, is correct for all positive integers n
when $y = \sin x$ 5

(b) A particle is projected under gravity with speed u m s^{-1} and at an angle $\frac{\pi}{4}$, from a point O on horizontal ground. It strikes the ground at P, where $OP = R$.

(i) Taking the x and y axes through 0, show that the equation of the trajectory is given by $y = x - g\frac{x^2}{u^2}$ 2

(ii) Hence, or otherwise, show that $R = \frac{u^2}{g}$ 2

(iii) A ball is fired from O with velocity 30 m s^{-1} at an angle $\frac{\pi}{4}$ to the horizontal. Find the speed of the ball when it has travelled a horizontal distance of 15 m from its starting point. (Take $g = 10$ m s^{-2})
(Answer correct to 1 decimal place) 3

End of Paper

YEAR 12 EXT. 1 MATHS TRIAL 2003, SOLUTIONS

QUESTION 1:

a) $f[g(9)] - g[f(9)] = (-\sqrt{9})^2 - \sqrt{9^2}$ ①

$= 9 + 9 = 18$ ①

b)(i) $y = \frac{1}{2}x + b$

Inv. $x = \frac{1}{2}y + b$ ①

$2x = y + c$

$\therefore y = 2x - c$ ①

(ii) $m_{inv.} = 2$ ①

(c) $\int \frac{2}{3\sqrt{16 - x^2}}\,dx = \frac{2}{3}\int \frac{1}{\sqrt{4^2 - x^2}}\,dx$

$= \frac{2}{3}\sin^{-1}\frac{x}{4} + C$ ①

(d) $A(-1, 3)$ $B(2, 8)$

$-3 : 2$

Point $= \left(\frac{2\times -1 + -3\times 2}{-3+2}, \frac{2\times 3 + -3\times 8}{-3+2}\right)$ ①

$= (8, 18)$ ①

(e) $\frac{1}{\sin A\cos A} = \frac{1}{\frac{1}{2}\sin 2A}$ ①

$= \frac{1}{\frac{1}{2}\times\frac{1}{2}}$

$= 4$ ①

(f) $y = t^2$, $x = \sqrt{t}$

$\therefore t = x^2$ ①

$\therefore y = (x^2)^2 = x^4$ ①

QUESTION 2:

(a)(i) $y = 2\sin^{-1}\frac{x}{3}$

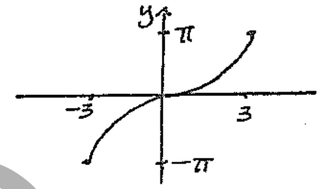
Dom. $-1 \leq \frac{x}{3} \leq 1$

$\therefore -3 \leq x \leq 3$ ①

Range: $-\frac{\pi}{2} \leq \sin^{-1}\frac{x}{3} \leq \frac{\pi}{2}$

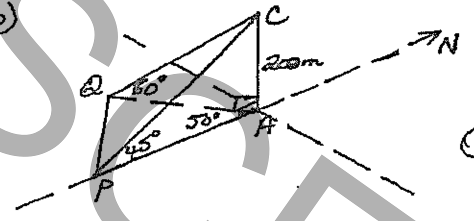
$\therefore -\pi \leq y \leq \pi$ ①

(ii)

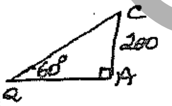


①

(b)

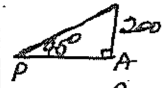


①

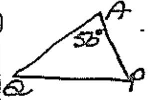


$QA = \frac{200}{\tan 60^\circ}$

$\doteqdot 115{\cdot}47$



$PA = 200$ (isos. ⊥ Δ)



$QP^2 = (115{\cdot}47)^2 + 200^2 - 2\times 115{\cdot}47\times 200\times\cos 50^\circ$

$= 23644{\cdot}24678$

$\therefore QP \doteqdot 154$ m (nearest m) ①

(c) $y = \frac{x^2}{1 - x^2}$

(i) Vert. asym. $x \neq \pm 1$ ①

(ii) $\lim\limits_{x\to\pm\infty}\frac{x^2}{1-x^2} = \lim\limits_{x\to\pm\infty}\frac{1}{\frac{1}{x^2} - 1}$

$= -1$ ①

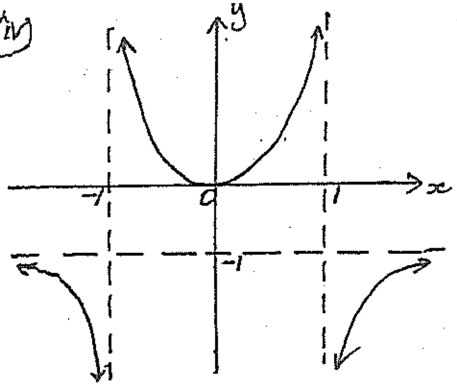
(iii) $f(x) = \frac{x^2}{1 - x^2}$

$f(-x) = \frac{(-x)^2}{1 - (-x)^2}$ ①

$= \frac{x^2}{1 - x^2} = f(x)$.

$\therefore$ Even

2(c)(iv)



QUESTION 3:

(a) Let $y = x\cos^{-1}x$

① rule
① ans.

$\frac{dy}{dx} = \cos^{-1}x + x\times -\frac{1}{\sqrt{1-x^2}}$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$

(b) $\sin 3x = 3\sin x - 4\sin^3 x$ ①

$\therefore \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\int_0^{\pi}\sin^3 x\,dx = \frac{1}{4}\int_0^{\pi}(3\sin x - \sin 3x)\,dx$

①

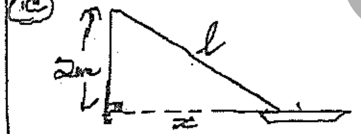
$= \frac{1}{4}\left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\pi}$

$= \frac{1}{4}\left\{\left(-3\cos\pi + \frac{1}{3}\cos 3\pi\right) - \left(-3\cos 0 + \frac{1}{3}\cos 0\right)\right\}$

$= \frac{1}{4}\left\{\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right\}$

$= \frac{4}{3}$ ①

(c)



$l = \sqrt{x^2 + 4}$, $\frac{dl}{dt} = -1$, $l = 3$, $x = \sqrt{5}$ ①

$\frac{dl}{dx} = \frac{1}{2}(x^2+4)^{-\frac{1}{2}}\times 2x$

$= \frac{x}{\sqrt{x^2+4}}$ ①

$\frac{dl}{dt} = \frac{dl}{dx}\times\frac{dx}{dt}$

$-1 = \frac{\sqrt{5}}{\sqrt{5+4}}\frac{dx}{dt}$ ①

$\therefore \frac{dx}{dt} = -\frac{3}{\sqrt{5}}$

$\doteqdot -1{\cdot}3416$

$\therefore$ Boat app. jetty at $1{\cdot}3\,ms^{-1}$

(d)(i) $x^2 + x + 1 = x^2 + x + \left(\frac{1}{2}\right)^2 + 1 -$

$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

OR

$x^2 + x + 1 \equiv x^2 - 2Ax + A^2 + B$

$\therefore -2A = -1 \Rightarrow A = -\frac{1}{2}$

$\therefore \left(-\frac{1}{2}\right)^2 + B = 1 \Rightarrow B = \frac{3}{4}$

$\therefore x^2 + x + 1 \equiv \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

(ii) $\int\frac{dx}{x^2+x+1} = \int\frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$

$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}$

QUESTION 4:

(a) At $x = 1$, $y = -4\sqrt{x}$

$\frac{dy}{dx} = -2x^{-\frac{1}{2}}$

$\therefore m_1 = -2$ (at $x = 1$)

$y = 2x - 6$

$m_2 = 2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-2-2}{1+-2\times 2}\right|$

$= \left|\frac{-4}{-3}\right| = \frac{4}{3}$

$\therefore \theta \doteqdot 53^\circ 8'$

4.(b) $\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\frac{1-(1-2\sin^2\theta)}{1+(1-2\sin^2\theta)}}$ ①

$= \sqrt{\frac{2\sin^2\theta}{2-2\sin^2\theta}}$

$= \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}$ $\Big\}$ ①

$= \tan\theta$

$= \frac{2t}{1-t^2}$ ①

(c) $\cos 105^\circ = \cos(60^\circ+45^\circ)$ ①

$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$ ①

(d) $\sqrt{2}\cos x - \sin x \equiv R\cos(x+\alpha)$

$R = \sqrt{(\sqrt{2})^2+1^2} = \sqrt{3}$ ①

$\cos\alpha = \sqrt{2}/\sqrt{3}$

$\therefore \alpha = 35^\circ 16'$ ①

$\therefore \sqrt{3}\cos(x+35^\circ 16') = \frac{3}{2}$

$\cos(x+35^\circ 16') = \frac{\sqrt{3}}{2}$ ①

$\therefore x+35^\circ 16' = 30^\circ, 330^\circ, 390^\circ, \ldots$

$\therefore x = 294^\circ 44'$ or $354^\circ 44'$ ①

QUESTION 5:

(a) $T = 15 + Ae^{-kt}$

$t=0$, $T=75$ $\therefore 75 = 15 + Ae^0$

$\therefore A = 60$ ①

$t=10$, $T=55$ $\therefore 55 = 15 + 60e^{-10k}$ A7.1

$\therefore \frac{2}{3} = e^{-10k}$

$k = -\frac{1}{10}\ln\frac{2}{3}$ ①

$\left(\text{or } \frac{1}{10}\ln\frac{3}{2}\right)$

$T = 15+60e^{\frac{t}{10}\ln\frac{2}{3}}$

$t=15$, $T = 15+60e^{\frac{15}{10}\ln\frac{2}{3}}$ ①

(b)(i) $v^2 = -kx^2 + c$

$\frac{1}{2}v^2 = -\frac{k}{2}x^2 + c_1$ ①

$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = -kx$

(ii) $\frac{d^2x}{dt^2} = -\frac{x}{9}$

i.e. $\frac{d(\frac{1}{2}v^2)}{dx} = -\frac{x}{9}$

$\therefore \frac{1}{2}v^2 = -\frac{x^2}{18} + c$ ①

$v=0$, $x=2$ $\therefore 0 = -\frac{4}{18} + c$ ①

$\therefore c = \frac{4}{18}$

$\therefore \frac{1}{2}v^2 = -\frac{x^2}{18} + \frac{4}{18}$

$\therefore v^2 = -\frac{x^2}{9} + \frac{4}{9}$ ① meth.

$= \frac{1}{9}(4-x^2)$

Max v when $x=0$ $\therefore v^2 = \frac{4}{9}$

$v = \pm\frac{2}{3}$

$\therefore$ Max speed $\frac{2}{3}$ cms^{-1} ①

(c) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$

$\left\{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \\ x=\frac{\pi}{3}, u=\frac{1}{2} \\ x=\frac{\pi}{2}, u=0 \end{array}\right.$ ①

$= \int_{\frac{1}{2}}^{0} -u^2\,du$

$= \left[-\frac{u^3}{3}\right]_{\frac{1}{2}}^{0}$ ①

$= 0 - -\frac{\frac{1}{8}}{3} = \frac{1}{24}$ ①

QUESTION 6:

(a) ${}^nC_j - {}^nC_{n-j} = 0$ ①

(b) $\left(x^3+\frac{5}{x}\right)^8$

$T_{r+1} = {}^8C_r\, x^{3(8-r)}\left(\frac{5}{x}\right)^r$ ①

$= \binom{8}{r} 5^r x^{24-4r}$ ①

$\therefore 24-4r = 0$ for T indep. of x

$\therefore r = 6$

$\therefore T_7 = \binom{8}{6}5^6$ ①

$= 437\,500$

6.(c) $(1+x)^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}x^k$

Differentiating both sides wrt x:

$2n(1+x)^{2n-1} = \sum_{k=1}^{2n} k\binom{2n}{k}x^{k-1}$ ② 1 each side

Let $x=1$, $2n \times 2^{2n-1} = \sum_{k=1}^{2n} k\binom{2n}{k}$ ①

$\therefore n2^{2n} = \sum_{k=1}^{2n} k\binom{2n}{k}$

$\therefore \sum_{k=1}^{2n} k\binom{2n}{k} = n4^n$ $\Big\}$ ①

(d) $(2+3x)^{30}$:

$\frac{\text{coeff. } T_{r+1}}{\text{coeff. } T_r} = \frac{n-r+1}{r} \times \frac{b}{a}$

$= \frac{31-r}{r} \times \frac{3}{2}$ ①

If $T_{r+1} > T_r$ then:

$93-3r > 2r$

$5r < 93$

$r < 18\frac{3}{5}$ ② method

$\therefore r = 18, 17, 16, \ldots$

$\therefore T_{19} > T_{18} > T_{17} > \ldots$

If $T_{r+1} < T_r$ then:

$93-3r < 2r$

$5r > 93$

$r > 18\frac{3}{5}$

$\therefore r = 19, 20, \ldots$

$\therefore T_{20} < T_{19}, T_{21} < T_{20}, \ldots$

$\therefore T_{19}$ coeff. is greatest.

coeff. $T_{19} = \binom{30}{18}2^{30-18}3^{18}$

$= \binom{30}{18}2^{12}3^{18}$ ①

QUESTION 7:

(a) STEP 1: Prove true for $n=1$

$y = \sin x$

$\frac{dy}{dx} = \cos x = \sin\left(x+\frac{\pi}{2}\right)$ (1

$\therefore$ True for $n=1$

STEP 2: Assume true for $n=k$

i.e. assume $\frac{d^ky}{dx^k} = \sin\left(x+\frac{k\pi}{2}\right)$ (

7. (a) cont. STEP 3: Prove true for $n=k+1$

i.e. Prove $\frac{d^{k+1}y}{dx^{k+1}} = \sin\left[x+\frac{(k+1)\pi}{2}\right]$

Now $\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$

$= \frac{d}{dx}\sin\left(x+\frac{k\pi}{2}\right)$ from assump.

$= \cos\left(x+\frac{k\pi}{2}\right)$

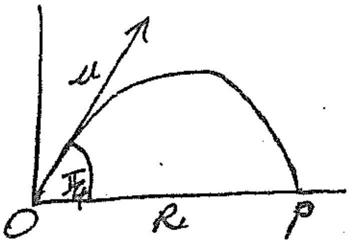
$= \sin\left(x+\frac{k\pi}{2}+\frac{\pi}{2}\right)$

$= \sin\left[x+\frac{(k+1)\pi}{2}\right]$

③ including 1 for conclusion if it follows from proof.

Hence if true for $n=k$, then true for $n=k+1$.
But true for $n=1$ ∴ true for $n=2$, $n=3$ and all positive integers n.

(b) (i)



Initially: $\dot{x}=u\cos\frac{\pi}{4}$, $\dot{y}=u\sin\frac{\pi}{4}$
$=\frac{u}{\sqrt{2}}$, $=\frac{u}{\sqrt{2}}$
$x=0$, $y=0$

Horiz.

$\ddot{x}=0$

$\dot{x}=C$

$=\frac{u}{\sqrt{2}}$

$x=\frac{u}{\sqrt{2}}t+C_1$

$\overset{t=0}{x=0}$ ∴ $C=0$

∴ $x=\frac{u}{\sqrt{2}}t$ ①

Vert.

$\ddot{y}=-g$

$\dot{y}=-gt+K$

$\overset{t=0}{\dot{y}=\frac{u}{\sqrt{2}}}$ ∴ $K=\frac{u}{\sqrt{2}}$

∴ $\dot{y}=-gt+\frac{u}{\sqrt{2}}$

$y=-\frac{gt^2}{2}+\frac{u}{\sqrt{2}}t+K_1$

$\overset{t=0}{y=0}$ ∴ $K=0$ ∴ $y=-\frac{gt^2}{2}+\frac{u}{\sqrt{2}}t$ ①

Now: $t=\frac{\sqrt{2}x}{u}$

∴ $y=-\frac{g}{2}\left(\frac{\sqrt{2}x}{u}\right)^2+\frac{u}{\sqrt{2}}\times\frac{\sqrt{2}x}{u}$ ①

$=x-g\frac{x^2}{u^2}$

(ii) At P, $y=0$ ∴ $0=x\left(1-\frac{gx}{u^2}\right)$

∴ $x=0$ or $\frac{u^2}{g}$

∴ $OP=R=\frac{u^2}{g}$

OR

① for method

$R=\frac{V^2\sin 2\alpha}{g}$

$=\frac{u^2\sin\frac{\pi}{2}}{g}=\frac{u^2}{g}$

(iii) From (i) $15=\frac{30}{\sqrt{2}}t$

∴ $t=\frac{\sqrt{2}}{2}$

$\dot{x}=\frac{30}{\sqrt{2}}=15\sqrt{2}$

$\dot{y}=-10t+\frac{30}{\sqrt{2}}$

$=-5\sqrt{2}+15\sqrt{2}=10\sqrt{2}$

∴ Speed $=\sqrt{(15\sqrt{2})^2+(10\sqrt{2})^2}$

$=\sqrt{450+200}$

$=\sqrt{650}$

$\doteqdot 25.5\ ms^{-1}$

Question Two (Start a NEW booklet)

Marks

(a) Evaluate $\lim_{x \to 0} \frac{\tan 2x}{4x}$ 1

(b) Solve the equation 4

$\sin\theta + \sqrt{3}\cos\theta = 1$ for $0 \le \theta \le 2\pi$

(c) Air is being pumped into a spherical balloon at the rate of 450cm^3 s^{-1}. Calculate the rate at which the radius of the balloon is increasing at the instant when the radius reaches 15cm. $\left[V = \frac{4}{3}\pi r^3\right]$ 3

(d) Let $f(x) = \cos x - \ln x$ 4

(i) Show that a root to $f(x) = 0$ lies between 0·5 and 1·5.

(ii) Starting with a value of $x = 1$, use one application of Newton's method to find a better approximation to this root of $f(x) = 0$.

Question Three (Start a NEW booklet)

Mar

(a) The region R is bounded by the curve $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$ and the x-axis.

(i) Sketch R.

(ii) Find the exact volume of the solid generated when the region R is rotated about the x-axis.

(b) If α, β, γ, are the roots of the cubic polynomial equation $x^3 + 4x^2 - 6x - 8 = 0$ find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 3.

(c) Find the term which is independent of x in the expansion of $\left(2x^3 + \frac{1}{3x^2}\right)^5$ 3

(d) The remainder when $x^3 + ax + b$ is divided by $(x-2)(x+3)$ is $2x+1$. Find the values of a and b. 3

THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2003

MATHEMATICS

EXTENSION 1

Time Allowed: Two hours (plus 5 minutes reading time)

Teacher Responsible: Mr D Price

SPECIAL INSTRUCTIONS:

- This paper contains 7 questions. ALL questions to be attempted.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question in the booklets provided.
- Start each question in a new booklet.
- A table of standard integrals is supplied at the back of this paper.
- Board approved calculators may be used.
- Hand up your paper in ONE bundle, together with this question paper.
- ALL HSC course outcomes are being assessed in this task. The Course Outcomes are listed on the back of this sheet.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

Question One Marks

(a) Evaluate $\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$ 2

(b) Differentiate $\cos^3 x$ 2

(c) Find the point which divides the line joining (4, 6) to (13, 5) externally in the ratio 4:1 2

(d) Write down the equation of the vertical asymptote of $y = \frac{2x}{3x-1}$ 1

(e) Solve for x: $\frac{3}{x+5} \leq 1$ 2

(f) Evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3}{\sqrt{1-x^4}}\,dx$ using the substitution $u = x^4$ 3

Question Six (Start a NEW booklet) **Marks**

(a) Prove by induction that, for all integers $n \geq 1$, 3

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

(b) Let $f(x)=x^2+6x$ for $x \geq -3$ 6

(i) Write down the range of $f(x)$.

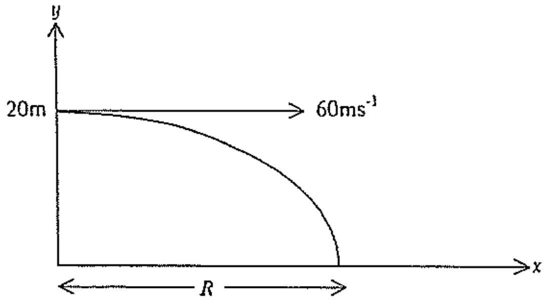
(ii) Briefly explain why the inverse function $f^{-1}(x)$ exists. Write down the domain and range of $f^{-1}(x)$.

(iii) Find $f^{-1}(x)$. Sketch the graph of $y=f^{-1}(x)$.

(c) Sketch the graph of $y=3\cos^{-1}(\frac{x}{2}-1)$. 3

Question Seven (Start a NEW booklet) Ma

(a) 8



An arrow is fired horizontally with a speed of 60ms^{-1} from the top of a 20m high wall on level ground as represented in the diagram above.

It is given that $\ddot{x}=0$ and $\ddot{y}=-10$ where (x, y) is the position of the arrow at time t seconds after firing.

(i) Using calculus, show that $x=60t$ and $y=20-5t^2$.

(ii) Find the time taken for the arrow to hit the ground.

(iii) Find the distance R metres from base of the wall where the arrow hits the ground.

(iv) Find the acute angle to the horizontal at which the arrow hits the ground.

(b) It is given that:

$$(1+x)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}x^k$$

(i) Show that $\sum_{k=0}^{2n}\binom{2n}{k}=4^n$

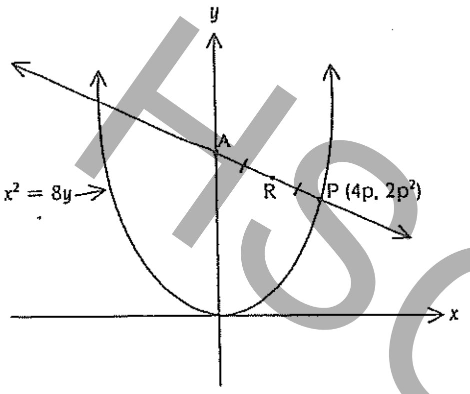
(ii) $\sum_{k=0}^{2n}\binom{2n}{k}\frac{1}{k+1}=\frac{4^{n+1}-2}{4n+2}$

END OF EXAMINATION

Question Four (Start a NEW booklet)

Marks

(a) 7



P (4p, $2p^2$) is a variable point on the parabola $x^2 = 8y$ as shown in the diagram above.

The normal at P cuts the y-axis at A and R is the midpoint of AP.

(i) Show that the normal at P has equation $x + py = 4p + 2p^3$

(ii) Show that R has coordinates $(2p,\ 2p^2 + 2)$

(iii) Show that the locus of R is a parabola and show that the vertex of this parabola is the focus of the parabola $x^2 = 8y$.

(b) (i) Evaluate $\int_1^3 \frac{dx}{x}$ 5

(ii) Use Simpson's rule with 3 function values to approximate $\int_1^3 \frac{dx}{x}$

(iii) Use your results to parts (i) and (ii) to obtain an approximation for e. Give your answer correct to 3 decimal places.

Question Five (Start a NEW booklet)

Marks

(a) Evaluate $\cos\left[\tan^{-1}(-\frac{1}{\sqrt{3}})\right]$ 1

(b) When the temperature T of a certain body is 65°C it is cooling at the rate of 1°C per minute. 8

Assuming Newton's law of cooling: $\frac{dT}{dt} = -k(T - S)$ where

T is the temperature of the body at time t minutes
S is the temperature of the surrounding medium, assumed constant
k is a constant

(i) Show that $T = S + Ae^{-kt}$ is a solution of the given differential equation, where A is also a constant.

(ii) Show that the value of k is 0·02 given that S is 15°C.

(iii) Find T when $t = 20$ minutes, giving your answer to the nearest degree. (You may assume that initially T = 65)

(iv) How long will it take for the temperature of the body to fall to 35°C?

(b) The acceleration of a particle P, moving along a straight line has an acceleration given by 3

$$\frac{d^2x}{dt^2} = -4\left(x + \frac{16}{x^3}\right)$$

Given that P is initially at rest at the point $x = 2$, show that the velocity v at any time is given by

$$v^2 = 4\left(\frac{16 - x^4}{x^2}\right)$$

rial HSC EXT1 2003

(a)

$\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$

$= \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^{2\sqrt{3}}$ by table of standard integrals

$= \frac{1}{2}\tan^{-1}\sqrt{3} - 0$

$= \frac{\pi}{6}$ ②

(b)

$\frac{d}{dx}(\cos^3 x)$

$= 3\cos^2 x \cdot (-\sin x)$

$= -3\cos^2 x \sin x$ ②

(c)

$(4, 6)$ $\quad$ 4

$(13, 5)$ $\quad$ -1

Let (x_r, y_r) be this pt.

$x_r = \frac{-4 + 52}{3}$

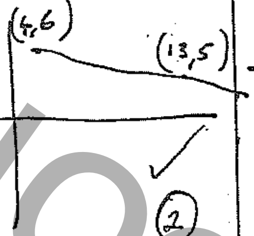
$= \frac{48}{3} = 16$

$y_r = \frac{-6 + 20}{3}$

$= +\frac{14}{3}$ or $+4\frac{2}{3}$

$\therefore$ pt is $(16, +4\frac{2}{3})$

CHECK



②

(d) $y = \frac{2x}{3x-1}$

Vertical asymptote is $x = \frac{1}{3}$ ①

(e)

method 1

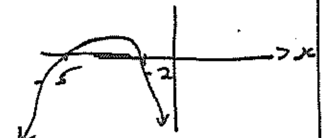
$\frac{3}{x+5} \le 1$ $\boxed{x \ne -5}$

$\times (x+5)^2$

$3(x+5) \le (x+5)^2$

$(x+5)\left[3-(x+5)\right] \le 0$

$(x+5)(-2-x) \le 0$



$x \ne -5$

$\therefore$ ~~[illegible]~~

$\therefore$ $x < -5$ or $x \ge 2$

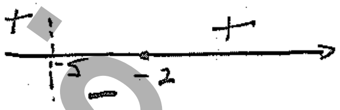
method 2 Sign diagram

$\frac{3}{x+5} \le 1$

$\Rightarrow \frac{3}{x+5} - 1 \le 0$

$\frac{-2-x}{x+5} \le 0$

$\frac{x+2}{x+5} \ge 0$



solution is

~~[illegible]~~

$x < -5$ or $x \ge -2$ ②

(f)

$u = x^4$

$du = 4x^3\,dx$

$\Rightarrow \frac{1}{2}du = 2x^3\,dx$

$x = 0 \Rightarrow u = 0$

$x = \frac{1}{\sqrt{2}} \Rightarrow u = \frac{1}{4}$

$I = \int_0^{1/4} \frac{1}{\sqrt{1-u}} \cdot \frac{1}{2}\,du$

$= \left[-\sqrt{1-u}\right]_0^{1/4}$

$= \sqrt{1} - \sqrt{\frac{3}{4}}$

$= 1 - \frac{\sqrt{3}}{2}$ ③

$\lim_{x \to 0} \frac{\tan 2x}{4x}$

$= \frac{1}{2}\lim_{x \to 0} \frac{\tan 2x}{2x}$

$= \frac{1}{2} \times 1$

$= \frac{1}{2}$ ①

(b)

method 1

Let $\sin\theta + \sqrt{3}\cos\theta$

$\equiv R\sin(\theta + \alpha)$

$\underline{\underline{\theta = 0}}$ $\quad R > 0 \quad -\pi < \alpha \le \pi$

$R\sin\alpha = \sqrt{3}$

$\underline{\underline{\theta = \pi/2}}$ $\quad R\cos\alpha = 1$ *

$\therefore R^2\sin^2\alpha + R^2\cos^2\alpha$

$= 3 + 1$

$R^2 = 4 \Rightarrow R = 2$

also

$0 < \alpha < \pi/2$ from *

and $\tan\alpha = \sqrt{3}$

$\Rightarrow \alpha = \pi/3$

$\therefore$ equation becomes

$2\sin(\theta + \pi/3) = 1$

$\sin(\theta + \pi/3) = 1/2$

Now $0 \le \theta \le 2\pi$

$\Rightarrow \pi/3 \le \theta + \pi/3 \le 2\pi + \frac{\pi}{3}$

So

$\theta + \frac{\pi}{3} = \frac{5\pi}{6}, \ 2\pi + \pi/6$

$\therefore \theta = \frac{\pi}{2}, \ \frac{11\pi}{6}$ ④

method 2 t-formulas

Let $t = \tan\theta/2$

TEST $\underline{\underline{\theta = \pi}}$ (why?)

LHS $= 0 - \sqrt{3}$

RHS $= 1$

$\theta \ne \pi$

probably the more difficult method, but it can be done!

(c)

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt}$

$450 = 4\pi \times 15^2 \frac{dr}{dt}$

~~$\frac{dr}{dt} = \frac{4\pi \times 15^2}{450}$ $\frac{450}{4\pi 15^2}$~~

~~$= 2\pi$ cm s^{-1}~~

~~or 6.2831 . . .~~

$\frac{dr}{dt} = \frac{450}{4\pi \times 15^2}$

$= \frac{1}{2\pi}$ or 0.1591... cms^{-1}

③

(d)

(i) $f(0.5) = \cos 0.5$

$= 1.57$

$f(1.5) = \cos 1.5 - \ell$

$= -0.334$

$f(x)$ is continuo

for $0.5 < x < 1.5$

$\Rightarrow$ there exists a x in this (open interval for $f(x)$

①

(ii)

Formula is

$x_1 = x_0 - \frac{f(x}{f'(x_0}$

(or derive formula quickly)

$x_1 = 1 - \frac{f(1)}{f'(1)}$

~~[illegible]~~

$f(1) = \cos 1 - \ln$

$= \cos 1$

$f'(x) = -\sin x - \frac{1}{x}$

$f'(1) = -\sin 1 - 1$

$x_1 = 1 - \frac{\cos 1}{-\sin 1}$

$= 1 + \frac{\cos 1}{\sin 1 + 1}$

$= 1.2934$

check is this a better approx (NOT required to show)

$f(1) = \cos 1 > 0$

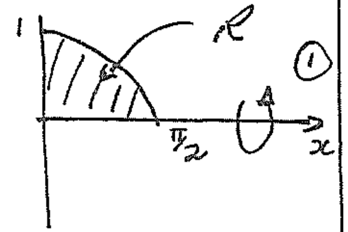
$f''(x) = -\cos x + \frac{1}{x^2}$

$f''(1) = -\cos 1 + 1 > 0$

So $x = 1.293\ldots$ is a better approx.

3.

(i)



(ii)

$V = \pi \int_0^{\pi/2} \cos^2 x \, dx$

$2\cos^2 x = \cos 2x + 1$

$\therefore V = \pi \int_0^{\pi/2} (1 + \cos 2x)\,dx$.

$= \pi \left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2}$

$= \pi\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right)$.

$= \frac{\pi^2}{2}$ unit^3 ②

(b)

$x^3 + 4x^2 - 6x - 8 = 0$

$\alpha + \beta + \gamma = -4$

$\alpha\beta + \beta\gamma + \gamma\alpha = -6$

$\alpha\beta\gamma = 8$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-6}{8}$

$= -3/4$ ③

(c)

$\left(2x^3 + \frac{1}{3x^2}\right)^5$

method 1

General term

$= \binom{5}{r} (2x^3)^{5-r} \left(\frac{1}{3x^2}\right)^r$

$= \binom{5}{r} \frac{2^{5-r}}{3^r} \frac{x^{15-3r}}{x^{2r}}$

$= \binom{5}{r} \frac{2^{5-r}}{3^r} x^{15-5r}$

choose $r = 3$

term $= \binom{5}{3} \frac{2^2}{3^3}$

$= \frac{40}{27}$

method 2

$\left(2x^3 + \frac{1}{3x^2}\right)^5$

$= (x^3)^5 \left(2 + \frac{1}{3x^5}\right)^5$

~~Term~~

$= x^{15}\left(2 + \frac{1}{3x^5}\right)^5$

General term

$= \binom{5}{3} \frac{2^2}{3^3} = \frac{40}{27}$

(d)

$x^3 + ax + b$

$= (x-2)(x+3)Q(x) + 2x + 1$

Let $P(x) = x^3 + ax + b$

$P(2) = 8 + 2a + b$

$= 2 \times 2 + 1$

$\boxed{2a + b = -3}$

$P(-3) = -27 - 3a + b$

$= 2 \times -3 + 1$

$\boxed{-3a + b = 22}$

Solving gives

$a = -5$

$b = 7$

4.

(i)

$x^2 = 8y \Rightarrow y = \frac{1}{8}x^2$

$y' = \frac{1}{4}x$

M_{tangent} at $(4p, 2p^2)$

is $\frac{1}{4} \times 4p = p$

$M_{\perp} = -\frac{1}{p}$

$y - 2p^2 = -\frac{1}{p}(x - 4p)$

$py - 2p^3 = -x + 4p$

$x + py = 4p + 2p^3$ ③

(ii)

$A = (0, 4 + 2p^2)$

$(p \neq 0)$

$R = \left(\frac{0 + 4p}{2}, \frac{4 + 2p^2 + 2p^2}{2}\right)$

$= (2p, 2p^2 + 2)$ ①

(iii)

Let $x = 2p$

$y = 2p^2 + 2$

$= 2 \times \left(\frac{x}{2}\right)^2 + 2$

$= \frac{1}{2}x^2 + 2$

$\therefore y - 2 = \frac{1}{2}x^2$

parabola

Vertex $= (0, 2)$

The focus of $x^2 = 8y$

ie, $x^2 = 4 \times 2 \times y$

ie focus $= (0, 2)$ ③

ie, Hence result.

(b)

(i) $\int_1^3 \frac{dx}{x}$

$= [\ln x]_1^3$

$= \ln 3$ ①

(ii)

x	1	2	3
$1/x$	1	$1/2$	$1/3$

$\int_1^3 \frac{dx}{x} \doteqdot \frac{h}{3}[y_0 + 4y_1 + y_2]$

$= \frac{1}{3}\left[1 + 4 \times \frac{1}{2} + \frac{1}{3}\right]$

$= \frac{10}{9}$ ②

(iii) $\ln 3 \doteqdot \frac{10}{9}$

$e^{10/9} \doteqdot 3$

$\left(e^{10/9}\right)^{9/10} \doteqdot 3^{9/10}$

$e \doteqdot 3^{0.9}$

$= 2.6878\ldots$

Q5.

(a) $\cos\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$.

$= \cos\left(-\frac{\pi}{6}\right)$

$= \cos\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}$ ①

(b)

(i) LHS =

$\frac{dT}{dt} = \frac{d}{dt}(5 + Ae^{-kt})$

$= 0 + A \times (-k)e$

$= -Ake^{-kt}$

RHS =

$-k(T - 5)$

$= -k(5 + Ae^{-kt} - 5$

$= -Ake^{-kt}$

(ii) ~~When t = 0~~

$\frac{dT}{dt} = -k(T -$

$T = 65$, $\frac{dT}{dt} =$

$-1 = -k(65 -$

$k = \frac{1}{50} = 0$

ii)
$T = 15 + Ae^{-0.02t}$

when $t = 0$ $T = 65$

$\therefore 65 = 15 + A$

$A = 50$

$\therefore T = 15 + 50e^{-0.02t}$

$t = 20$

$T = 15 + 50e^{-0.02\times 20}$

$= 15 + 50e^{-0.4}$

$\cancel{=} = 48.51\ldots$

$\doteqdot 49°$ ②

iv)
$T = 15 + 50e^{-0.02t}$

$T = 35$ $t = ?$

$35 = 15 + 50e^{-0.02t}$

$e^{-0.02t} = \frac{2}{5}$

$t = \frac{-1}{0.02} \ln 0.4$

$= 45.81\ldots$

$\doteqdot 46$ minutes

③

(b) $\frac{d^2x}{dt^2} = -4\left(x + \frac{16}{x^3}\right)$

$\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4(x + 16x^{-3})$

$\frac{1}{2}v^2 = -4\left(\frac{1}{2}x^2 + 16\frac{x^{-2}}{-2}\right) + c$

$v^2 = -4\left(x^2 - \frac{16}{x^2}\right) + c'$

$v = 0$, $x = 2$

$\Rightarrow$

$0 = -4\left(2^2 - \frac{16}{2^2}\right) + c'$

$= 0 + c'$

$\Rightarrow c' = 0$

$\therefore v^2 = -4\left(x^2 - \frac{16}{x^2}\right)$

$= 4\left(\frac{16 - x^4}{x^2}\right)$

③

6.
(a) $\frac{1}{1\times 2} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

For $n = 1$

LHS $= \frac{1}{1\times 2} = \frac{1}{2}$

RHS $= \frac{1}{1+1} = \frac{1}{2}$

$\therefore$ true for $n = 1$

Suppose statement true for $n = k$

i.e. $\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Consider $n = k+1$

$\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

by inductive hypothesis.

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

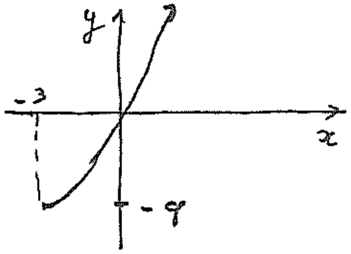
$= \frac{k+1}{k+2}$

So result true for $n = k$ provided true for $n = k+1$

$\therefore$ by M.I. true for $n = 1, 2, 3, \ldots$

5(b)

(i) $f(x) = x(x+6)$



Range: $f(x) \geq -9$ ①

(ii) $f^{-1}(x)$ exists as for $x \geq -3$ $f(x)$ is increasing. ①

Domain of $f^{-1}(x)$ ~~[illegible]~~ is $x \geq -9$ ①

(iii) $y = x(x+6)$

$= x^2 + 6x$

$x \geq -3$, $y \geq -9$

$f^{-1}(x)$: $x \leftrightarrow y$

$x = y^2 + 6y$

$x \geq -9$ and $y \geq -3$

$y^2 + 6y - x = 0$

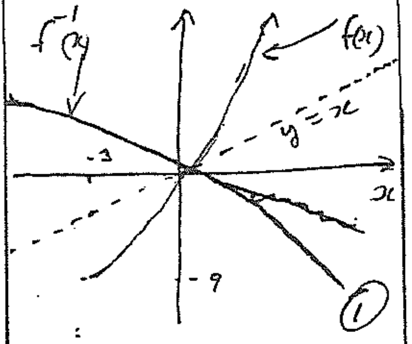
$y = \frac{-6 \pm \sqrt{36 - 4x}}{2}$

$= -3 \pm \sqrt{9 - x}$

but $y \geq -3$ f

so $y = -3 + \sqrt{9 - x}$

$\Rightarrow f^{-1}(x) = -3 + \sqrt{9 - x}$ ②



①

(c)
$y = 3\cos^{-1}\left(\frac{x}{2} - 1\right)$

one only

Domain

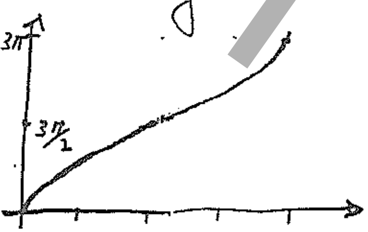
$-1 \leq \frac{x}{2} - 1 \leq 1$

$0 \leq \frac{x}{2} \leq 2$

$0 \leq x \leq 4$

Range

$0 \leq y \leq 3\pi$



⑦

(i)
$\ddot{x} = 0$

$\therefore \dot{x} = c$

at $t = 0$ $\dot{x} = 6$

$\therefore \dot{x} = 60$

$x = 60t + c$

at $t = 0$, $x = 0$

$\Rightarrow c = 0$

$\therefore \boxed{x = 60t}$

$\ddot{y} = -10$

$\dot{y} = -10t +$

at $t = 0$ $\dot{y} =$

$\Rightarrow c = 0$

$\therefore \dot{y} = -10$

$y = -5t^2 +$

at $t = 0$, y

$\Rightarrow c = 20$

$\therefore y = 20 - 5$

(ii) at ground

$(x, y) = (R, 0$

Solve $y = 0$

$20 - 5t^2 =$

$t^2 = 4$

$t > 0 \Rightarrow t$

(iii) at $(R, 0)$

$\therefore x(2)$

v) at ground
vector diagram

$t = 2$

$\dot{y} = -10 \times 2$

$= -20$

$\dot{x} = 60$

$\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$

$= \frac{20}{60}$

$= 1/3$

$\theta = \tan^{-1} 1/3$ ②

$\doteqdot 18 \cdot 43 \ldots$

$\underline{\underline{18^\circ}}$

(b)

$(1+x)^{2n} = \sum_0^{2n} \binom{2n}{k} x^k$

(i) put $x = 1$

$2^{2n} = \sum_0^{2n} \binom{2n}{k} 1^k$

$4^n = \sum_0^{2n} \binom{2n}{k}$

①

(ii) integrating

$\frac{(1+x)^{2n+1}}{2n+1} + c = \sum_0^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1}$

put $x = 0$

$\frac{1}{2n+1} + c = 0$

$\Longrightarrow c = -\frac{1}{2n+1}$

$\therefore \sum_0^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1}$

$= \frac{(1+x)^{2n+1}}{2n+1} - \frac{1}{2n+1}$

put $x = 1$

$\sum_0^{2n} \binom{2n}{k} \frac{1}{k+1}$

$= \frac{2^{2n+1} - 1}{2n+1}$

$= \frac{2 \times 4^n - 1}{2n+1} \times \frac{2}{2}$

$= \frac{4 \times 4^n - 2}{4n+2}$

$= \frac{4^{n+1} - 2}{4n+2}$

③

THE KING'S SCHOOL

2003
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) P(5,7) divides the interval AB externally in the ratio $m:n$

If $A=(-1,-5)$ and $B=(0,-3)$, find $m:n$. **2**

(b) Find the derivative of $\tan^{-1}\left(1+x^2\right)$ **2**

(c) In how many ways can 8 rowers be divided into two groups of 5 and 3 rowers? **2**

(d) Find the acute angle between the lines $y=2x+1$ and $y=7-3x$. **2**

(e) $x+1$ is a factor of the polynomial

$$P(x)=x^{2n+1}-x^{2n}+b\text{, } n \text{ a positive integer}$$

Find the value of b. **2**

(f) Evaluate $\displaystyle\sum_{n=1}^{9}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Show that the function $f(x)=5x-\sin 4x-12$

increases for all values of x **3**

(b) (i) Find $R>0$ and α, $0<\alpha<\frac{\pi}{2}$, so that

$R\sin(x-\alpha)=\sin x-\sqrt{3}\cos x$ **2**

(ii) Solve $\sin x-\sqrt{3}\cos x=\sqrt{2}$, $0<x<2\pi$, exactly **2**

(c) (i) Use the substitution $u=4-x^2$ to evaluate

$$\int_0^{\sqrt{3}}\frac{x}{\sqrt{4-x^2}}\,dx$$ **3**

(ii) Evaluate $\displaystyle\int_0^{\sqrt{3}}\frac{4-x}{\sqrt{4-x^2}}\,dx$ **2**

Marks

Question 3 **(12 marks)** Use a SEPARATE writing booklet.

(a) The function $f(x)=e^{x^3}-x-3$ has a zero near $x=1.2$.

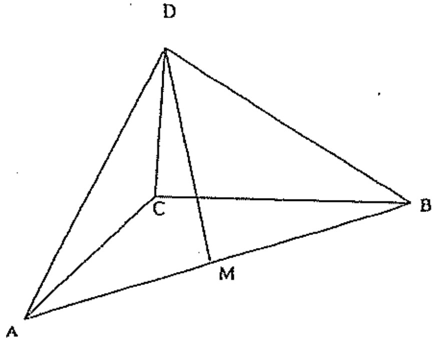
Use Newton's Method once to find a two decimal place approximation to this zero. 3

(b) (i) Write $\sin 2A$ in terms of t, where $t=\tan A$ 1

(ii) Prove the identity $\operatorname{cosec} 2A-3\cot 2A=2\tan A-\cot A$ 3

(c) A, B, C are three points in a horizontal plane and M is the mid-point of AB. CD is at right-angles to the horizontal plane ABC.

The length of AB = 20 metres.
The angles of elevation from A, M, B to D are 30°, 45°, 30°, respectively.



(i) Copy the diagram into your booklet and include on it the information given. 1

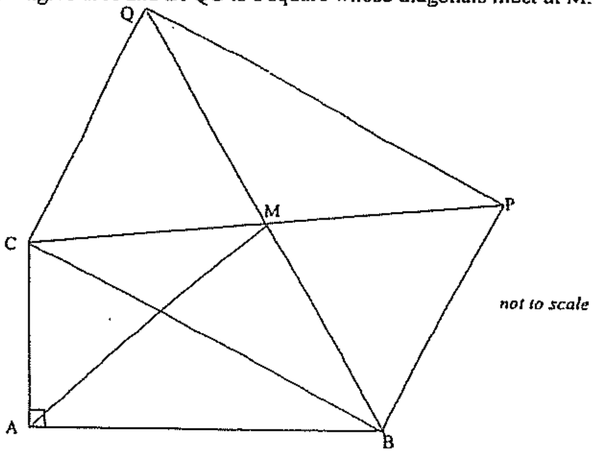
(ii) If $CD=x$, show that $AC=\sqrt{3}x$ 1

(iii) Find the exact value of x. 3

Marks

Question 4 **(12 marks)** Use a SEPARATE writing booklet.

(a) ΔABC is right-angled at A and BPQC is a square whose diagonals meet at M.



(i) Copy the diagram into your booklet.

(ii)Prove that ABMC is a cyclic quadrilateral. 2

(iii) Prove that MA bisects $\angle$BAC 2

(b) A particle is moving in simple harmonic motion on the x-axis. Its displacement, x metres, at any time t seconds, where $t \geq 0$, is given by $x=10\cos nt$, n a constant.

(i) Show that the particle is initially at rest at $x=10$. 2

(ii) The period of the motion is T seconds and after $\frac{T}{3}$ seconds the particle is at position $x=b$.

Find the value of b. 2

(iii) The speed at $x=b$ is $20\sqrt{3}$ m/s. Find the period of the motion. 2

(c) Simplify $\binom{n}{3}\div\binom{n-1}{2}$, $n\geq 3$ 2

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) $P\left(2ap, ap^2\right)$, $Q\left(2aq, aq^2\right)$ are the end points of a focal chord of the parabola $x^2 = 4ay$.

(i) Show that the equation of the chord PQ is

$$y - \frac{1}{2}(p+q)x + apq = 0$$ 2

(ii) Deduce that $q = -\frac{1}{p}$ 1

(iii) Show that $PQ = 2a + a\left(p^2 + \frac{1}{p^2}\right)$ 2

(iv) A circle is drawn with PQ as its diameter.

Prove that the directrix is a tangent to this circle. 3

(b) A cube is expanding in such a manner that it maintains its cubic shape. Initially, each edge is 10cm and the surface area is expanding at a constant rate of 12.6 cm²/s.

(i) Find an expression in terms of t for the surface area of the cube after t seconds. 1

(ii) Hence, or otherwise, find the rate at which the volume of the cube is increasing after 10 seconds. 3

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Prove by mathematical induction for $n \geq 0$ that $E(n) = 9^{n+2} - 4^n$ is a multiple of 5. 3

(b) (i) With the aid of a diagram, or otherwise,

solve $\left(x^2 - 1\right)\left(x^2 - 4\right) \leq 0$ 2

(ii) A particle is moving along the x-axis with its acceleration at position x given by

$$\ddot{x} = 10x - 4x^3$$

When $x = \sqrt{2}$ its velocity $v = 2$

(α) Prove that the expression $\frac{1}{2}v^2 + x^4 - 5x^2$ is a constant for the motion and find this constant.

(v is its velocity at position x) 2

(β) Describe the motion. 2

(c) A random sample of 10 people is made. Assuming that either sex is equally likely, find the probability that

(i) there is an equal number of each sex 1

(ii) there are more females 2

Give your answers correct to three decimal places.

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Solve $\frac{-2x}{x+1} > 0$ 2

(b) Consider the curve $y = \ln\left(\frac{-2x}{x+1}\right)$

(i) Use (a) to explain why

$$\ln\left(\frac{-2x}{x+1}\right) = \ln(-2x) - \ln(x+1)$$ 1

(ii) Show that the curve has no stationary points. 2

(iii) Sketch the curve, showing the x-intercept. 2

(iv) Find the inverse function of $y = \ln\left(\frac{-2x}{x+1}\right)$, expressing your answer with y as subject. 3

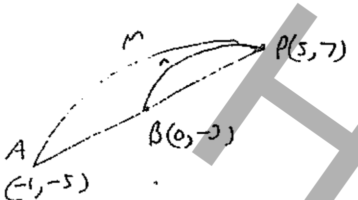
(v) Find the area of the region bounded by $y = \ln\left(\frac{-2x}{x+1}\right)$ and the y-axis and the lines $y = 0$ and $y = 2$ 2

End of Paper

Solutions TKS Trial HSC 2003 Extension 1

Qu 1

(a)



$\Rightarrow \boxed{M:n = 6:5}$

(b) $\frac{d}{dx}\tan^{-1}(1+x^2) = \frac{1}{1+(1+x^2)^2} \times 2x = \boxed{\frac{2x}{1+(1+x^2)^2}}$

(c) $\binom{8}{5}$ or, of course, $\binom{8}{3} = \boxed{56}$

(d) gradients of lines are 2 and -3

$\therefore \tan\alpha = \left|\frac{2-(-3)}{1+2(-3)}\right| = \frac{5}{5} = 1$

$\therefore$ acute angle is $\boxed{45^0}$

(e) $P(-1) = 0 \Rightarrow (-1)^{2n+1} - (-1)^{2n} + b = 0$

i.e. $-1 - 1 + b = 0 \therefore \boxed{b=2}$

(f) $\sum_{n=1}^{9}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \dots + \left(\frac{1}{8}-\frac{1}{9}\right) + \left(\frac{1}{9}-\frac{1}{10}\right)$

$= 1 - \frac{1}{10}$

$= \boxed{\frac{9}{10}}$

Qu 2

(a) $f'(x) = 5 - 4\cos 4x$

$\geq 5 - 4(1)$ since $-1 \leq \cos 4x \leq 1$

$\Rightarrow f'(x) \geq 1 > 0 \quad \forall x$

$\therefore f(x)$ increases $\forall x$

(b) (i) $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$

$\equiv \sin x - \sqrt{3}\cos x$

$\Rightarrow R\cos\alpha = 1$
$R\sin\alpha = \sqrt{3}$

$\therefore \tan\alpha = \sqrt{3}$, $\boxed{\alpha = \frac{\pi}{3}}$

and $R = \sqrt{1^2 + \sqrt{3}^2} = \boxed{2}$

(ii) From (i), $2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{2}$

$\therefore \sin\left(x-\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\therefore x - \frac{\pi}{3} = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

i.e. $x = \boxed{\frac{7\pi}{12} \text{ or } \frac{13\pi}{12}}$

(c) (i) $u = 4 - x^2$

$\frac{du}{dx} = -2x$ or $du = -2x\,dx$

$x = 0,\ u = 4$
$x = \sqrt{3},\ u = 1$

$\therefore I = -\frac{1}{2}\int_4^1 \frac{du}{\sqrt{u}} = \frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\cdot 2\left[u^{\frac{1}{2}}\right]_1^4$

$= 2 - 1 = 1$

(ii) $I = \int_0^{\sqrt{3}} \frac{4}{\sqrt{4-x^2}} - \frac{x}{\sqrt{4-x^2}}\,dx = 4\cdot\left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}} - 1$, from (i)

$= 4\frac{\pi}{3} - 1$

(a) $f'(x) = 2x e^{x^2} - 1$

$\therefore x_1 = 1.2 - \dfrac{e^{1.44} - 1.2 - 3}{2.4e^{1.44} - 1} = 1.1977\ldots\ldots$

$\therefore$ two decimal approx^n = $\boxed{1.20}$

(b) (i) $\sin 2A = \dfrac{2t}{1+t^2}$

(ii) put $t = \tan A$,

then $\operatorname{cosec} 2A - 3\cot 2A = \dfrac{1+t^2}{2t} - 3 \cdot \dfrac{1-t^2}{2t}$

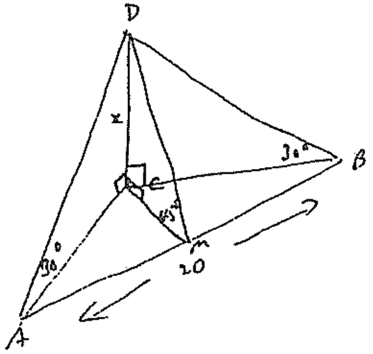
$= \dfrac{1+t^2-3+3t^2}{2t}$

$= \dfrac{4t^2-2}{2t}$

$= 2t - \dfrac{1}{t}$

$= 2\tan A - \cot A$

(c) (i)

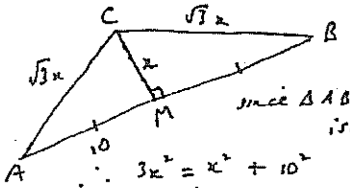


(ii) In ΔACD,

$\tan 30° = \dfrac{x}{AC} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow AC = \sqrt{3}\,x$

(iii) From (ii), we have



since ΔABC is isosceles

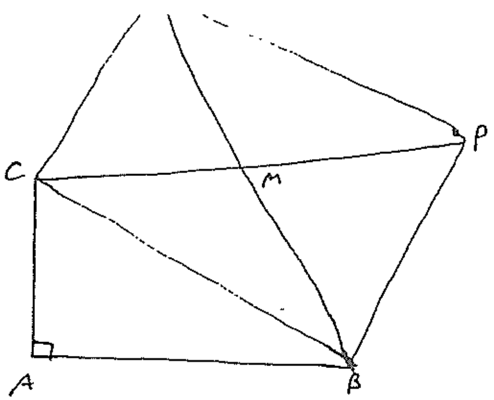
$\therefore 3x^2 = x^2 + 10^2$

$2x^2 = 100$

$x^2 = 50$

$\therefore x = \sqrt{50}$ or $5\sqrt{2}$

(i)



(ii) $\angle CMB = 90°$, the diagonals of a square meet at right angles

$\therefore$ In ◇ABMC, $\angle A + \angle M = 180°$

$\Rightarrow$ ◇ABMC is cyclic, opposite angles are supplementa

(iii) $MC = MB$, equal diagonals in a square bisect each other.

$\therefore \angle CAM = \angle BAM$, $\angle$s at the circumference of a circle standing on equal arcs.

$\therefore MA$ bisects $\angle BAC$

(d) (i) $t = 0$, $x = 10\cos 0 = 10$

$\dot{x} = -10n \sin nt = -10n \sin 0$ at $t = 0$

$= 0$

∴ particle is initially at rest at $x = 10$

(ii) $T = \dfrac{2\pi}{n}$ $\therefore n = \dfrac{2\pi}{T}$

$\therefore x = 10\cos\left(\dfrac{2\pi}{T} \cdot \dfrac{T}{3}\right) = 10\cos\left(\dfrac{2\pi}{3}\right)$

ie $x = 10\left(-\dfrac{1}{2}\right) = -5$

(iii) $\dot{x} = -10n \sin nt = -10 \cdot \frac{2\pi}{T} \sin\left(\frac{2\pi t}{T}\right)$

$\therefore \; -20\sqrt{3} = -\frac{20\pi}{T} \sin\left(\frac{2\pi}{3}\right) = -\frac{20\pi}{T} \cdot \frac{\sqrt{3}}{2}$.

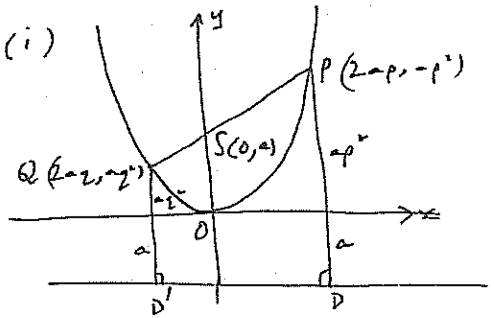
$\therefore \; T = \frac{\pi}{2}$ (seconds) i.e. period is $\frac{\pi}{2}$ s

(c) $\binom{n}{3} \div \binom{n-1}{2} = \frac{n!}{(n-3)!\,3!} \div \frac{(n-1)!}{(n-3)!\,2!}$

$= \frac{n!}{(n-3)!\,3!} \times \frac{(n-3)!\,2!}{(n-1)!} = \frac{n}{3}$

Q 5

(a) (i)



gradient PQ

$= \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p-q)(p+q)}{2a(p-q)}$

$= \frac{p+q}{2}$

$\therefore$ chord PQ is $y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$

or $y - ap^2 = \frac{1}{2}(p+q)x - ap(p+q)$

$\Rightarrow \; y - \frac{1}{2}(p+q)x + apq = 0$

(ii) Since S (0, a) is on PQ, then

$a - 0 + apq = 0$

i.e. $pq = -1$ or $q = -\frac{1}{p}$

(iii) $PQ = PS + QS = PD + QD'$, focus-directrix defn (see diagram)

$= a + ap^2 + a + aq^2$, $q = -\frac{1}{p}$

$= 2a + a\left(p^2 + \frac{1}{p^2}\right)$

(iv) If PQ is a diameter, the radius is $a + \frac{a}{2}\left(p^2 + \frac{1}{p^2}\right)$, from (iii)

& the centre is $\left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a\left(p - \frac{1}{p}\right), \frac{a}{2}\left(p^2 + \frac{1}{p^2}\right)\right)$, $q = -\frac{1}{p}$

$\therefore$ distance from centre to directrix $y = -a$ is

$\frac{a}{2}\left(p^2 + \frac{1}{p^2}\right) + a$ = radius

$\therefore$ directrix is a tangent to the circle

(iv) (i) $A = 6\times10^2 + 12\cdot6t = 600 + 12\cdot6t$

(ii) If an edge is x, $A = 6x^2$, $V = x^3$

$\therefore$ From (i), $6x^2 = 600 + 12\cdot6t$

$x^2 = 100 + 2\cdot1t$

$\therefore V = (100 + 2\cdot1t)^{3/2}$

& so, $\dfrac{dV}{dt} = \dfrac{3}{2}(100 + 2\cdot1t)^{\frac{1}{2}}(2\cdot1)$

$= 3\cdot15 \times \sqrt{121}$ cm³/s when $t = 10$

$= 34\cdot65$ cm³/s

Alternatively, using $A = 6x^2$, $V = x^3$, $\dfrac{dA}{dt} = 12\cdot6$,

we have $\dfrac{dV}{dt} = \dfrac{dV}{dx}\cdot\dfrac{dx}{dt}$

$= \dfrac{dV}{dx}\cdot\dfrac{dx}{dA}\cdot\dfrac{dA}{dt}$

$= 3x^2\cdot\dfrac{1}{12x}\cdot(12\cdot6)$

$= 3\cdot15x$

But, when $t = 10$, $A = 6\times10^2 + 12\cdot6\times10 = 726$

$\therefore 6x^2 = 726 \Rightarrow x = 11$

$\therefore \dfrac{dV}{dt} = 3\cdot15\times11$ cm³/s $= 34\cdot65$ cm³/s

(a) $E(0) = 9^2 - 4^0 = 80$ is a multiple of 5

$\therefore$ assume $E(n) = 9^{n+2} - 4^n = 5q$, q an integer, $n \geq 0$

Then, $E(n+1) = 9^{n+3} - 4^{n+1}$

$= 9(9^{n+2}) - 4^{n+1}$

$= 9(5q + 4^n) - 4^{n+1}$, using the assumption

$= 5(9q) + 4^n(9-4)$

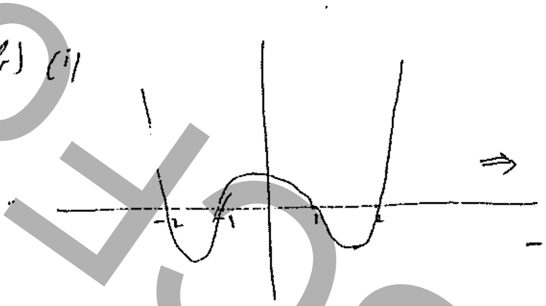
$= 5(9q + 4^n)$ is a multiple of 5 since $9q + 4^n$ is an integer

$\therefore$ if $E(n)$ is a multiple of 5, so is $E(n+1)$

But, $E(0)$ is a multiple of 5

$\therefore E(n)$ is a multiple of 5 by induction

(b) (i)



$\Rightarrow (x^2-1)(x^2-4) \leq 0$ has solutions $-2 \leq x \leq 1$ or $1 \leq x \leq 2$

(ii) (α) $\dfrac{d(\frac{1}{2}v^2)}{dx} = 10x - 4x^3$

$\therefore \frac{1}{2}v^2 = 5x^2 - x^4 + c$, c a constant

ie. $\frac{1}{2}v^2 + x^4 - 5x^2 = c$

When $x = \sqrt{2}$, $v = 2$

$\therefore 2 + 4 - 10 = c = -4$

(β) we have $\frac{1}{2}v^2 = 5x^2 - x^4 - 4$

or $\frac{1}{2}v^2 = -(x^4 - 5x^2 + 4)$

$= -(x^2-1)(x^2-4)$

Now, $\frac{1}{2}v^2 \geq 0 \Rightarrow (x^2-1)(x^2-4) \leq 0$

∴ Using (b)(i) and when $x = \sqrt{2}$, $v = 2$, we

have the particle oscillates between $x = 1$ and $x = 2$

(c) (i) $P(5 \text{ males}, 5 \text{ females}) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5$

$= 0.246$, 3 d.p.

(ii) P(more females) = P(more males), since $P(M) = P(F) = \frac{1}{2}$

∴ using (i), $P(\text{more females}) = \frac{1 - 0.246}{2} = 0.377$

Qn 7

(a) if $x+1 > 0$, then $-2x > 0$

i.e. $x > -1$ i.e. $x < 0$

∴ solution is $-1 < x < 0$

if $x < -1$, we'd have $x > 0$ ⇒ no further solutions

∴ $-1 < x < 0$

(b)(i) We need $\frac{-2x}{x+1} > 0$ and $-2x > 0$ and $x+1 > 0$.

All 3 inequalities are "true" from (a)

(ii) From (i), $y = \ln(-2x) - \ln(x+1)$

$\therefore \frac{dy}{dx} = \frac{-2}{-2x} - \frac{1}{x+1}$

$= \frac{1}{x} - \frac{1}{x+1} = \frac{x+1-x}{x(x+1)}$

$= \frac{1}{x(x+1)} \neq 0$ for any x

∴ there are no stationary points

(iii) Since $-1 < x < 0$, then $x(x+1) < 0$

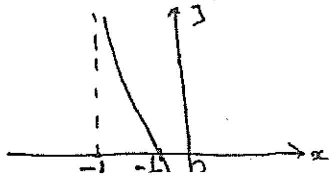
∴ From (ii) $\frac{dy}{dx} < 0$ for $-1 < x < 0$

i.e. curve is decreasing for $-1 < x < 0$

When $y = 0$, $\frac{-2x}{x+1} = 1$

∴ $-2x = x+1$

⇒ x intercept is $-\frac{1}{3}$



is $x = \ln\left(\frac{-2y}{y+1}\right)$

$\therefore \ -\frac{2y}{y+1} = e^x$

$-2y = e^x y + e^x$

or $y(e^x+2) = -e^x$

ie inverse function is $y = -\frac{e^x}{e^x+2}$

(v) $A = \left| \int_0^2 x \, dy \right| = \int_0^2 \frac{e^y}{e^y+2} \, dy$, from (iv)

$= \left[\ln(e^y+2)\right]_0^2$

$= \ln(e^2+2) - \ln 3 \quad u^2$

Question 3 12 marks **Start a New Booklet** marks

a) Find the general solution for $\sqrt{3}\sin 2\theta = \cos 2\theta$. 3

b) The region bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{12}$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution about the x-axis. Find the volume of the solid so formed. 3

c) Two points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x^2 = 4y$. 6
- (i) Show that the equation of the tangent to the parabola at P is $y = px - p^2$.
- (ii) The tangent at P and the line through Q parallel to the y axis intersect at T. Find the coordinates of T.
- (iii) Write down the coordinates of M, the midpoint of PT.
- (iv) Determine the locus of M when $pq = -1$.

Question 4 12 marks **Start a New Booklet**

a) If $\tan A$ and $\tan B$ are the roots of the equation $3x^2 - 5x - 1 = 0$, find the value of $\tan(A + B)$. 3

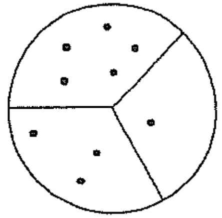
b) A particle is moving with simple harmonic motion. When it is at a distance d from the centre of motion, its speed is V. If its speed is $\frac{V}{2}$ when the distance from the centre is $2d$, show that the period of the motion is $\frac{4\pi d}{V}$ and the amplitude is $d\sqrt{5}$. 5

c) The rate at which a body cools in air is assumed to be proportional to the difference between its temperature T and the constant temperature S of the surrounding air. This can be expressed by the differential equation $\frac{dT}{dt} = k(T - S)$ where t is the time in hours and k is a constant. 4
- (i) Show that $T = S + Be^{kt}$, where B is a constant, is a solution of the differential equation.
- (ii) A heated body cools from 80 °C to 40 °C in 2 hours. The air temperature S around the body is 20 °C. Find the temperature of the body after one further hour has elapsed. Give your answer correct to the nearest degree.

Q5 ... Page 3

Question 5 12 marks **Start a New Booklet** marks

a) Nine points lie inside a circle. No three of the points are collinear. Five of the points lie in sector 1, three lie in sector 2, and the other point lies in sector 3. 5



- (i) Show that 84 triangles can be made using these points as vertices.
- (ii) One triangle is chosen at random from all the possible triangles. Find the probability that the vertices of the triangle chosen lie one in each sector.
- (iii) Find the probability that the vertices of the triangle chosen lie all in the same sector.

b) Find the roots of the equation $x^3 - 12x^2 + 12x + 80 = 0$ given that they are three consecutive terms in an Arithmetic Series. 3

c) Consider the binomial expansion $1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$. 4
- (i) Show that $1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$.
- (ii) Show that $1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \ldots + (-1)^n\frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}$.

Question 6 12 marks **Start a New Booklet**

a) Colour-blindness affects 5% of all men. What is the probability that any random sample of 20 men should contain: 5
- (i) no colour-blind men.
- (ii) only one colour-blind man.
- (iii) two or more colour-blind men.

Q6 cont. ... Page 4

NEWINGTON COLLEGE

IN FIDE SCIENTIAM

Trial Examination

12 MATHEMATICS

2003

Extension 1

Time allowed: 2 hours (plus five minutes reading time)

DIRECTIONS TO CANDIDATES

- All questions may be attempted.
- In every question, show all necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Approved silent calculators may be used.
- A table of standard integrals is provided for your convenience.
- The answers to the questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2, etc.
- Each bundle must show the candidate's computer number.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Unless otherwise stated, candidates should leave their answers in simplest exact form.

Question 1 12 marks marks

a) Differentiate $\tan^{-1}\frac{x}{3}$. 2

b) Evaluate: 6

(i) $\int_1^{\sqrt{2}} \frac{x}{\sqrt{4-x^2}}\,dx$ using the substitution $u = 4 - x^2$.

(ii) $\int_0^1 \sqrt{1-x^2}\,dx$ using the substitution $x = \sin\theta$.

c) Solve the equation $3\sin\theta + 4\cos\theta = 2.5$ for values of θ between 0° and 360°. Give your answer correct to the nearest minute. 4

Question 2 12 marks **Start a New Booklet**

a) (i) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dt}$. 5

(ii) The acceleration of a particle moving in a straight line is given by $\ddot{x} = -2e^{-x}$ where x metres is the displacement from the origin. Initially, the particle is at the origin with velocity 2 ms^{-1}.

Prove that $v = 2e^{\frac{-x}{2}}$.

(iii) What happens to v as x increases without bound?

b) (i) By considering the graph of $y = e^x$, show that the equation $e^x + x + 1 = 0$ has only one real root and that this root is negative. 4

(ii) Taking $x = -1{\cdot}5$ as a first approximation to this root, use one application of Newton's method to find a better approximation.

c) In how many ways can the letters of the word *GEOMETRY* be arranged in a straight line if the vowels must occupy the 2nd, 4th and 6th places. 3
(NOTE: The vowels in the English alphabet are the letters *A, E, I, O, U*).

Q3 … Page 2

Solutions ① YR 12 TRIAL EXT 1

1/(a) $\frac{\frac{1}{3}}{1+\frac{x^2}{9}} = \frac{3}{x^2+9}$

(b)(i) $\int_1^{\sqrt{2}} \frac{x}{\sqrt{4-x^2}}\,dx$ $u = 4-x^2$ $x=1,\ u=3$

$\frac{du}{dx} = -2x$ $x=\sqrt{2},\ u=2$

$x\,dx = -\frac{du}{2}$

$= -\int_3^2 \frac{du}{2\sqrt{u}}$

$= -\int_3^2 \frac{1}{2}u^{-\frac{1}{2}}\,du$

$= \left[u^{\frac{1}{2}}\right]_2^3$

$= \sqrt{3}-\sqrt{2}$

(ii) $\int_0^1 \sqrt{1-x^2}\,dx$ $x = \sin\theta$ $x=0,\ \theta=0$

$dx = \cos\theta\,d\theta$ $x=1,\ \theta=\frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,.\cos\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

$\left(\frac{\pi}{2}+0-(0+0)\right)$

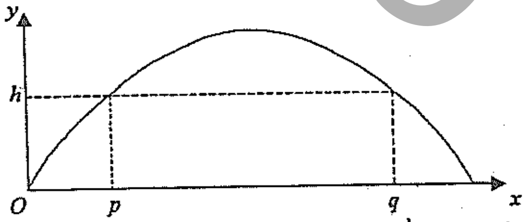
marks

b) When $(3 + 2x)^n$ is expanded in increasing powers of x, it is found that the coefficients of x^5 and x^6 have the same value. Find the value of n and show that the two coefficients mentioned are greater than all other coefficients in the expansion. 7

Question 7 12 marks **Start a New Booklet**

a) Prove by induction that $2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2n^2(n+1)^2$. 5

b) 7



A particle is projected with velocity V ms^{-1} from a point O at an angle of elevation α. Axes Ox and Oy are taken horizontally and vertically through O. The particle just clears two vertical chimneys of height h meters at horizontal distances of p metres and q metres from O. The acceleration due to gravity is taken as 10 ms^{-2} and air resistance if ignored.

(i) Write down expressions for the horizontal displacement x and the vertical displacement y of the particle after time t seconds.

(ii) Show that $V^2 = \dfrac{5p^2(1+\tan^2\alpha)}{p\tan\alpha - h}$.

(iii) Show that $\tan\alpha = \dfrac{h(p+q)}{pq}$.

END OF PAPER

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②

c) $5\left(\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right) = 2\cdot 5$

$\sin(\theta + \alpha) = \frac{1}{2}$, $\alpha = \sin^{-1}\frac{4}{5}$

$\theta + \alpha = 30°, 150°, 390°$

$\theta = 30° - \alpha, 150° - \alpha, 390° - \alpha$

$= -23°8', 96°52', 336°52'$

$= 96°52', 336°52'$ (nearest minute)

2/

(a) (i) $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$

$= v\frac{dv}{dx}$

$= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(ii) $\ddot{x} = -2e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\frac{1}{2}v^2 = 2e^{-x} + c$

$x=0, v=2$

$2 = 2e^0 + c, \quad c = 0$

$\frac{1}{2}v^2 = 2e^{-x}$

$v^2 = 4e^{-x}$

$v = \pm 2e^{-\frac{x}{2}}$

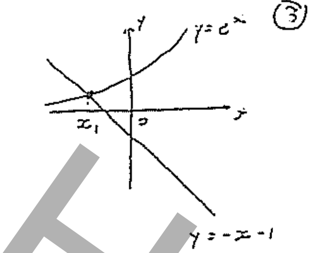
Initially $v > 0$ and $v^2 \neq 0$ ∴ reject $-$ve v

$v = 2e^{-\frac{x}{2}}$

(iii) as $x \to \infty$, $v \to 0$

b) $y = e^x$ ③

$y = -x-1$



$f(x) = e^x + x + 1$

$f'(x) = e^x + 1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= -1\cdot5 - \frac{e^{-1\cdot5} - 1\cdot5 + 1}{e^{-1\cdot5} + 1}$

$= -1\cdot27$ (correct to 2 dec. pl)

c) — V — V — V — —

No. of arrangements $= \frac{3!}{2!} 5!$

$= 360$

3(a)

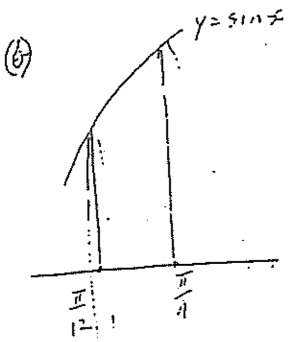
i) $\tan 2\theta = \frac{1}{\sqrt{3}}$

$2\theta = n\pi + \frac{\pi}{6}$

$\theta = n\frac{\pi}{2} + \frac{\pi}{12}$

Q 3 cont. ④

(b)



$V = \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2 x \, dx$

$= \frac{\pi}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1 - \cos 2x)\, dx$

$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2} - \left(\frac{\pi}{12} - \frac{1}{2}\cdot\frac{1}{2}\right)\right)$

$= \frac{\pi}{2}\left(\frac{\pi}{6} - \frac{1}{4}\right)$

$= \frac{\pi(2\pi - 3)}{24}$ units^3

(c)(i) $\frac{dy}{dx} = \frac{x}{2y}$ at $P(2p, p^2)$

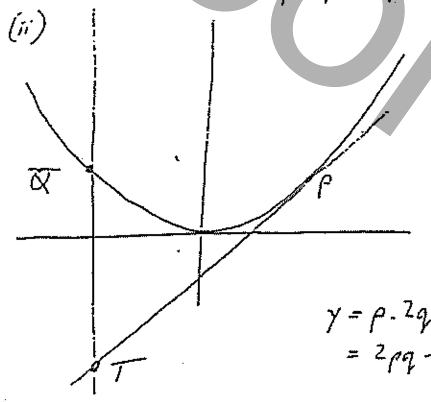
$= p$

required equation: $y - p^2 = p(x - 2p)$

$= px - 2p^2$

$y = px - p^2$

(ii)



$y = p\cdot 2q - p^2$

$= 2pq - p^2$

3 cont

(c)

(iii) $M(p+q, pq)$

(iv) $y = -1$

✓ (a) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan A + \tan B = \frac{5}{3}$

$\tan A \tan B = -\frac{1}{3}$

$\therefore \tan(A+B) = \frac{\frac{5}{3}}{1 - -\frac{1}{3}}$

$= \frac{5}{4}$

(b) $\ddot{x} = -n^2 x$

$\frac{1}{2}v^2 = -\frac{n^2x^2}{2} + C$

$v = V, \quad x = d$

$\therefore C = \frac{1}{2}V^2 + \frac{n^2d^2}{2}$

$\frac{1}{2}v^2 = -\frac{n^2x^2}{2} + \frac{1}{2}V^2 + \frac{n^2d^2}{2}$

$v^2 = V^2 + n^2(d^2 - x^2)$

$v = \frac{V}{2} \quad x = 2d$

$\left(\frac{V}{2}\right)^2 = V^2 + n^2(d^2 - 4d^2)$

$n^2 = \left(V^2 - \frac{V^2}{4}\right) \div 3d^2$

$= \frac{3V^2}{4} \times \frac{1}{3d^2}$

$= \frac{V^2}{[illegible]}$

Q4 cont

(b)

Period $= \frac{2\pi}{n}$

$= 2\pi \cdot \frac{2d}{V}$

$= \frac{4\pi d}{V}$

When $v = 0$, $V^2 + n^2(d^2 - x^2) = 0$

$V^2 + \frac{V^2}{4d^2}(d^2 - x^2) = 0$

$x^2 - d^2 = V^2 \cdot \frac{4d^2}{V^2}$

$x^2 = 5d^2$

$\therefore$ amplitude $= \sqrt{5d^2}$

$= d\sqrt{5}$

(c)

(i) $\frac{dT}{dt} = kBe^{kt}$

$= k(S + Be^{kt} - S)$

$= k(T - S)$

(ii) $t = 0, T = 80°$

$80° = 20° + Be^0$

$B = 60$

$T = 20 + 60e^{kt}$

$40 = 20 + 60e^{2k}$

$e^{2k} = \frac{1}{3}$

$2k = \ln\left(\frac{1}{3}\right)$

$k = \frac{1}{2}\ln\left(\frac{1}{3}\right)$

$t = 3, \quad T = 20 + 60e^{\frac{1}{2}\ln(\frac{1}{3}) \cdot 3}$

$= 20 + 60^{\ln(\frac{1}{3})^{3/2}}$

$= 20 + 60 \cdot 3^{-3/2}$

5/(a) (i) ${}^9C_3 = 84$

(ii) $\frac{{}^5C_1\,{}^3C_1}{84} = \frac{15}{84}$

$= \frac{5}{28}$

(iii) $\frac{{}^5C_3 + 1}{84} = \frac{11}{84}$

(b) $x^3 - 12x^2 + 12x + 80 = 0$

Let roots be $\alpha - m, \alpha, \alpha + m$

sum of roots: $3\alpha = 12$

$\alpha = 4$

product of roots: $(4-m)4(4+m) = {}^-80$

$16 - m^2 = {}^-20$

$m = \pm 6$

$\therefore$ roots are $-2, 4, 10$

(c) $1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$

(i) $x = -1$, $1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$

(ii) integrate both sides wrt x

$x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \ldots + \frac{1}{n+1}\binom{n}{n}x^{n+1} = \frac{1}{n+1}(1+x)^{n+1} + C$

Let $x = 0$ $\therefore \frac{1}{n+1} + C = 0$, $C = -\frac{1}{n+1}$

$x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \ldots + \frac{1}{n+1}\binom{n}{n}x^{n+1} = \frac{1}{n+1}(1+x)^{n+1} - \frac{1}{n+1}$

et $x = -1$,

$-1 + \frac{1}{2}\binom{n}{1} - \frac{1}{3}\binom{n}{2} + \ldots + (-1)^{n+1}\frac{1}{n+1}\binom{n}{n} = -\frac{1}{n+1}$

$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + (-1)^n\frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}$

6(a) $p = \frac{1}{20}$, $q = \frac{19}{20}$

(i) $(q+p)^{20} = \sum_{r=0}^{20}\binom{20}{r}q^{20-r}p^r$

$P(\text{no colour-blind}) = \binom{20}{0}\left(\frac{19}{20}\right)^{20}$

$= \left(\frac{19}{20}\right)^{20}$

(ii) $P(\text{one colour-blind}) = \binom{20}{1}q^{19}p$

$= \cancel{20}\cdot\left(\frac{19}{20}\right)^{19}\cdot\frac{1}{\cancel{20}}$

$= \left(\frac{19}{20}\right)^{19}$

(iii) $P(\text{at least 2 colour-blind})$

$= 1 - \left(\left(\frac{19}{20}\right)^{20} + \left(\frac{19}{20}\right)^{19}\right)$

(b) $(3 + 2x)^n = \sum_{r=0}^{n}\binom{n}{r}3^{n-r}(2x)^r$

$\binom{n}{5}3^{n-5}(2\cancel{x})^5 = \binom{n}{6}3^{n-6}(2\cancel{x})^6$

$\binom{n}{5} \div \binom{n}{6} = \frac{3^{n-6}}{3^{n-5}}\cdot\frac{2^6}{2^5}$

$\frac{\cancel{n!}}{(n-5)!\,5!}\cdot\frac{(n-6)!\,6!}{\cancel{n!}} = \frac{2}{3}$

$\frac{6}{n-5} = \frac{2}{3}$

$2n - 10 = 18$

$n = 14$

6(b) cont

$$\frac{t_{r+1}}{t_r} = \frac{{}^{14}C_r\, 3^{14-r}(2x)^r}{{}^{14}C_{r-1}\, 3^{15-r}(2x)^{r-1}}$$ where t_r = rth term in the expansion

$$= \frac{14!}{(14-r)!\,r!} \cdot \frac{(15-r)!\,(r-1)!}{14!} \cdot \frac{2x}{3}$$

Let c_r = coefficient of the rth term

$$\frac{C_{r+1}}{C_r} = \frac{15-r}{r} \cdot \frac{2}{3}$$

$$= \frac{30-2r}{3r}$$

$\frac{C_{r+1}}{C_r} = 1$ when $r = 6$ $\therefore C_6 = C_7$

$\frac{C_{r+1}}{C_r} > 1$ when $r < 6$ $\therefore C_1 < C_2 < C_3 < C_4 < C_5 < C_6$

$\frac{C_{r+1}}{C_r} < 1$ when $r > 6$ $\therefore C_7 > C_8 > C_9 > C_{10} > C_{11} > C_{12} > C$

C_6 & C_7 are greatest coefficients

7(a) Step1 Let $n=1$

$LHS = (2\times1)^3$

$= 8$ $\therefore$ true for $n=1$

$RHS = 2\times1^2(1+1)^2$

$= 8$

Step 2 Assume result true for $n=k$, k is a positive integer

ie. $2^3 + 4^3 + 6^3 + \dots + (2k)^3 = 2k^2(k+1)^2$

Step 3 Prove result true for $n=k+1$

$\therefore$ prove $2^3 + 4^3 + 6^3 + \dots + (2k)^3 + (2(k+1))^3 = 2(k+1)^2((k+1)+1)^2$

7(a) cont.

$LHS = 2k^2(k+1)^2 + (2(k+1))^3$ from assumption

$= 2(k+1)^2(k^2+4(k+1))$

$= 2(k+1)^2(k+2)^2$

$= RHS$

Step 4 Result is true for $n=1$, Hence it is true for $n=1+1=2$, $n=$ etc. $\therefore$ The result is true for all positive integer.

(b) (i) $x = vt\cos\alpha$

$y = vt\sin\alpha - \frac{1}{2}gt^2$

(ii) $p = vt\cos\alpha$

$t = \frac{p}{v\cos\alpha}$

$$h = v.\frac{p}{v\cos\alpha}\cdot\sin\alpha - \frac{1}{2}g\frac{p^2}{v^2\cos^2\alpha}$$

$$= p\tan\alpha - \frac{5p^2}{v^2\cos^2\alpha}$$

$$= p\tan\alpha - \frac{5p^2(\tan^2\alpha+1)}{v^2}$$

$$5p^2(\tan^2\alpha+1) = v^2(p\tan\alpha - h)$$

$$v^2 = \frac{5p^2(\tan^2\alpha+1)}{p\tan\alpha - h}$$

(iii) $$v^2 = \frac{5q^2(\tan^2\alpha+1)}{q\tan\alpha - h}$$

$$\therefore \frac{\cancel{5}p^2\cancel{(\tan^2\alpha+1)}}{p\tan\alpha - h} = \frac{\cancel{5}q^2\cancel{(\tan^2\alpha+1)}}{q\tan\alpha - h}$$

$$p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$$

$$\tan\alpha(p^2q - q^2p) = p^2h - q^2h$$

$$\tan\alpha = \frac{h\cancel{(p-q)}(p+q)}{} = \underline{h(p+q)}$$