Higher School Certificate Practise Papers

Independent papers 1999-2001

MATHEMATICS EXTENSION 1

Student Number: ……………………………………

2001

HIGHER SCHOOL CERTIFICATE
Sample Examination Paper

MATHEMATICS
Extension 1

General Instructions
Reading time - 5 minutes
Working time - 2 hours

- Attempt ALL questions
- Show all necessary working, marks may be deducted for careless or untidy work
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- Additional Answer Booklets are available

Directions to School or College

Question One

(a) If $y = \sin^{-1} 2x$, find

$\frac{2}{\sqrt{1-4x}}$

(i) $\frac{dy}{dx}$ 2

(ii) $\frac{d^2y}{dx^2}$ 1

(b) Use the definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find the derivative of $2x^3 + x$ 2

(c) Solve the equation $\sin 2\theta + 1 - 4\cos^2\theta = 0$ for $0 \le \theta \le 2\pi$ 3

(d) How many arrangements of the letters of the word SUBSTITUTE are possible? 2

(e) Solve $\frac{4}{x+2} \ge \frac{1}{x}$ 2

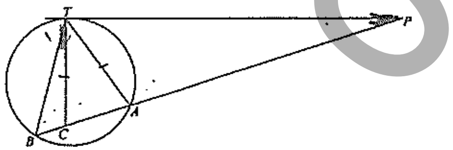
Question Two

(a) Use the substitution $u = 3 + x$ *to find* $\int \frac{x+1}{\sqrt{3+x}}\,dx$ 3

(b) By writing ${}^nC_r$ in factorial form, find the value of n if
${}^nC_3 + {}^{n+1}C_2 + {}^{n+2}C_1 = 72$ 3

(c) Prove by mathematical induction that
$1 + (2+3) + (3+4+5) + ... + [n + (n+1) + (n+2) + ... + (2n-1)]$
$= \frac{n^2}{2}(n+1) \qquad (n \geq 1)$ 3

(d)



PT is a tangent to a circle and TAB is a secant. C is a point on BA such that $TC = TA$. Prove that $\angle BTC = \angle TPA$. 3

Question Three

(a) The curve $y = \frac{\sqrt{2}}{\sqrt{4+x^2}}$ is rotated about the x-axis. Find the exact value of the volume of the solid enclosed between $x = \frac{2}{\sqrt{3}}$ and $x = 2$. 3

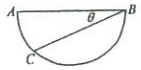
(b) Determine the value of the constant term in the expansion of
$(1+3x)^4\left(1-\frac{1}{x^2}\right)^5$ 3

(c) (i) Show that the equation of the normal to the parabola $x^2 = 4y$ at the point $P(2p, p^2)$ is $x + py = 2p + p^3$ 2

(ii) S is the focus and T is a point on the normal such that ST is perpendicular to the normal. Prove that the locus of T is a parabola with vertex $(0,1)$ and with focal length ¼ that of the parabola $x^2 = 4y$ 4

Question Four

(a)



AB is the diameter of a semi-circle and BC is a chord which makes an angle of θ with AB such that the area of the semi-circle is bisected by the chord.

(i) Show that $2\sin 2\theta + 4\theta = \pi$ 4

(ii) Prove that a root of this equation lies between $\theta = 0.4$ and $\theta = 0.5$ 1

(iii) By putting $f(\theta) = 2\sin 2\theta + 4\theta - \pi$ and taking $\theta = 0.4$ as a first approximation, use Newton's method once to obtain a better approximation of the root to the equation. 2

(b) A gardener is holding a hose at ground level and spraying water in a parabolic path 2 metres high and 6 metres long. Taking $g = 10m/s^2$,

(i) find the initial speed and angle of elevation, and the time each droplet of water is in the air; and 4

(ii) find the maximum distance the gardener can water. 1

$P = 3000 + Ae^{kt}$

$\frac{dP}{dt} = kAe^{kt}$

kt

$\frac{dP}{dt} = k(P-3000)$

$P-3000 = Ae^{kt}$

$\therefore P = 3000 + Ae^{kt}$

Question Five

(a) During the early summer months the rate of increase of the population P of fruit flies is proportional to the excess of the population over 3000. $\frac{dP}{dt} = k(P-3000)$ where k is a constant. At the beginning of summer the population is 4000 and 1 month later it is 10 000.

(i) Show that $P = 3000 + Ae^{kt}$ is a solution of the differential equation, A is a constant 1

(ii) Find the value of A and that of k. 2

(iii) Find to the nearest 100, the population after 2½ months. 1

(iv) After how many weeks does the population reach ½ million? 1

(b) Consider the function $y = x^3e^{-x}$

(i) State the greatest possible domain of the function. 1

(ii) Find the maximum value of the function in the domain. 2

(iii) Show that there are 3 points of inflexion and that one of them has a horizontal tangent. 3

(iv) Sketch the curve for $-1 \le x \le 6$ 1

Question Six

(a) A factory manufactures white T-shirts and coloured T-shirts in the ratio 4 : 1 . The T-shirts are randomly placed in large cartons. One in every 25 white T-shirts is discarded for various reasons, while on average 10% of the coloured T-shirts are discarded because the colour has faded.

(i) What is the probability that a single T-shirt selected at random from a carton will have to be discarded? *2*

(ii) If 6 T-shirts are selected from a carton, find the probability that

(I) all will be satisfactory. *1*

(II) one will have to be discarded. *1*

(III) the majority will be satisfactory. *2*

(b) In quadrilateral $ABCD$, $\angle A$, $\angle B$ and $\angle D$ are obtuse. A line AF parallel to BC meets BD produced in F and a line BE parallel to AD meets AC produced in E. The diagonals AC and BD meet in M.

(i) State with reasons two pairs of triangles which are similar. *3*

(ii) Hence, or otherwise, prove that EF is parallel to CD. *3*

Question Seven

(a) A weight on a spring is moving in simple harmonic motion with a vertical velocity v cm/s given by $v^2 = -8 + 24y - 4y^2$ where y is the vertical displacement in cm.

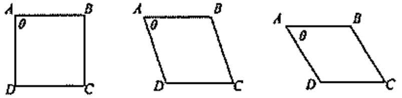
(i) Find the acceleration of the weight in terms of y. *2*

(ii) Find the mean position of the weight. *1*

(iii) Find the period of the oscillation. *1*

(b) By choosing an appropriate value for x in the expansion of $(1+x)^n$, prove that $\sum_{k=0}^{n} 2^{3k}\binom{n}{k} = 3^{2n}$ *2*

(c)



A square ABCD of side 1 unit is gradually 'pushed over' to become a rhombus. The angle at A (θ) decreases at a constant rate of 0.1^c/second.

(i) At what rate is the area of the rhombus ABCD decreasing when $\theta = \frac{\pi}{6}$? *3*

(ii) At what rate is the shorter diagonal of the rhombus ABCD decreasing when $\theta = \frac{\pi}{3}$? *3*

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\; x \neq 0, \textit{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Question One

(a) (i) $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-4x^2}}$

(ii) $\dfrac{d^2y}{dx^2} = 2\left(-\dfrac{1}{2}\right)(1-4x^2)^{-\frac{3}{2}}(-8x)$

$= \dfrac{8x}{(1-4x^2)^{\frac{3}{2}}}$

(b) $f(x) = 2x^3 + x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{2(x+h)^3+x+h-(2x^3+x)}{h} \\ &= \lim_{h\to 0}\frac{2(x^3+3x^2h+3xh^2+h^3)+x+h-2x^3-x}{h} \\ &= \lim_{h\to 0}\frac{2x^3+6x^2h+6xh^2+2h^3+x+h-2x^3-x}{h} \\ &= \lim_{h\to 0}\frac{h(6x^2+6xh+2h^2+1)}{h} \\ &= 6x^2+1 \end{aligned}$$

(c)

$$\begin{aligned} \sin 2\theta + 1 - 4\cos^2\theta &= 0 \\ 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta - 4\cos^2\theta &= 0 \\ \sin^2\theta + 2\sin\theta\cos\theta - 3\cos^2\theta &= 0 \\ (\sin\theta - \cos\theta)(\sin\theta + 3\cos\theta) &= 0 \end{aligned}$$

$\sin\theta - \cos\theta = 0$, $\sin\theta + 3\cos\theta = 0$

$\sin\theta = \cos\theta$, $\sin\theta = -3\cos\theta$

$\tan\theta = 1$, $\tan\theta = -3$

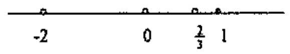
$\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \pi - 1.249, 2\pi - 1.249$

i.e. $\dfrac{\pi}{4}^c, \dfrac{5\pi}{4}^c, 1.89^c, 5.03^c$

(d) Number of arrangements $= \dfrac{10!}{2!3!2!} = 151200$

(e)

$$\begin{aligned} \frac{4}{x+2} &\geq \frac{1}{x} \\ 4(x+2)x^2 &\geq (x+2)^2 x \\ 4(x+2)x^2 - x(x+2)^2 &\geq 0 \\ x(x+2)[4x-(x+2)] &\geq 0 \\ x(x+2)(3x-2) &\geq 0 \end{aligned}$$



Test $x = 1$: valid

$\therefore -2 < x < 0$ *or* $x \geq \dfrac{2}{3}$

Question Two

(a) $\int \frac{x+1}{\sqrt{3+x}}\,dx = \int \frac{u-2}{\sqrt{u}}\,du$

Let $u = 3+x$
$x = u-3$
$du = dx$

$= \int (u^{\frac{1}{2}} - 2u^{-\frac{1}{2}})\,du$

$= \frac{2}{3}u^{\frac{3}{2}} - 2.2u^{\frac{1}{2}} + C$

$= \frac{2}{3}(3+x)^{\frac{3}{2}} - 4(3+x)^{\frac{1}{2}} + C$

(b) ${}^nC_3 + {}^{n+1}C_2 + {}^{n+2}C_1 = 72$

$\frac{n!}{(n-3)!3!} + \frac{(n+1)!}{(n-1)!2!} + n + 2 - 72 = 0$

$\frac{n(n-1)(n-2)}{3!} + \frac{(n+1)n}{2} + n - 70 = 0$

$n(n-1)(n-2) + 3n(n+1) + 6n - 420 = 0$

$n^3 - 3n^2 + 2n + 3n^2 + 3n + 6n - 420 = 0$

$n^3 + 11n - 420 = 0$

Let $f(n) = n^3 + 11n - 420$

$f(8) = 180 \neq 0$

$f(7) = 0$

$\therefore n-7$ is a factor

$\therefore\ n-7 = 0$

$n = 7$

(c) If $n = 1$, LHS = 1 and RHS = $\frac{1}{2}(1+1) = 1$

$\therefore$ The statement is true for $n = 1$

Assume statement is true for $n = k$, *i.e.* assume

$$1 + (2+3) + (3+4+5) + ... + [k + (k+1) + (k+2) + ... + (2k-1)] = \frac{k^2}{2}(k+1)$$

Hence prove statement is true for $n = k+1$, *i.e.* prove

$$1 + (2+3) + (3+4+5) + ... + [k + (k+1) + (k+2) + ... + (2k-1)] + [(k+1) + (k+2) + (k+3) + ... + (2k+1)] = \frac{(k+1)^2(k+2)}{2}$$

Now LHS = $\frac{k^2}{2}(k+1) + [(k+1) + (k+2) + (k+3) + ... + (2k+1)]$

$(k+1) + (k+2) + (k+3) + ... + (2k+1)$ is an A.P. where

$a = k+1,\ d = 1,\ l = 2k+1,\ n = 2k+1-(k+1)+1 = k+1$

$\therefore S_n = \frac{n}{2}(a+l)$

$= \left(\frac{k+1}{2}\right)(k+1+2k+1)$

$= \left(\frac{k+1}{2}\right)(3k+2)$

$\therefore$ LHS $= \frac{k^2}{2}(k+1) + \left(\frac{k+1}{2}\right)(3k+2)$

$= \frac{(k+1)}{2}[k^2 + 3k + 2]$

$= \frac{(k+1)}{2}(k+1)(k+2)$

$= \frac{(k+1)^2}{2}(k+2)$

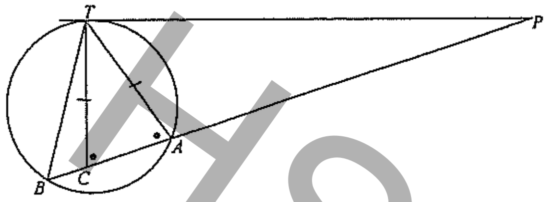
= RHS

$\therefore$ If the statement is true for $n = k$, it is also true for $n = k+1$. But it is true for $n = 1$.

$\therefore$ It is true for $n = 1+1 = 2$ and so on.

$\therefore$ It is true for all integers $n \geq 1$.

(d)



$\angle CAT = \angle TPA + \angle ATP$ *(exterior angle of ΔATP)*
$\angle ACT = \angle TBC + \angle BTC$ *(exterior angle of ΔBTC)*
But $\angle CAT = \angle ACT$ *(base angles of isosceles ΔTCA)*
$\therefore \angle TPA + \angle ATP = \angle TBC + \angle BTC$
But $\angle ATP = \angle TBC$ *(angle between tangent and chord)*
$\therefore \angle TPA = \angle BTC$

Question Three

(a)

$$V = \pi\int_a^b y^2\,dx \qquad y = \frac{\sqrt{2}}{\sqrt{4+x^2}}$$

$$= \pi\int_{\frac{2}{\sqrt{3}}}^{2} \frac{2}{4+x^2}dx$$

$$= \pi\left[\tan^{-1}\frac{x}{2}\right]_{\frac{2}{\sqrt{3}}}^{2}$$

$$= \pi\left[\tan^{-1}1 - \tan^{-1}\frac{1}{\sqrt{3}}\right]$$

$$= \pi\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$$

$$= \frac{\pi^2}{12}u^3$$

(b)

$$(1+3x)^4\left(1-\frac{1}{x^2}\right)^5$$

$$= \left[1+4(3x)+6(3x)^2+4(3x)^3+(3x)^4\right]$$
$$\times\left[1-5(x^{-2})+10(x^{-2})^2-10(x^{-2})^3+5(x^{-2})^4-(x^{-2})^5\right]$$
$$= \left[1+12x+54x^2+108x^3+81x^4\right]\left[1-5x^{-2}+10x^{-4}-10x^{-6}+5x^{-8}-x^{-10}\right]$$

The terms producing constants arise from:
1×1, $54x^2\times(-5x^{-2})$ and $81x^4\times10x^{-4}$
i.e. $1-270+810=541$, the value of the constant term.

(c) (i) $x^2 = 4y$ (focal length $a = 1$)

$$y = \frac{x^2}{4}$$

$$\frac{dy}{dx} = \frac{x}{2}$$

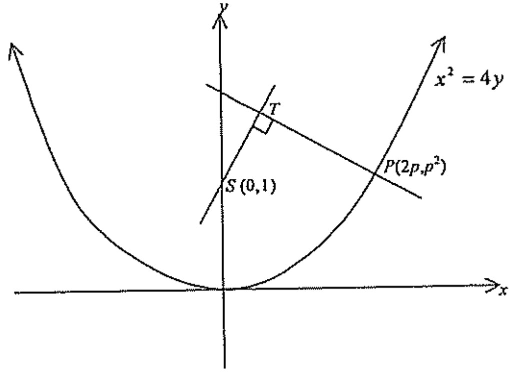
At P, $x = 2p$, $\frac{dy}{dx} = \frac{2p}{2} = p$ (gradient of tangent at P)

$\therefore$ Gradient of normal $= -\frac{1}{p}$

Equation of normal:

$$y - y_1 = m(x - x_1)$$
$$y - p^2 = -\frac{1}{p}(x - 2p)$$
$$py - p^3 = -x + 2p$$
$$x + py = 2p + p^3$$

(ii)



Gradient of $ST = p$
Equation of ST: $y = px + 1$
Since T is the intersection of ST and PT

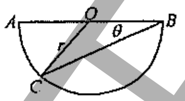
$$\left.\begin{aligned} y &= px + 1 \\ x + py &= 2p + p^3 \end{aligned}\right\} \begin{aligned} &\text{..........(1)} \\ &\text{..........(2)} \end{aligned}$$

$$\underline{py = p^2x + p}\text{............}1 \times p\text{......(3)}$$
$$x = p + p^3 - p^2x\text{....(2)} - \text{(3)}$$
$$x(1 + p^2) = p(1 + p^2)$$
$$x = p$$
$$y = p^2 + 1$$

$\therefore$ Cartesian equation of T: $y = x^2 + 1$, *i.e.* $x^2 = y - 1$
Comparing with $(x - h)^2 = 4a(y - k)$
Vertex $(0,1)$, focal length $a = \frac{1}{4}$ i.e. $\frac{1}{4}$ of that of $x^2 = 4y$.

Question Four

(a)



(i) Let O be the centre of the semi-circle, radius r.
Join CO.
Then $\angle OCB = \theta$ (OC = OB)
and $\angle COB = \pi - 2\theta$

$\therefore$ Area of segment cut off by CB $= \frac{1}{2}r^2(\pi - 2\theta - \sin(\pi - 2\theta))$

Area of segment = ½ area of semi-circle

$$\frac{1}{2}r^2(\pi - 2\theta - \sin 2\theta) = \frac{1}{2}\left(\frac{1}{2}\pi r^2\right)$$

$$\therefore\ \pi - 2\theta - \sin 2\theta = \frac{\pi}{2}$$

$$2\pi - 4\theta - 2\sin 2\theta = \pi$$

$$2\sin 2\theta + 4\theta = \pi$$

(ii) Let $f(\theta) = 2\sin 2\theta + 4\theta - \pi$

$f(0.4) = -0.107 < 0$

$f(0.5) = +0.541 > 0$

Change in sign shows root lies between 0.4 and 0.5

(iii) $f(\theta) = 2\sin 2\theta + 4\theta - \pi$

$f(0.4) = -0.107$

$f'(\theta) = 4\cos 2\theta + 4$

$f'(0.4) = 6.787$

By Newton's Method:

$$\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)}$$

$$= 0.4 - \frac{(-0.107)}{6.787}$$

$$= 0.416$$

(b) (i) Maximum height $= y_{max} = \frac{v^2 \sin^2 \alpha}{2g}$

Range $= \frac{v^2 \sin 2\alpha}{g}$

$\therefore\ 2 = \frac{v^2 \sin^2 \alpha}{2g}$ ....(1)

$6 = \frac{v^2 \sin 2\alpha}{g}$ ....(2)

$\therefore\ \frac{1}{3} = \frac{\sin^2 \alpha}{2\sin 2\alpha}$

$\frac{1}{3} = \frac{\sin^2 \alpha}{4\sin\alpha\cos\alpha}$

$\tan\alpha = \frac{4}{3}$

$\alpha = 53°08'$ the initial angle of elevation

From (1):

$$2 = \frac{v^2 \sin^2 53°08'}{2g}$$

$$v^2 = \frac{40}{\sin^2 53°08'}$$

$$v = 7.9 \text{ m/s}$$

Given $y = vt\sin\alpha - \frac{1}{2}gt^2$

Droplet strikes ground when $y = 0$

i.e. $7.9t\sin 53°08' - 5t^2 = 0$

$t(7.9\sin 53°08' - 5t) = 0$

$t = \frac{7.9\sin 53°08'}{5}$

≈ 1.26 s the time each droplet is in the air.

(ii) Maximum range: when $\alpha = 45°$

$R\max = \frac{v^2}{g}$

$= \frac{62.4}{10}$

$= 6.24$ m

Question Five

(a) (i)

$$\begin{aligned} P &= 3000 + Ae^{kt} \\ \frac{dP}{dt} &= kAe^{kt} \\ &= k(P - 3000) \end{aligned}$$

(ii) When $t = 0$, $P = 3000 + A$

i.e. $4000 = 3000 + A$

$A = 1000$

When P = 10 000 and $t = 1$, $10\,000 = 3000 + 1000e^{k}$

$$\begin{aligned} 7000 &= 1000e^{k} \\ e^{k} &= 7 \\ k &= \ln 7 \\ &\approx 1.946 \end{aligned}$$

(iii) When t = 2.5, $P = 3000 + 1000e^{1.946 \times 2.5} \approx 132600$

(iv)

$$\begin{aligned} P &= 500000 \\ 500000 &= 3000 + 1000e^{1.946t} \\ 497000 &= 1000e^{1.946t} \\ e^{1.946t} &= 497 \\ t &= 3.2 \text{ months} \\ &= 12.8 \text{ weeks} \end{aligned}$$

(b) (i) x may be any real number ($x \in R$)

(ii) Let $f(x) = x^3e^{-x}$

$f'(x) = 3x^2e^{-x} - x^3e^{-x} = 0$ for stationary points

$$\begin{aligned} x^2e^{-x}(3 - x) &= 0 \\ x &= 0 \text{ or } 3 \end{aligned}$$

$$\begin{aligned} f''(x) &= 6xe^{-x} - 3x^2e^{-x} - (3x^2e^{-x} - x^3e^{-x}) \\ &= 6xe^{-x} - 6x^2e^{-x} + x^3e^{-x} \\ &= xe^{-x}(x^2 - 6x + 6) \end{aligned}$$

$f''(0) = 0$ so possible point of inflexion with horizontal tangent at (0,0)

$f''(3) = 3e^{-3}(9 - 18 + 6) < 0$

$\therefore$ maximum turning point at $x = 3$

$$\begin{aligned} f(3) &= 3^3e^{-3} \\ &= \frac{27}{e^3} \end{aligned}$$

$\therefore$ maximum value of the function in the domain is $\frac{27}{e^3} \approx 1.34$

(iii) Possible points of inflexion when $f''(x) = 0$, i.e.:

$xe^{-x}(x^2 - 6x + 6) = 0$

$x = 0$ *or* $x^2 - 6x + 6 = 0$

$$\begin{aligned} x &= \frac{6 \pm \sqrt{12}}{2} \\ &= 3 \pm \sqrt{3} \\ &\approx 4.73 \text{ or } 1.27 \end{aligned}$$

To test change in concavity:

$f''(-1) = -1e^{+1}(1 - 6 + 6) < 0$

$f''(1) = 1e(1 - 6 + 6) > 0$

$\therefore$ Point of inflexion at (0,0)

Since $f'(0) = 0$, this point of inflexion has a horizontal tangent.

$$\left.\begin{aligned} f''(1) &> 0 \text{ already done} \\ f''(2) = 2e^{-2}(4 - 12 + 6) &< 0 \end{aligned}\right\} \text{change in concavity}$$

$f(1.27) \approx 0.58$

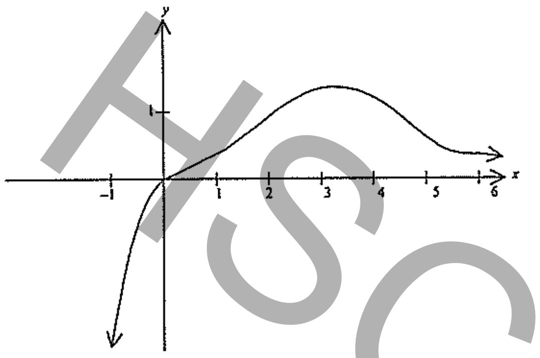
$\therefore$ Point of inflexion at (1.27,0.58)

$$\left.\begin{aligned} f''(4) &= 4e^{-4}(16 - 24 + 6) < 0 \\ f''(5) &= 5e^{-5}(25 - 30 + 6) > 0 \end{aligned}\right\} \text{change in concavity}$$

$f(4.73) \approx 0.93$

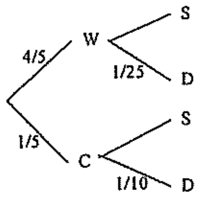
$\therefore$ Point of inflexion at (4.73,0.93)

(iv) $f(-1) \approx -2.72$, $f(6) \approx 0.54$



Question Six

(a)



W = White

C = Coloured

S = Satisfactory

D = Discarded

(i)

$$P(E) = \left(\frac{4}{5} \times \frac{1}{25}\right) + \left(\frac{1}{5} \times \frac{1}{10}\right)$$
$$= \frac{4}{125} + \frac{1}{50}$$
$$= \frac{13}{250}$$

(ii) Let p = success, a satisfactory T-shirt: $\frac{237}{250} = 0.948$

Let q = failure, a discarded T-shirt: $\frac{13}{250} = 0.052$

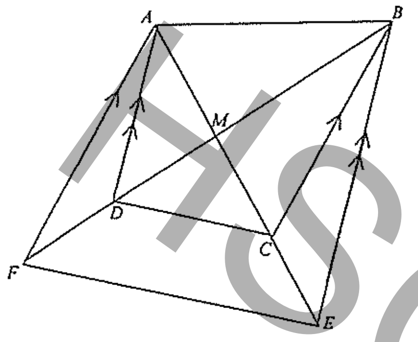
$$(q+p)^6 = \sum_{k=0}^{6} \binom{6}{k} q^{6-k} p^k$$

I For all satisfactory, $k = 6$:
$P(all\ satisfactory) = \binom{6}{6}(0.052)^0(0.948)^6 = 0.726$

II When one is discarded, $k = 5$:
$P(one\ discarded) = \binom{6}{5}(0.052)^1(0.948)^5 = 0.239$

III When majority are satisfactory, $k = 4$, 5 and 6:
$P(most\ satisfactory) = \binom{6}{4}(0.052)^2(0.948)^4 + 0.239 + 0.726$
$= 0.998$

(b)



(i) In Δs AFM and CBM:

$\angle AFM = \angle CBM$ *(alternative angles: AF // BC)*

$\angle AMF = \angle CMB$ *(vertically opposite angles)*

$\therefore$ Δs AFM and CBM are equiangular

$\therefore \Delta AFM \ ||| \ \Delta CBM$

Similarly $\Delta ADM \ ||| \ \Delta EBM$

(ii) $\frac{FM}{BM} = \frac{AM}{CM}$ *(sides of Δs AFM, CBM are in proportion)*

$\therefore FM = \frac{AM \cdot BM}{CM}$

$\frac{EM}{AM} = \frac{BM}{DM}$ *(sides of Δs ADM, EBM are in proportion)*

$\therefore EM = \frac{AM \cdot BM}{DM}$

$$\therefore \frac{FM}{EM} = \frac{\frac{AM \cdot BM}{CM}}{\frac{AM \cdot BM}{DM}}$$

$$= \frac{DM}{CM}$$

$\therefore \frac{FM}{DM} = \frac{EM}{CM}$

$\therefore EF \ // \ CD$ *(line drawn parallel to one side of a triangle divides the other sides in proportion)*

Question Seven

(a) $v^2 = -8 + 24y - 4y^2$

$\frac{1}{2}v^2 = -4 + 12y - 2y^2$

(i) $\therefore \ddot{y} = \frac{d}{dy}(-4 + 12y - 2y^2)$

$= 12 - 4y$

$= -4(y - 3)$

acceleration $\ddot{y} = -4(y - 3)$

(ii) Mean position $= 3\text{cm}$

(iii) $n^2 = 4$

$n = 2$

$\therefore T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ sec

(b) $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

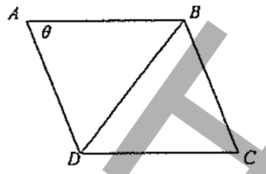
Put $x = 8$,

$9^n = \sum_{k=0}^{n} \binom{n}{k} 8^k$

i.e. $(3^2)^n = \sum_{k=0}^{n} (2^3)^k \binom{n}{k}$

i.e. $3^{2n} = \sum_{k=0}^{n} 2^{3k} \binom{n}{k}$

(c)



Join BD. Rhombus $ABCD$ consists of two congruent triangles.

Area of $\Delta ABD = \frac{1}{2}(1)(1)\sin\theta$

$\therefore$ Area of rhombus $= \sin\theta$

(i)

$$A = \sin\theta$$

$$\frac{dA}{d\theta} = \cos\theta$$

$$\frac{d\theta}{dt} = -0.1^c / s$$

$$\frac{dA}{dt} = \frac{dA}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= \cos\theta\cdot(-0.1)$$

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \cos\frac{\pi}{6}(-0.1)$

$$= -0.087\,\text{u}^2/\text{s}$$

(ii) Let $BD = l$:

$$l^2 = 1^2 + 1^2 - 2(1)(1)\cos\theta \quad \text{(cosine rule)}$$

$$= 2 - 2\cos\theta$$

$l = (2-2\cos\theta)^{\frac{1}{2}}$ *and* $\frac{dl}{d\theta} = \frac{1}{2}(2-2\cos\theta)^{-\frac{1}{2}}.2\sin\theta$

$$\frac{dl}{dt} = \frac{dl}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= \frac{\sin\theta}{\sqrt{2-2\cos\theta}}\cdot(-0.1)$$

When $\theta = \frac{\pi}{3}$,

$$\frac{dl}{d\theta} = \frac{\sin\frac{\pi}{3}(-0.1)}{\sqrt{2-2(\frac{1}{2})}}$$

$$= -0.087\,\text{u/s}$$

Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2001
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 1

Afternoon Session
Thursday 9 August 2001

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks (84)

- Attempt Questions 1 – 7
- All questions are of equal value

2603 - 1

Marks

Question 1 **Begin a new page**

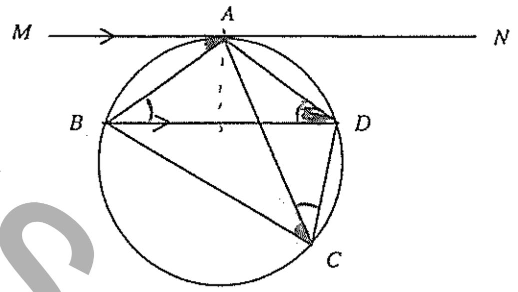
(a) Find the value of $\sum_{k=1}^{4}(-1)^k k!$ 2

(b) $A(-2,-5)$ and $B(1,4)$ are two points. Find the acute angle θ between the line AB and the line $x+2y+1=0$, giving the answer correct to the nearest minute. 3

(c) The polynomial $P(x)=x^5+ax^3+bx$ leaves a remainder of 5 when it is divided by $(x-2)$, where a and b are numerical constants.

(i) Show that $P(x)$ is odd. 1
(ii) Hence find the remainder when $P(x)$ is divided by $(x+2)$. 2

(d)



$ABCD$ is a cyclic quadrilateral. The tangent at A to the circle through A, B, C and D is parallel to BD.

(i) Copy the diagram.
(ii) Give a reason why $\angle ACB = \angle MAB$. 1
(iii) Give a reason why $\angle ACD = \angle ABD$. 1
(iv) Hence show that AC bisects $\angle BCD$. 2

Marks

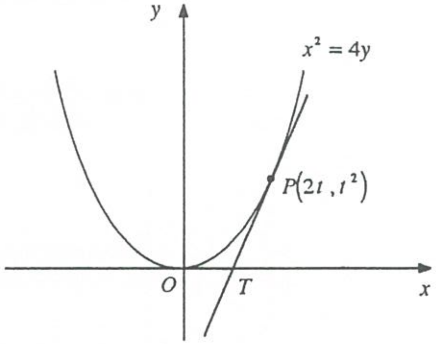
uestion 2 Begin a new page

a) Find $\frac{d^2}{dx^2} e^{x^2}$. 2

b) $A(-1,4)$ and $B(x,y)$ are two points. The point $P(14,-6)$ divides the interval AB externally in the ratio $5:3$. Find the coordinates of B. 3

c) Find the number of ways in which the letters of the word EXTENSION can be arranged in a straight line so that no two vowels are next to each other. 3

d)



$P(2t, t^2)$ is a variable point which moves on the parabola $x^2 = 4y$. The tangent to the parabola at P cuts the x axis at T. M is the midpoint of PT.

(i) Show that the tangent PT has equation $tx - y - t^2 = 0$. 1

(ii) Show that M has coordinates $\left(\frac{3t}{2}, \frac{t^2}{2}\right)$. 2

(iii) Hence find the Cartesian equation of the locus of M as P moves on the parabola. 1

Marks

Question 3 Begin a new page

(a) (i) By expanding $\cos(2A + A)$, show that $\cos 3A = 4\cos^3 A - 3\cos A$. 2

(ii) Hence show that if $2\cos A = x + \frac{1}{x}$ then $2\cos 3A = x^3 + \frac{1}{x^3}$. 3

(b) The function $f(x)$ is given by $f(x) = \sqrt{x+6}$ for $x \geq -6$.

(i) Find the inverse function $f^{-1}(x)$ and find its domain. 2

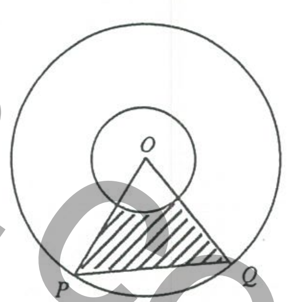
(ii) On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the intercepts on the coordinate axes. Draw in the line $y = x$. 3

(iii) Show that the x coordinates of any points of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfy the equation $x^2 - x - 6 = 0$. Hence find any points of intersection of the two graphs. 2

Question 4 Begin a new page

(a) Use Mathematical Induction to show that $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n. 5

(b)



Two concentric circles with centre O have radii 2 cm and 4 cm . The points P and Q lie on the larger circle and $\angle POQ = x$, where $0 < x < \frac{\pi}{2}$.

(i) If the area A cm^2 of the shaded region is $\frac{1}{16}$ the area of the larger circle, show that x satisfies the equation $8\sin x - 2x - \pi = 0$. 3

(ii) Show that this equation has a solution $x = \alpha$, where $0 \cdot 5 < \alpha < 0 \cdot 6$. 2

(iii) Taking $0 \cdot 6$ as a first approximation for α , use one application of Newton's Method to find a second approximation, giving the answer correct to two decimal places. 2

Marks

Question 5

Begin a new page

(a) Evaluate $\int_1^{49} \frac{1}{4(x+\sqrt{x})}\,dx$ using the substitution $u^2 = x$, $u > 0$. Give the answer in simplest exact form. **4**

(b) At any point on the curve $y = f(x)$, the gradient function is given by $\frac{dy}{dx} = \sin^2 x$. **4**

Find the value of $f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)$.

(c) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds it has velocity v metres per second, and displacement x metres from a fixed point O on the line, where $x = 5\cos\frac{\pi t}{2}$.

(i) Find the period of the motion. **1**

(ii) Find an expression for v in terms of t, and hence show that $v^2 = \frac{\pi^2}{4}(25 - x^2)$. **3**

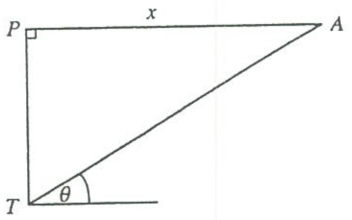
Find the speed of the particle when it is 4 metres to the right of O.

Marks

Question 6

Begin a new page

(a)



A person on horizontal ground is looking at an aeroplane A through a telescope T. The aeroplane is approaching at a speed of $80\ \text{ms}^{-1}$ at a constant altitude of 200 metres above the telescope. When the horizontal distance of the aeroplane from the telescope is x metres, the angle of elevation of the aeroplane is θ radians.

(i) Show that $\theta = \tan^{-1}\frac{200}{x}$. **1**

(ii) Show that $\frac{d\theta}{dt} = \frac{16000}{x^2 + 40000}$. **2**

(iii) Find the rate at which θ is changing when $\theta = \frac{\pi}{4}$, giving the answer in degrees per second correct to the nearest degree. **2**

(b) A particle moves in a straight line. At time t seconds its displacement is x metres from a fixed point O on the line, its acceleration is $a\ \text{ms}^{-2}$, and its velocity is $v\ \text{ms}^{-1}$ where v is given by $v = \frac{32}{x} - \frac{x}{2}$.

(i) Find an expression for a in terms of x. **1**

(ii) Show that $t = \int \frac{2x}{64 - x^2}\,dx$, and hence show that $x^2 = 64 - 60\,e^{-t}$. **3**

(iii) Sketch the graph of x^2 against t and describe the limiting behaviour of the particle. **3**

Marks

Question 7 **Begin a new page**

(a) Four fair dice are rolled. Any die showing 6 is left alone, while the remaining dice are rolled again.

(i) Find the probability (correct to 2 decimal places) that after the first roll of the dice, exactly one of the four dice is showing 6. **1**

(ii) Find the probability (correct to 2 decimal places) that after the second roll of the dice exactly two of the four dice are showing 6. **4**

(b) A particle is projected from a point O with speed $50\ \text{ms}^{-1}$ at an angle of elevation θ, and moves freely under gravity, where $g = 10\ \text{ms}^{-2}$.

(i) Write down expressions for the horizontal and vertical displacements of the particle at time t seconds referred to axes Ox and Oy. **1**

(ii) Hence show that the equation of the path of the projectile, given as a quadratic equation in $\tan\theta$, is $x^2\tan^2\theta - 500x\tan\theta + (x^2 + 500y) = 0$. **2**

(iii) Hence show that there are two values of θ, $0 < \theta < \frac{\pi}{2}$, for which the projectile passes through a given point (X, Y) provided that $500\,Y < 62500 - X^2$. **2**

(iv) If the projectile passes through the point (X, X) whose coordinates satisfy this inequality, and the two values of θ are α and β, find expressions in terms of X for $\tan\alpha + \tan\beta$ and $\tan\alpha\tan\beta$, and hence show that $\alpha + \beta = \frac{3\pi}{4}$. **2**

Question 1

(a) **Outcomes Assessed: H5, H9**

Marking Guidelines

Criteria	Marks
• one mark for simplification of sum • one mark for value of sum	2

Answer:

$$\sum_{k=1}^{4}(-1)^k k! = -1! + 2! - 3! + 4! = 19$$

(b) **Outcomes Assessed: P4**

Marking Guidelines

Criteria	Marks
• one mark for values of gradients • one mark for value of $\tan\theta$ • one mark for size of angle	3

Answer:

$$\left.\begin{array}{r} AB \text{ has gradient } m_1 = 3 \\ x+2y+1=0 \text{ has gradient } m_2 = -\frac{1}{2} \end{array}\right\} \Rightarrow \tan\theta = \left|\frac{3-\left(-\frac{1}{2}\right)}{1+3\left(-\frac{1}{2}\right)}\right| = 7 \qquad \therefore \theta = 81°\,52'$$

(c) **Outcomes Assessed: (i) P5 (ii) PE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for showing $P(x)$ is odd (ii) • one mark for showing remainder is $-P(2)$ • one mark for value of remainder	3

Answer:

(i)

$$\begin{aligned} P(-x) &= (-x)^5 + a(-x)^3 + b(-x) \\ &= -x^5 - a x^3 - b x \\ &= -(x^5 + a x^3 + b x) \\ &= -P(x) \quad \text{for all } x \end{aligned}$$

$\therefore P(x)$ is odd.

(ii) When $P(x)$ is divided by $(x+2)$, remainder is $P(-2) = -P(2)$ since $P(x)$ is odd

$= -5$ since $P(2) = 5$

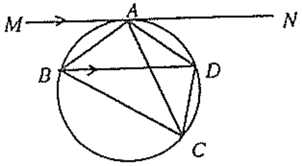
(d) **Outcomes Assessed:** (i) (ii) PE3 (iii) PE3 (iv) H5, PE2, PE3

Marking Guidelines

Criteria	Marks
(i) • no marks for copying diagram (ii) • one mark for reason (iii) • one mark for reason (iv) • one mark for showing $\angle MAB = \angle ABD$ • one mark for showing $\angle ACB = \angle ACD$	4

Answer:

(i)



(ii) $\angle ACB = \angle MAB$ because the angle between the tangent MA and the chord AB through the point of contact A is equal to the angle ACB in the alternate segment.

(iii) $\angle ACD = \angle ABD$ because the angles subtended in the same segment at B and C by the arc AD are equal.

(iv)

$\angle MAB = \angle ABD$ (equal alternate angles, $MN \parallel BD$)

$\angle ACB = \angle ACD$ ($\angle MAB = \angle ACB$, $\angle ABD = \angle ACD$)

$\therefore$ AC bisects $\angle BCD$

Question 2

(a) **Outcomes Assessed:** P7, PE5

Marking Guidelines

Criteria	Marks
• one mark for first derivative • one mark for second derivative using product rule.	2

Answer:

$$\frac{d}{dx} e^{x^2} = 2x e^{x^2} \qquad \frac{d^2}{dx^2} e^{x^2} = \frac{d}{dx} 2x e^{x^2} = 2\left(e^{x^2}\right) + (2x)\left(2x e^{x^2}\right) = 2\left(1+2x^2\right)e^{x^2}$$

(b) **Outcomes Assessed:** P4

Marking Guidelines

Criteria	Marks
• one mark for equation in x • one mark for equation in y • one mark for coordinates of B	3

Answer:

$$\frac{5x-3\times(-1)}{5-3} = 14 \Rightarrow 5x+3=28 \qquad \therefore x=5$$

$$\therefore B(5,0)$$

$$\frac{5y-3\times(4)}{5-3} = -6 \Rightarrow 5y-12=-12 \qquad \therefore y=0$$

(c) **Outcomes Assessed:** PE3

Marking Guidelines

Criteria	Marks
• one mark for number of arrangements of vowels • one mark for number of arrangements of consonants • one mark for total number of arrangements	3

Answer:

The vowels (E, E, I, O) can be arranged in positions 2, 4, 6, 8 in $\frac{4!}{2!} = 12$ ways.

The consonants (N,N, S, T, X) can be arranged in positions 1, 3, 5, 7, 9 in $\frac{5!}{2!} = 60$ ways.

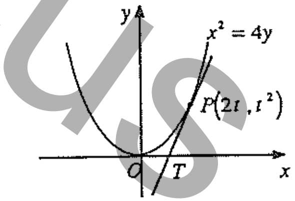
Hence the total number of arrangements is $12 \times 60 = 720$.

(d) **Outcomes Assessed:** (i) PE3, PE4 (ii) PE3 (iii) PE3

Marking Guidelines

Criteria	Marks
(i) • one mark for equation of tangent (ii) • one mark for coordinates of T • one mark for coordinates of M (iii) • one mark for equation of locus	4

Answer:



(i) $y = \frac{1}{4}x^2 \Rightarrow \frac{dy}{dx} = \frac{1}{2}x$

$\therefore$ tangent at $P(2t, t^2)$ has gradient $\frac{1}{2}(2t) = t$

and equation $y - t^2 = t(x-2t)$

$tx - y - t^2 = 0$

(ii) At T, $y = 0 \Rightarrow tx - 0 - t^2 = 0 \Rightarrow x = t$

Hence T has coordinates $(t, 0)$, and

M is the midpoint of $P(2t, t^2)$ and $T(t, 0)$,

with coordinates $\left(\frac{2t+t}{2}, \frac{t^2+0}{2}\right) \equiv \left(\frac{3t}{2}, \frac{t^2}{2}\right)$.

(iii) At M, $x = \frac{3t}{2} \Rightarrow t = \frac{2x}{3}$

$\therefore y = \frac{1}{2}t^2 = \frac{1}{2}\left(\frac{2x}{3}\right)^2 = \frac{2x^2}{9}$

Hence the locus has equation $2x^2 = 9y$.

stion 3

Outcomes Assessed: (i) P4 (ii) PE3

Marking Guidelines

Criteria	Marks
(i) • one mark for expansion and expressions for $\cos 2A$, $\sin 2A$	5
• one mark for simplification to obtain final expression for $\cos 3A$ in terms of $\cos A$	
(ii) • one mark for expressing $2\cos 3A$ in terms of $\left(x+\frac{1}{x}\right)$	
• one mark for binomial expansion of $\left(x+\frac{1}{x}\right)^3$	
• one mark for simplification to obtain final expression for $\cos 3A$ in terms of x	

wer:

os $3A = \cos(2A+A)$
$= \cos 2A \cos A - \sin 2A \sin A$
$= \left(2\cos^2 A - 1\right)\cos A - (2\sin A\cos A)\sin A$
$= 2\cos^3 A - \cos A - 2\sin^2 A\cos A$
$= 2\cos^3 A - \cos A - 2\left(1-\cos^2 A\right)\cos A$
$= 4\cos^3 A - 3\cos A$

(ii)
$2\cos 3A = 8\cos^3 A - 6\cos A$
$= (2\cos A)^3 - 3(2\cos A)$
$= \left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right)$
$= x^3 + 3x + \frac{3}{x} + \left(\frac{1}{x}\right)^3 - 3x - \frac{3}{x}$
$= x^3 + \frac{1}{x^3}$

utcomes Assessed: (i) P5, HE4 (ii) P5, HE4 (iii) P4

Marking Guidelines

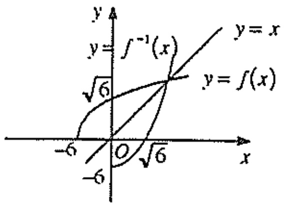
Criteria	Marks
(i) • one mark for finding the inverse function	7
• one mark for the domain of the inverse function	
(ii) • one mark for the graph of $y=f(x)$ and intercepts	
• one mark for the graph of $y=f^{-1}(x)$ and intercepts	
• one mark for the line $y=x$ passing through the point of intersection	
(iii) • one mark for the equation	
• one mark for the coordinates of the point of intersection	

ver:

$= \sqrt{x+6}$ Interchanging x and y
$= x+6$ gives $y = x^2 - 6$
$= y^2 - 6$ $\therefore f^{-1}(x) = x^2 - 6$

nge of $f(x)$ is $\{y : y \geq 0\}$ $\Rightarrow$ Domain of $f^{-1}(x)$ is $\{x : x \geq 0\}$

(ii)



/here $y=f(x)$, $y=f^{-1}(x)$, $y=x$ intersect,
$f^{-1}(x) = x \Rightarrow \quad x^2 - 6 = x$
$x^2 - x - 6 = 0$
$(x+2)(x-3) = 0$
ıt $x \neq -2$ (outside domain). $\therefore x = 3$

Hence intersection point of the curves is $(3,3)$.

Question 4

(a) Outcomes Assessed: HE2

Marking Guidelines

Criteria	Marks
• one mark for establishing the truth of $S(1)$	5
• one mark for $S(k)$ true $\Rightarrow 5^k + 2\left(11^k\right) = 3M$ for some integer M.	
• one mark for $5^{k+1} + 2\left(11^{k+1}\right) = 5\left(5^k\right) + 22\left(11^k\right)$	
• one mark for deducing $S(k)$ true $\Rightarrow S(k+1)$ true	
• one mark for deducing $S(n)$ true for all integers $n \geq 1$	

Answer:

Define the sequence of statements $S(n)$: $5^n + 2\left(11^n\right)$ *is a multiple of* 3, $n = 1, 2, 3, \ldots$

Consider $S(1)$: $5^1 + 2\left(11^1\right) = 27 = 3 \times 9 \quad \therefore S(1)$ is true.

If $S(k)$ *is true*, then $5^k + 2\left(11^k\right) = 3M$ for some integer M. **

Consider $S(k+1)$: $5^{k+1} + 2\left(11^{k+1}\right) = 5\left(5^k\right) + 22\left(11^k\right) = 5\left\{\left(5^k\right) + 2\left(11^k\right)\right\} + 12\left(11^k\right)$

$\therefore 5^{k+1} + 2\left(11^{k+1}\right) = 5(3M) + 12\left(11^k\right) = 3\left\{5M + 4\left(11^k\right)\right\}$ *if* $S(k)$ *is true, using* **

But M and k integral $\Rightarrow \left\{5M + 4\left(11^k\right)\right\}$ is an integer.

$\therefore S(k)$ true $\Rightarrow S(k+1)$ true, $k = 1, 2, 3, \ldots$

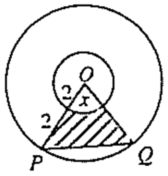
Hence $S(1)$ is true, and if $S(k)$ is true, then $S(k+1)$ is true. $\therefore S(2)$ is true, and then $S(3)$ is true, and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers n.

(b) Outcomes Assessed: (i) H5 (ii) P5, H2 (iii) PE3

Marking Guidelines

Criteria	Marks
(i) • one mark for areas of small circle sector and triangle OPQ	7
• one mark for equating expression for shaded area to $\frac{1}{16}$ of large circle area	
• one mark for simplification to find equation in required form	
(ii) • one mark for showing $f(0{\cdot}5), f(0{\cdot}6)$ have opposite signs	
• one mark for using continuity of $f(x)$ to deduce $0{\cdot}5 < \alpha < 0{\cdot}6$	
(iii) • one mark for expression for second approximation	
• one mark for calculation of second approximation	

Answer:



(i)
Area of $\Delta POQ = \frac{1}{2}\left(4^2\right)\sin x$
Area small circle sector$= \frac{1}{2}\left(2^2\right)x$
$\therefore$ shaded area $= 8\sin x - 2x$
$\therefore 8\sin x - 2x = \frac{1}{16}\pi\left(4^2\right) = \pi$
$8\sin x - 2x - \pi = 0$

(ii) Let $f(x) = 8\sin x - 2x - \pi$. Then $f(0{\cdot}5) \approx -0{\cdot}31 < 0$ and $f(0{\cdot}6) \approx 0{\cdot}18 > 0$. Hence, since $f(x)$ is continuous, $f(\alpha) = 0$ for some $0{\cdot}5 < \alpha < 0{\cdot}6$.

(iii) Taking a first approximation $\alpha \approx 0{\cdot}6$,
Newton's method gives a second approximation

$$\alpha \approx 0{\cdot}6 - \frac{f(0{\cdot}6)}{f'(0{\cdot}6)}$$
$$= 0{\cdot}6 - \frac{8\sin(0{\cdot}6) - 2(0{\cdot}6) - \pi}{8\cos(0{\cdot}6) - 2}$$
$$\approx 0{\cdot}56 \text{ to 2 decimal places.}$$

Question 5

(a) **Outcomes Assessed: HE6**

Marking Guidelines

Criteria	Marks
• one mark for change of limits • one mark for change of variable • one mark for integration • one mark for evaluation	4

Answer:

Let $I = \int_1^{49} \frac{1}{4\left(x+\sqrt{x}\right)}\, dx$

$u^2 = x\ ,\ \ u > 0$

$2u = \frac{dx}{du} \Rightarrow dx = 2u\, du$

$x = 1 \Rightarrow u = 1, \quad x = 49 \Rightarrow u = 7$

Then $I = \int_1^7 \frac{1}{4\left(u^2+u\right)}\, 2u\, du$

$= \int_1^7 \frac{1}{2\left(u+1\right)}\, du$

$= \frac{1}{2}\left[\ln(u+1)\right]_1^7$

$\therefore\ I = \frac{1}{2}(\ln 8 - \ln 2) = \frac{1}{2}\ \ln 4 = \ln 2$

(b) **Outcomes Assessed: H5**

Marking Guidelines

Criteria	Marks
• one mark for expressing $\sin^2 x$ in terms of $\cos 2x$ • one mark for integration, including constant of integration • one mark for evaluation of $f\left(\frac{\pi}{4}\right),\ f\left(\frac{3\pi}{4}\right)$ • one mark for value of difference	4

Answer:

$\frac{dy}{dx} = \sin^2 x$

$= \frac{1}{2}(1 - \cos 2x)$

$y = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c, \quad c \text{ constant}$

$f(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

$\therefore\ f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)$

$= \left(\frac{3\pi}{8} - \frac{1}{4}\sin\frac{3\pi}{2} + c\right) - \left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + c\right)$

$= \left(\frac{3\pi}{8} + \frac{1}{4} + c\right) - \left(\frac{\pi}{8} - \frac{1}{4} + c\right)$

$= \frac{\pi}{4} + \frac{1}{2}$

(c) **Outcomes Assessed: (i) HE3 (ii) H5, HE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for finding the period of the motion (ii) • one mark for expressing v^2 in terms of t • one mark for expressing v^2 in terms of x • one mark for the value of the speed.	4

Answer:

(i) Period is $2\pi \div \frac{\pi}{2} = 4$ seconds

(ii)

$x = 5\cos\frac{\pi}{2}t$

$v = \frac{dx}{dt} = 5\left(-\frac{\pi}{2}\sin\frac{\pi}{2}t\right)$

$v^2 = \left(\frac{\pi^2}{4}\right).25\sin^2\frac{\pi}{2}t$

$v^2 = \left(\frac{\pi^2}{4}\right).25\left(1 - \cos^2\frac{\pi}{2}t\right)$

$= \frac{\pi^2}{4}\left(25 - 25\cos^2\frac{\pi}{2}t\right)$

$v^2 = \frac{\pi^2}{4}\left(25 - x^2\right)$

$x = 4 \Rightarrow v^2 = \frac{\pi^2}{4}(25-16) = \frac{9\pi^2}{4}$

Speed is $\frac{3\pi}{2}$ ms^{-1}

Question 6

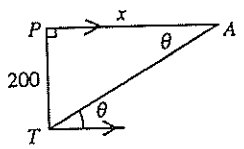
(a) **Outcomes Assessed: (i) P4, HE4 (ii) HE4, HE5 (iii) H5**

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for θ (ii) • one mark for expression for $\frac{d\theta}{dx}$ • one mark for expression for $\frac{d\theta}{dt}$ (iii) • one mark for value of $\frac{d\theta}{dt}$ • one mark for value of θ	5

Answer:

(i)



$\angle TAP = \theta$

(alt. $\angle$ s, parallel lines)

$\tan\theta = \frac{200}{x}$

$\theta = \tan^{-1}\frac{200}{x}$

(ii)

$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{200}{x}\right)^2}\left(-\frac{200}{x^2}\right) = \frac{-200}{x^2+40000}$

$\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt} = \frac{-200}{x^2+40000}\ (-80)$

$\therefore\ \frac{d\theta}{dt} = \frac{16\,000}{x^2+40\,000}$

(iii) When $\theta = \frac{\pi}{4}$, $TP = AP \Rightarrow x = 200$, and $\frac{d\theta}{dt} = \frac{16000}{(200)^2+40000} = 0{\cdot}2$ radians per second.

Hence θ is increasing at 11° s^{-1} (correct to the nearest degree)

(b) **Outcomes Assessed: (i) HE5 (ii) H3, H5, HE4 (iii) HE3, HE7**

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for a in terms of x (ii) • one mark for expressing t as an integral with respect to x • one mark for integration to find t in terms of x • one mark for expression for x^2 in terms of t (iii) • one mark for graph of x^2 as a function of t • one mark for limiting values of $x,\ v,\ a$ • one mark for description of limiting behaviour in words	7

Answer:

(i)

$v^2 = \left(\frac{32}{x} - \frac{x}{2}\right)^2 = \frac{1024}{x^2} - 32 + \frac{x^2}{4}$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}\frac{d}{dx}\left(\frac{1024}{x^2} - 32 + \frac{x^2}{4}\right)$

$\therefore\ a = \frac{-1024}{x^3} + \frac{x}{4}$

(ii)

$\frac{dx}{dt} = v = \frac{32}{x} - \frac{x}{2} = \frac{64-x^2}{2x}$

$\therefore\ \frac{dt}{dx} = \frac{2x}{64-x^2}$

$t = \int \frac{2x}{64-x^2}\, dx$

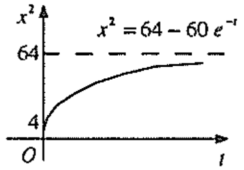
(ii) Cont.

$$t = -\ln\left(64 - x^2\right) + c\ , \quad \left.\begin{matrix} t=0 \\ x=2 \end{matrix}\right\} \Rightarrow c = \ln 60$$

$$-t = \ln\left(\frac{64-x^2}{60}\right)\ , \qquad e^{-t} = \frac{64-x^2}{60}$$

$$\therefore x^2 = 64 - 60\,e^{-t}$$

(iii)



As $t \to \infty$, $x \to 8^-$, $v \to \frac{32}{8} - \frac{8}{2} = 0^+$, $a \to \frac{-1024}{512} + \frac{8}{4} = 0^-$

Hence the particle is moving right and slowing down as it approaches its limiting position 8 metres to the right of O.

Question 7

(a) **Outcomes Assessed:** (i) **HE3** (ii) **HE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for value of probability (ii) • one mark for expression for probability of two 6's on first roll and no 6's on second • one mark for expression for probability of one 6 on first roll and one 6 on second • one mark for expression for probability of no 6's on first roll and two 6's on second • one mark for value of probability	5

Answer:

(i) $P(\textit{one 6 on first roll}) = {}^4C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 \approx 0\cdot39$ (to 2 decimal places)

(ii) $P(\textit{two 6 s on first roll and no 6 s on second roll}) = {}^4C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 \times {}^2C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^2 \approx 0\cdot0804$

$P(\textit{one 6 on first roll and one 6 on second roll}) = {}^4C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 \times {}^3C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 \approx 0\cdot1340$

$P(\textit{no 6 s on first roll and two 6 s on second roll}) = {}^4C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 \times {}^4C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 \approx 0\cdot0558$

$\therefore\ P(\textit{two 6 s overall}) \approx 0\cdot0804 + 0\cdot1340 + 0\cdot0558 \approx 0\cdot27$ (to 2 decimal places)

(b) **Outcomes Assessed:** (i) **HE3** (ii) **HE3** (iii) **P4, H2** (iv) **P4, H2**

Marking Guidelines

Criteria	Marks
(i) • one mark for expressions for x and y in terms of θ and t (ii) • one mark for expression for y in terms of x • one mark for rearrangement as quadratic in $\tan\theta$ (iii) • one mark for discriminant in terms of X and Y • one mark for using discriminant > 0 to give required inequality (iv) • one mark for the values of the sum and product of $\tan\alpha$, $\tan\beta$ in terms of X • one mark for the value of $\alpha+\beta$	7

Answer:

(i) $x = 50\,t\cos\theta$ and $y = 50\,t\sin\theta - 5\,t^2$

(ii)

$$t = \frac{x}{50\cos\theta} \Rightarrow y = x\frac{\sin\theta}{\cos\theta} - \frac{5\,x^2}{2500\cos^2\theta}$$

$$500\,y = 500\,x\tan\theta - x^2\sec^2\theta$$

$$= 500\,x\tan\theta - x^2\left(1+\tan^2\theta\right)$$

$$= 500\,x\tan\theta - x^2 - x^2\tan^2\theta$$

$$\therefore x^2\tan^2\theta - 500\,x\tan\theta + \left(x^2 + 500\,y\right) = 0$$

(iii) Projectile passes through the point (X, Y) if $\tan\theta$ satisfies the quadratic equation

$$X^2\tan^2\theta - 500\,X\tan\theta + \left(X^2 + 500\,Y\right) = 0$$

This equation has two distinct solutions for $\tan\theta$, and hence for θ , provided its discriminant $\Delta > 0$.

$$\Delta = (-500\,X)^2 - 4\,X^2\left(X^2 + 500\,Y\right)$$

$$= 4X^2\left(62\,500 - X^2 - 500\,Y\right)$$

$\therefore\ \Delta > 0$ provided $500\,Y < 62\,500 - X^2$

(iv) If the projectile passes through the point (X, X) where $500\,X < 62\,500 - X^2$, then the equation $X^2\tan^2\theta - 500\,X\tan\theta + \left(X^2 + 500\,X\right) = 0$ has two distinct real roots $\tan\alpha$, $\tan\beta$ where

$$\tan\alpha + \tan\beta = \frac{500\,X}{X^2} = \frac{500}{X} \quad \text{and} \quad \tan\alpha\,\tan\beta = \frac{X^2 + 500\,X}{X^2} = 1 + \frac{500}{X}$$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} = \frac{500}{X} \div \left(-\frac{500}{X}\right) = -1$$

Since $0 < \alpha + \beta < \pi$, $\alpha + \beta = \frac{3\pi}{4}$.

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HSC Trial Examination 2001

Mathematics
Extension 1

This paper must be kept under strict security and may only be used on or after the afternoon of Thursday 9 August, 2001, as specified in the NEAP Examination Timetable.

General Instructions

Reading time 5 minutes

Working time 2 hours

Write using blue or black pen.

Board-approved calculators may be used.

A table of standard integrals is provided on page 10.

All necessary working should be shown in every question.

Examination structure

Total marks 84

Attempt all questions.

All questions are of equal value.

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QUESTION 1. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$. 3

(b) Solve $\frac{3}{x^2-1} \geq 0$. 3

(c) Use the substitution $u^2 = 4 + x$ to evaluate $\int_0^5 \frac{x}{\sqrt{4+x}} dx$. 4

(d) Simplify $\frac{{}^nP_{r+1}}{{}^nP_r}$. 2

QUESTION 2. (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Differentiate $x\tan^{-1}x$. 1

(ii) Hence evaluate $\int_0^1 \tan^{-1}x\,dx$. 3

(b) (i) Sketch the graph of $y = \cos x, -\pi \leq x \leq \pi$ and use this graph to show that $\cos x + x = 0$ has only one solution. 2

(ii) Use Newton's method with a first approximation of $x = -1$ to find a second approximation to the root of $\cos x + x = 0$. 2

(c) There are 12 videotapes arranged in a row on a shelf in a video shop. There are 3 identical copies of *Gone with the Wind*, 4 of *Tootsie* and 5 of *Gladiator.*

(i) How many different arrangements of the videotapes are there? 1

(ii) How many different arrangements are there if videos with the same title are grouped together? 1

(iii) The 12 videotapes are arranged at random in a row on the shelf. Find the probability that the arrangement has a copy of *Gone with the Wind*, at one end of the row, and a copy of *Gladiator* at the other end. 2

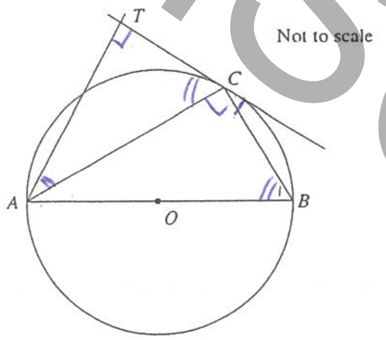
QUESTION 3. (12 marks) Use a SEPARATE writing booklet.

Marks

(a) (i) Write down an expression for $\tan(\alpha + \beta)$. 1

(ii) Hence evaluate $\alpha + \beta$, for $0 \le \alpha + \beta \le 2\pi$, if $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$. 2

(b) 3



In the diagram AOB is the diameter of a circle centre O, and C is the point of contact of the tangent TC such that AC bisects $\angle TAB$.

Prove that AT is perpendicular to TC.

(c) The acceleration of a particle moving in a straight line is given by $\frac{d^2x}{dt^2} = -e^{-2x}$, where x is the displacement, in metres, from the origin O and t is time elapsed in seconds. Initially the particle moves from the origin with velocity +2 metres per second. 3

Find the velocity of the particle at $x = 1$, correct to 2 decimal places.

(d) The coefficients of x^3 and x^4 in the expansion of $(1 + bx)^{12}$, $b \ne 0$, are equal. 3

Find the value of b.

QUESTION 4. (12 marks) Use a SEPARATE writing booklet.

Marks

(a) Consider the expression $P(x) = x^3 + ax^2 + bx + 6$, where a and b are constants.

When $P(x)$ is divided by $(x-2)(x+1)$, the remainder is $4 - 4x$, and the quotient is $x + k$.

(i) By using the division transformation, or otherwise, find the values of a and b. 4

(ii) Find the value k for these values of a and b. 1

(b) (i) Write down an expression for $\cos 2A$ in terms of $\sin A$. 1

(ii) Hence show that $\sin^2\frac{x}{2} = \frac{1}{2}(1 - \cos x)$. 1

(iii) Sketch $y = \sin^2\frac{x}{2}$ for $0 \le x \le 2\pi$. 2

(iv) State the amplitude and period of $y = \sin^2\frac{x}{2}$. 1

(v) Find the exact area of the region bounded by the curve $y = \sin^2\frac{x}{2}$ and the x-axis between $x = 0$ and $x = \frac{\pi}{3}$. 2

QUESTION 5. (12 marks) Use a SEPARATE writing booklet. Marks

(a) Use mathematical induction to prove that **4**

$$\log 2+\log\left(\frac{3}{2}\right)+\log\left(\frac{4}{3}\right)+\ldots+\log\left(\frac{n+1}{n}\right)=\log(n+1) \text{ for } n\geq 1 .$$

(b) Newton's law of cooling states that the rate of change of the temperature T of a body at any time t is proportional to the difference in the temperature of the body and the temperature M of the surrounding medium.

i.e. $\frac{dT}{dt}=k(T-M)$, where k is a constant.

(i) Show that $T=M+Ae^{kt}$, where A is a constant, satisfies this equation. **1**

(ii) A freezer is maintained at a constant temperature of -8 °C. When water at 25°C is placed in the freezer, the temperature of the water falls to 15°C in 10 minutes. Find the temperature of the water after 10 more minutes, correct to the nearest degree. **3**

(c) Tony and Rosa each throw an unbiased coin five times.

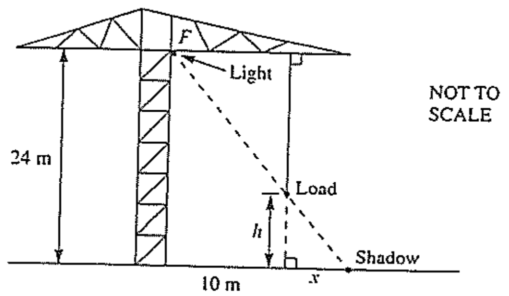
(i) Find the probability they both throw exactly three heads. **2**

(ii) Assuming they both throw at least one head, find the probability they both throw the same number of heads. **2**

QUESTION 6. (12 marks) Use a SEPARATE writing booklet. Marks

(a) A particle moves with simple harmonic motion and has a speed of 5 centimetres per second when passing through the centre O of its path. The period is π seconds. Find the speed of the particle when it is 1.5 centimetres from O. **2**

(b)



The diagram shows a crane that can lift loads vertically up to a height of 24 metres. When work is being done at night, a floodlight at point F casts a shadow of each load onto the level ground.

When loads are being lifted, their height above ground level, h metres, after t seconds, is given by $h=2t$, where $0\leq t\leq 12$. As shown in the diagram, each load is lifted from a starting point which is on the ground and which is 10 metres from the base of the crane. The shadow is cast x metres away from the load's starting point.

(i) Show that $x=\frac{10h}{24-h}$. **1**

(ii) Find the position of the shadow when $t=4$. **1**

(iii) Find the rate at which the shadow is moving when $t=4$. **3**

(c) Consider the function $f(x)=\frac{1}{x^2}-1, x>0$.

(i) Write down the equations of the horizontal and vertical asymptotes for $y=f(x)$. **1**

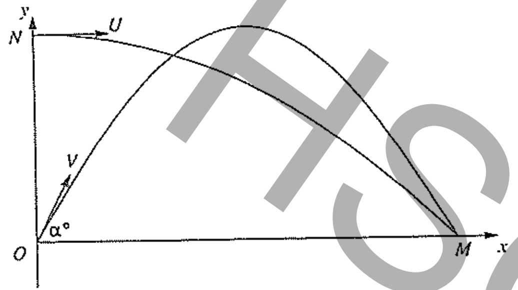
(ii) Show that $y=f(x)$ has no stationary points. **1**

(iii) Show that the curve of $y=f(x)$ is always concave up. **1**

(iv) Determine the inverse function, $f^{-1}(x)$. **1**

(v) Sketch $y=f^{-1}(x)$. **1**

QUESTION 7. (12 marks) Use a SEPARATE writing booklet. Marks



The diagram shows the path of two projectiles in the same plane.

Particle A is fired from a point O, with speed V metres per second, at an angle of elevation α radians to the horizontal. It reaches a maximum height, H metres, above O and lands at point M on the same horizontal level as O.

Particle B is fired from point N, H metres above O, with speed U metres per second in a horizontal direction, and also lands at point M.

With the above axes, you may assume that the velocity and the position of particle A, at time t seconds after it is fired, are given by

$$\dot{x} = V\cos\alpha \qquad \dot{y} = V\sin\alpha - gt$$
$$x = Vt\cos\alpha \qquad y = Vt\sin\alpha - \frac{1}{2}gt^2$$

where g is the acceleration due to gravity.

Also with the above axes, you may assume that the velocity and the position of particle B are given by

$$\dot{x} = U \qquad \dot{y} = -gt$$
$$x = Ut \qquad y = -\frac{1}{2}gt^2 + H$$

(a) Show that the greatest height reached by a particle A is given by $H = \dfrac{V^2\sin^2\alpha}{2g}$. 1

(b) Show that the range of particle A is given by $R = \dfrac{V^2\sin 2\alpha}{g}$. 2

(c) Show that the range of particle B is given by $R = \dfrac{UV\sin\alpha}{g}$. 2

QUESTION 7. (Cont.) Marks

(d) Show that the relationship between U and V can be given by $U = 2V\cos\alpha$. 1

(e) Show that the trajectory of particle B can be expressed as $y = \dfrac{V^2\sin^2\alpha}{2g} - \dfrac{g}{2U^2}x^2$. 1

(f) If particles A and B land at point M at the same instant, and if θ is the difference in angles of impact of the two particles, show that $\tan\theta = \dfrac{\tan\alpha}{2 + \tan^2\alpha}$, where θ is acute. 3

(g) By expressing the equation in part (f) as a quadratic in terms of $\tan\alpha$, or otherwise, find the range of values that θ can take, correct to the nearest minute. 2

Standard integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right).$$

Note: $\ln x = \log_e x, \quad x > 0$

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HSC Trial Examination 2001

Mathematics Extension 1

Solutions and suggested marking scheme

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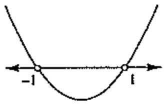
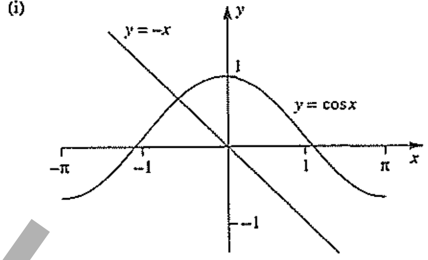
QUESTION 1

Sample Answer	Outcome listing and mark guide
(a) $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{3}}$ $= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$ $= \frac{\pi}{3} - \frac{\pi}{6}$ $= \frac{\pi}{6}$	HE4 • Gives correct answer ....3 • Gives correct integral and gives at least one correct $\sin^{-1}$ evaluation ....2 • Gives correct integral 1
(b) $\frac{3}{x^2-1} \geq 0$ $x^2 - 1 \neq 0$ i.e. $x \neq 1$ or $x \neq -1$ $(x^2 - 1) > 0$ $(x-1)(x+1) > 0$ $x < -1$ or $x > 1$  	PE3 • Gives correct solution ....3 • Gives solution of $x \leq -1, x \geq 1$ ....2 • Gives correct factorisation 1
(c) $u^2 = 4 + x$ $x = u^2 - 4$ $dx = 2u\,du$ If $x = 5$, $u^2 = 9$ $\therefore u = 3$ If $x = 0$, $u^2 = 4$ $\therefore u = 2$ $\int_0^5 \frac{x}{\sqrt{4+x}}dx = \int_2^3 \frac{u^2-4}{\sqrt{u^2}}2u\,du$ $= 2\int_2^3 (u^2-4)du$ $= 2\left[\frac{1}{3}u^3 - 4u\right]_2^3$ $= 2\left\{[9-12] - \left[\frac{8}{3} - 8\right]\right\}$ $= \frac{14}{3}$ or $4\frac{2}{3}$ or 4.67 (2 d.p.)	HE6 • Gives correct solution ....4 • Finds correct integral.................3 • Gives correct expression to be integrated .2 • Shows significant progress in finding the correct expression to be integrated 1
(d) $\frac{{}^nP_{r+1}}{{}^nP_r} = \frac{n!}{(n-(r+1))!} + \frac{n!}{(n-r)!}$ $= \frac{n!}{(n-r-1)!} \times \frac{(n-r)(n-r-1)!}{n!}$ $= n - r$	PE3 • Gives correct answer ....2 • Gives correct expression for ${}^nP_{n+1}$ and ${}^nP_r$ 1

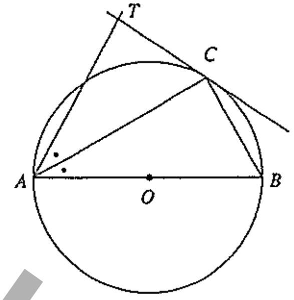
QUESTION 2

Sample Answer	Outcome listing and mark guide
(a) (i) $\frac{d}{dx}x\tan^{-1}x = \tan^{-1}x + \frac{x}{1+x^2}$	HE4, H5 • Gives correct derivative...............1
(ii) $\int_0^1 \tan^{-1}x\,dx = \int_0^1 \frac{d}{dx}x\tan^{-1}x\,dx - \int_0^1 \frac{x\,dx}{1+x^2}$ $= \left[x\tan^{-1}x\right]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ $= [(1)\tan(1) - 0]$ $-\left[\frac{1}{2}\ln(1+1) - \frac{1}{2}\ln(1+0)\right]$ $= \left[\frac{\pi}{4} - 0\right] - \left[\frac{1}{2}\ln 2 - 0\right]$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$	HE4, H5 • Gives correct answer.................3 • Gives the two correct integrals or • Gives the correct answer to one of the integrals..........................2 • Gives correct expression to integrate 1
(b) (i)  To solve: $\cos x + x = 0$ $\cos x = -x$ $\therefore$ draw $y = \cos x$ and $y = -x$, $-\pi \leq x \leq \pi$ There is only one point of intersection. $\therefore \cos x + x = 0$ has only one solution.	P4 • One mark for drawing two correct graphs. • One mark for giving a satisfactory explanation 2
(ii) Newton's method Let $f(x) = \cos x + x$ $f(-1) = \cos(-1) - 1$ $= -0.459697\ldots$ $f'(x) = -\sin x + 1$ $f'(-1) = -\sin(-1) + 1$ $= 1.84147\ldots$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $= -1 - \frac{-0.459697\ldots}{1.84147\ldots}$ $= -0.75036\ldots$ $= -0.75$ (to 2 decimal places)	PE3, H5 • Gives correct answer.................2 • Finds correct values of $f(-1)$ and $f'(-1)$ and quotes correct formula 1

QUESTION 2 (Continued)

Sample Answer	Outcome listing and mark guide
(c) (i) Number of arrangements $= \frac{12!}{3!4!5!}$ $= 27\ 720$	PE3 • Gives correct answer 1
(ii) Number of arrangements $= 3!$ $= 6$	PE3 • Gives correct answer 1
(iii) Number of arrangements with *Gone with the Wind* at one end and *Gladiator* at the other end $= 2! \times \frac{10!}{2! \times 4! \times 4!}$ $= 6300$ $P(\text{above arrangement}) = \frac{6300}{27\ 720}$ $= \frac{5}{22}$	PE3, H5 • Gives correct answer . . . . 2 • Gives correct answer for number of arrangements 1

QUESTION 3

Sample Answer	Outcome listing and mark guide
(a) (i) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	H5 • Gives correct expression 1
(ii) $\tan(\alpha+\beta) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}$ $= \frac{\frac{5}{6}}{\frac{5}{6}}$ $= 1$ $\therefore \alpha+\beta = \frac{\pi}{4}$ or $\alpha+\beta = \frac{5\pi}{4}$	H5 • One mark for finding $\tan(\alpha+\beta) = 1$ • One mark for the two answer 2
(b)  In ΔACT and ΔABC $\angle TAC = \angle BAC$ (AC bisects $\angle TAB$) $\angle TCA = \angle CBA$ (alternate segment theorem) $\Delta ACT \parallel\!\!\mid \Delta ABC$ (two pairs of corresponding angles equal) $\angle ATC = \angle ACB$ (corresponding angles of similar triangles) $\angle ACB = 90°$ (angle of a semi-circle) $\therefore \angle ATC = 90°$ $\therefore AT \perp TC$	PE2, PE3 • Gives correct proof with reasons. . . . . 3 • Gives partially correct proof with reasons or • Gives correct proof without reasons . . . . . 2 • Gives correct initial step to a solution, e.g. correctly uses alternate segment theorem 1

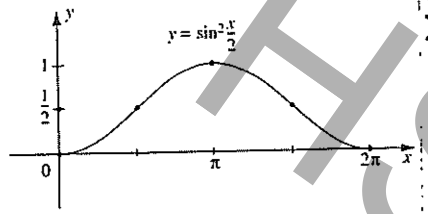
QUESTION 3 (Continued)

Sample Answer	Outcome listing and mark guide
(c) $\frac{d^2x}{dt^2} = -e^{-2x}$ $\frac{1}{2}v^2 = \int -e^{-2x}dx$ $= \frac{1}{2}e^{-2x} + c$ At $x = 0$, $v = +2$. $\frac{1}{2}(4) = \frac{1}{2}e^0 + c$ $c = \frac{3}{2}$ $\therefore v^2 = e^{-2x} + 3$ at $x = 1$, $v^2 = e^{-2} + 3$ $v^2 = 3.13533\ldots$ $v = 1.77068\ldots$ $\therefore$ velocity is 1.77 m/s (to 2 decimal places).	HE5 - One mark for correct primitive - One mark for correct expression for $v^2 = e^{-2x} + 3$ - One mark for correct answer. 3
(d) For $(1 + bx)^{12}$, $T_4 = {}^{12}C_3(bx)^3$; $T_5 = {}^{12}C_4(bx)^4$ $\therefore$ coefficient of x^3 is ${}^{12}C_3b^3$ and coefficient of x^4 is ${}^{12}C_4b^4$ $\therefore {}^{12}C_3b^3 = {}^{12}C_4b^4$ $\frac{b^4}{b^3} = \frac{{}^{12}C_3}{{}^{12}C_4}$ $\therefore b = \frac{\frac{12!}{3!9!}}{\frac{12!}{4!8!}}$ $= \frac{4!8!}{3!9!}$ $\therefore b = \frac{4}{9}$	PE3, HE6, HE7 - Gives correct answer. . . . . 3 - Gives correct expression for b. . . . . 2 - Gives correct expressions for coefficients of x^3 and x^4 1

QUESTION 4

Sample Answer	Outcome listing and mark guide
(a) (i) For polynomial $P(x)$, $P(x) = A(x)Q(x) + R(x)$ $A(x) =$ divisor $Q(x) =$ quotient $R(x) =$ remainder and degree $R(x) <$ degree $A(x)$ Here $x^3 + ax^2 + bx + 6$ $= (x - 2)(x + 1)(x + k) + (4 - 4x)$ Let $x = 2$ $8 + 4a + 2b + 6 = 0 + (-4)$ $4a + 2b = -18$ $2a + b = -9$ [Eq. 1] Let $x = -1$ $-1 + a - b + 6 = 0 + 8$ $a - b = 3$ [Eq. 2] On adding Eq. 1 and Eq. 2 $3a + 0 = -6$ $a = -2$ Then from Eq. 2, $b = -5$ $\therefore a = -2, b = -5$	PE3 - Writes the division transformation or uses the remainder theorem - Determines the 2 equations needed to find a and b - Gives the values for a and b . . . . 4 - Writes the division transformation or uses the remainder theorem - Determines the 2 equations in a and b - Attempts to solve equations or - Writes the division transformation, or uses the remainder theorem - Determines one correct equation in a and b - Finds a and b correctly from their equations. . . . . 3 - Writes the division transformation or uses the remainder theorem - Determines one correct equation in a and b and attempts to find the other . . . . 2 - Writes the division transformation or uses the remainder theorem and attempts to find the two equations. 1
(ii) Now $x^3 - 2x^2 - 5x + 6$ $= (x - 2)(x + 1)(x + k) + (4 - 4k)$ Let $x = 0$ $6 = -2 \times 1 \times k + 4$ $2 = -2k$ $k = -1$	PE3 - Finds the value of k which is correct for their values of a and b. 1
(b) (i) $\cos 2A = \cos^2 A - \sin^2 A$ $= 1 - 2\sin^2 A$	H5 - Gives correct expression 1
(ii) $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ $= \left(1 - \sin^2\frac{x}{2}\right) - \sin^2\frac{x}{2}$ $= 1 - 2\sin^2\frac{x}{2}$ $2\sin^2\frac{x}{2} = 1 - \cos x$ $\sin^2\frac{x}{2} = \frac{1}{2}(1 - \cos x)$	H5 - Uses a correct method. 1

QUESTION 4 (Continued)

Sample Answer	Outcome listing and mark guide
(iii)  (Note that the curve is more easily drawn by thinking of it as $y=\frac{1}{2}(1-\cos x)$ rather than as $y=\sin^2\frac{x}{2}$.)	P5, H5 • One mark for correct shape • One mark for correct values on x and y axes 2
(iv) Amplitude $=\frac{1}{2}$ Period $=2\pi$	H5 • Gives both correct values 1
(v) Area $=\int_0^{\frac{\pi}{3}}\sin^2\frac{x}{2}dx$ $=\int_0^{\frac{\pi}{3}}\frac{1}{2}(1-\cos x)dx$ $=\frac{1}{2}\Big[x-\sin x\Big]_0^{\frac{\pi}{3}}$ $=\frac{1}{2}\Big[\Big(\frac{\pi}{3}-\sin\frac{\pi}{3}\Big)-(0-0)\Big]$ $=\frac{1}{2}\Big[\frac{\pi}{3}-\frac{\sqrt{3}}{2}\Big]$ $=\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ $\therefore$ Area is $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ unit^2	H8 • One mark for writing correct integrated expression $\frac{1}{2}\Big[x-\sin x\Big]_0^{\frac{\pi}{3}}$ • One mark for correct answer, in any form..........2

QUESTION 5

Sample Answer	Outcome listing and mark guide
(a) To be proved: $\log 2+\log\Big(\frac{3}{2}\Big)+\log\Big(\frac{4}{3}\Big)...+\log\Big(\frac{n+1}{n}\Big)$ $=\log(n+1), n\geq 1$ *Step 1* Let $n=1$. The statement becomes $\log 2=\log 2$ which is true. *Step 2* Assume the statement is true for some positive integer k. That is, assume $\log 2+\log\Big(\frac{3}{2}\Big)+\log\Big(\frac{4}{3}\Big)+...+\log\Big(\frac{k+1}{k}\Big)$ $=\log(k+1)$ [statement 1] Now it has to be shown that the following statement, where $n=k+1$, is true. $\log 2+\log\Big(\frac{3}{2}\Big)+\log\Big(\frac{4}{3}\Big)+...+\log\Big(\frac{k+1}{k}\Big)+\log\Big(\frac{k+2}{k+1}\Big)$ $=\log(k+2)$ [statement 2] Now $\log 2+\log\Big(\frac{3}{2}\Big)+\log\Big(\frac{4}{3}\Big)+...+\log\Big(\frac{k+1}{k}\Big)+\log\Big(\frac{k+2}{k+1}\Big)$ $=\log(k+1)+\log\Big(\frac{k+2}{k+1}\Big)$ (by statement 1 above) $=\log(k+1)+\log(k+2)-\log(k+1)$ (by logarithm laws) $=\log(k+2)$, as required. Hence if statement 1 is true, then so is statement 2. Now, if the original statement is true when $n=k$, where k is a positive integer, then it is also true when $n=k+1$. But the statement is true when $n=1$. Therefore, the statement is true for all positive integers n.	PE2, HE2 • One mark for showing statement is true for $n=1$ • One mark for correctly using statement 1 to attempt to prove statement 2 • One mark for correctly proving the statement for $n=k+1$ • One mark for correctly stating conclusion 4
(b) (i) $T=M+Ae^{kt}$ $\therefore\frac{dT}{dt}=Ake^{kt}$ Now consider the equation $\frac{dT}{dt}=k(T-M)$ $\frac{dT}{dt}=Ake^{kt}$, from above $k(T-M)=k(M+Ae^{kt}-M)$, from above $=kAe^{kt}$ $\therefore T=M+Ae^{kt}$ satisfies the equation.	H2, H3, H5 • Gives correct derivative of T, and correctly substitutes expressions for T and $\frac{dT}{dt}$ into the given equation 1

QUESTION 5 (Continued)

		Sample Answer	Outcome listing and mark guide
	(ii)	$T = M + Ae^{kt}$ Here, $M = -8$, and at $t = 0$, $T = 25$ $25 = -8 + Ae^0$ $\therefore A = 33$ $\therefore T = -8 + 33e^{kt}$ Now at $t = 10$, $T = 15$ $15 = -8 + 33e^{10k}$ $23 = 33e^{10k}$ $e^{10k} = \frac{23}{33}$ $10k = \log_e \frac{23}{33}$ $k = \frac{1}{10}\log_e \frac{23}{33}$ $= -0.036101\ldots$ $= -0.0361$ (correct to 4 decimal places) $\therefore T = -8 + 33e^{-0.0361t}$ Now let $t = 20$ $T = -8 + 33e^{-0.0361 \times 20}$ $= 8.030$ $\therefore$ after 10 more minutes, the temperature of the water is 8°C.	H3, HE3 • One mark for correctly finding the value of A • One mark for correctly finding the value of k, given their value of A • One mark for correctly finding the temperature after 10 more minutes, given their values of A and k 3
(c)	(i)	Probability of one person throwing exactly 3 heads $= {}^5C_3[P(T)]^2[P(H)]^3$ $= {}^5C_3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$ $= {}^5C_3\left(\frac{1}{2}\right)^5$ $\therefore$ probability of *both* Tony and Rosa throwing 3 heads. $= {}^5C_3\left(\frac{1}{2}\right)^5 \times {}^5C_3\left(\frac{1}{2}\right)^5$ $= \left({}^5C_3\left(\frac{1}{2}\right)^5\right)^2$ $= \left(10 \times \left(\frac{1}{2}\right)^5\right)^2$ $= \frac{100}{1024}$ $= \frac{25}{256}$	HE3 • Finds the correct probability, in exact or decimal form . . . . 2 • Finds correct probability of one person throwing exactly 3 heads, i.e. ${}^5C_3\left(\frac{1}{2}\right)^5$ . . 1

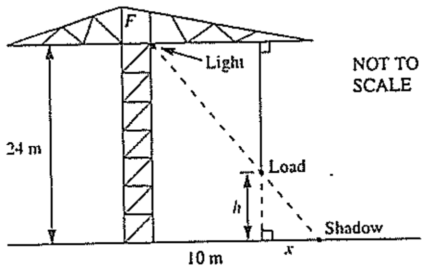
QUESTION 5 (Continued)

	Sample Answer	Outcome listing and mark guide
(ii)	Probability of Tony and Rosa throwing same number of heads $= [P(1H)]^2 + [P(2H)]^2 + [P(3H)]^2$ $+ [P(4H)]^2 + [P(5H)]^2$ $= \left({}^5C_1\left(\frac{1}{2}\right)^5\right)^2 + \left({}^5C_2\left(\frac{1}{2}\right)^5\right)^2 + \left({}^5C_3\left(\frac{1}{2}\right)^5\right)^2$ $+ \left({}^5C_4\left(\frac{1}{2}\right)^5\right)^2 + \left({}^5C_5\left(\frac{1}{2}\right)^5\right)^2$ $= [({}^5C_1)^2 + ({}^5C_2)^2 + ({}^5C_3)^2 + ({}^5C_4)^2 + ({}^5C_5)^2] \times \frac{1}{2^{10}}$ $= [5^2 + 10^2 + 10^2 + 5^2 + 1^2] \times \frac{1}{1024}$ $= \frac{251}{1024}$	HE3 • Shows that the probability is given by sum of the series with terms such as $\left({}^5C_1\left(\frac{1}{2}\right)^5\right)^2$ • Finds the correct probability in any form . . . . 2 • Attempts to find the probability by finding the sum of the series with terms such as $\left({}^5C_1\left(\frac{1}{2}\right)^5\right)^2$ 1

QUESTION 6

Sample Answer	Outcome listing and mark guide
(a) For simple harmonic motion. $v^2 = n^2(a^2 - x^2)$ Period = π sec, given $\therefore \frac{2\pi}{n} = \pi$ $n = 2$ At $x = 0, v = 5$ $\therefore 5^2 = 2^2(a^2 - 0^2)$ $25 = 4a^2$ $a = \frac{5}{2}$ $= 2.5$ Now let $x = 1.5$ $v^2 = 2^2(2.5^2 - 1.5^2)$ $v^2 = 4 \times 4$ $v = \pm 4$ ∴ speed of particle when it is 1.5 cm from 0 is 4 cm/s.	HE3 • Finds the correct speed . . . . 2 • Shows substantial progress towards finding the correct speed 1

(b)

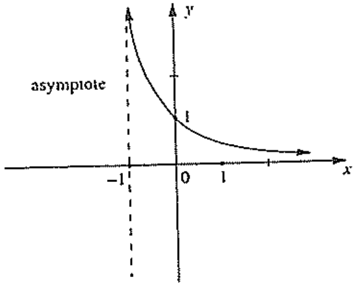


Sample Answer	Outcome listing and mark guide
(i) From similar triangles in diagram: $\frac{24}{10+x} = \frac{h}{x}$ $24x = 10h + hx$ $24x - hx = 10h$ $x(24 - h) = 10h$ $x = \frac{10h}{24-h}$	H5 • Shows appropriate ratios, based on similar triangles, as being equal 1
(ii) $h = 2t$ (given) ∴ when $t = 4, h = 8$ $\therefore x = \frac{10 \times 8}{24 - 8}$ $= 5$ ∴ shadow is 5 metres to the right of load's starting point.	P4, H9 • Correctly states position of shadow 1

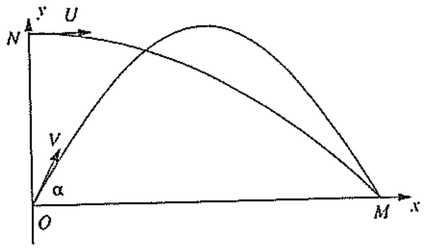
QUESTION 6 (Continued)

Sample Answer	Outcome listing and mark guide
(iii) $\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$ $\frac{dx}{dh} = \frac{(24-h)10 - 10h \times (-1)}{(24-h)^2}$ $= \frac{240 - 10h + 10h}{(24-h)^2}$ $= \frac{240}{(24-h)^2}$ $h = 2t$ $\frac{dh}{dt} = 2$ $\therefore \frac{dx}{dt} = \frac{240}{(24-h)^2} \times 2$ $= \frac{480}{(24-h)^2}$ At $t = 4, h = 8$ $\therefore \frac{dx}{dt} = \frac{480}{(24-8)^2}$ $= 1.875$ ∴ shadow is moving at a rate of 1.875 m/s to the right.	HE5 • One mark for correctly finding the expression for $\frac{dx}{dh}$ • One mark for correctly finding the expression for $\frac{dx}{dt}$ in terms of h • One mark for correctly stating the speed of the shadow 3
(c) (i) $f(x) = \frac{1}{x^2} - 1, x > 0$ To find horizontal asymptote, let $x \to \infty$ Then $f(x) \to -1$ ∴ horizontal asymptote is $y = -1$ Vertical asymptote is $x = 0$	P5 • Correctly states the horizontal and vertical asymptotes 1
(ii) $f'(x) = -2x^{-3}$ $= \frac{-2}{x^3}$ There is no value of x for which $\frac{-2}{x^3} = 0$ $\therefore y = f(x)$ has no stationary points.	H6 • Uses the derivative to show the result 1
(iii) $f''(x) = 6x^{-4}$ $= \frac{6}{x^4}$ $\frac{6}{x^4} > 0$, for all x, except $x = 0$ $\therefore y = f(x)$ is always concave up.	H6 • Uses the second derivative to show the result 1

QUESTION 6 (Continued)

Sample Answer	Outcome listing and mark guide
(iv) $y = \frac{1}{x^2} - 1,\ x > 0,\ y > -1$ Interchange x, y $x = \frac{1}{y^2} - 1,\ y > 0,\ x > -1$ $y^2x = 1 - y^2$ $y^2(x+1) = 1$ $y^2 = \frac{1}{x+1}$ $y = \frac{1}{\sqrt{x+1}}$, since $x > -1,\ y > 0$	HE4 • Determines the inverse function correctly. 1
(v) 	P5, HE4 • Draws the correct graph for their inverse function 1

QUESTION 7

Sample Answer	Outcome listing and mark guide
	
(a) $y = Vt\sin\alpha - \frac{1}{2}gt^2$ $\dot{y} = V\sin\alpha - gt$ Height is greatest when $\dot{y} = 0$ i.e. when $t = \frac{V\sin\alpha}{g}$ $\therefore y = V\sin\alpha \times \frac{V\sin\alpha}{g} - \frac{1}{2} \times g \times \frac{V^2\sin^2\alpha}{g^2}$ $= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$ $= \frac{V^2\sin^2\alpha}{2g}$ $\therefore H = \frac{V^2\sin^2\alpha}{2g}$	HE3 • Shows result by setting $\dot{y} = 0$, finding expression for t, and substituting into expression for y. 1
(b) $x = Vt\cos\alpha$ Range is obtained by letting $y = 0$ $Vt\sin\alpha - \frac{1}{2}gt^2 = 0$ $t\left(V\sin\alpha - \frac{1}{2}gt\right) = 0$ $t = 0$ or $t = \frac{2V\sin\alpha}{g}$ (ignore $t = 0$) $x = V\cos\alpha \times \frac{2V\sin\alpha}{g}$ $= \frac{V^2 \times 2\sin\alpha\cos\alpha}{g}$ $\therefore R = \frac{V^2\sin2\alpha}{g}$	HE3 • One mark for finding correct expression for t • One mark for correctly deriving the expression for R. 2

QUESTION 7 (Continued)

	Sample Answer	Outcome listing and mark guide
(c)	For particle B, $x = Ut$ Range is obtained by letting $y = 0$. $y = -\frac{1}{2}gt^2 + H$ $-\frac{1}{2}gt^2 + H = 0$ $\therefore t = \sqrt{\frac{2H}{g}}$ $\therefore R = U\sqrt{\frac{2H}{g}}$ $= U\sqrt{\frac{2}{g} \times \frac{V^2\sin^2\alpha}{2g}}$ $= U\sqrt{\frac{V^2\sin^2\alpha}{g^2}}$ $= \frac{UV\sin\alpha}{g}$	HE3 • One mark for finding correct expression for t • One mark for correctly deriving the expression for R. .2
(d)	To show relationship between U and V, equate the two different expressions for R. From (b), $R = \frac{V^2\sin 2\alpha}{g}$ [Eq. 1] From (c), $R = U\frac{V\sin\alpha}{g}$ [Eq. 2] Let Eq. 1 = Eq. 2 $\frac{V^2\sin 2\alpha}{g} = \frac{UV\sin\alpha}{g}$ $2\sin\alpha\cos\alpha V = U\sin\alpha$ $\therefore U = 2V\cos\alpha$	HE3 • Correctly equates the 2 different expressions for R and simplifies 1
(e)	For B, $x = Ut$ $y = \frac{1}{2}gt^2 + H$ Eliminate t $t = \frac{x}{U}$ $y = -\frac{1}{2}g\left(\frac{x}{U}\right)^2 + H$ $= -\frac{1}{2}g\left(\frac{x}{U}\right)^2 + \frac{V^2\sin^2\alpha}{2g}$ from (a) $\therefore y = \frac{V^2\sin^2\alpha}{2g} - \frac{g}{2U^2}x^2$	HE3 • Eliminates t to find equation of trajectory 1

QUESTION 7 (Continued)

	Sample Answer	Outcome listing and mark guide
(f)	Similarly for A. $y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$ For A: $\frac{dy}{dx} = \tan\alpha - \frac{gx}{V^2\cos^2\alpha}$ At M: $\frac{dy}{dx} = \tan\alpha - \frac{g\left(\frac{V^2 2\sin\alpha\cos\alpha}{g}\right)}{V^2\cos^2\alpha}$ $= \tan\alpha - 2\tan\alpha = -\tan\alpha$ For B: $\frac{dy}{dx} = -\frac{g}{U^2}x$ At M: $\frac{dy}{dx} = -\frac{g}{U^2} \times \frac{V^2\sin 2\alpha}{g}$ $= -\frac{g}{4V^2\cos^2\alpha} \times \frac{V^2 2\sin\alpha\cos\alpha}{g}$ $= -\frac{1}{2}\tan\alpha$ Now $\tan\theta = \left\|\frac{m_1 - m_2}{1 + m_1m_2}\right\|$ $= \left\|\frac{-\frac{1}{2}\tan\alpha - (-\tan\alpha)}{1 + (-\tan\alpha)\left(-\frac{1}{2}\tan\alpha\right)}\right\|$ $= \frac{\frac{1}{2}\tan\alpha}{1 + \frac{1}{2}\tan^2\alpha}$ $= \frac{\tan\alpha}{2 + \tan^2\alpha}$ $\therefore \tan\theta = \frac{\tan\alpha}{2 + \tan^2\alpha}$	HE3 • One mark for finding gradient of direction of impact of A • One mark for finding gradient of direction of impact of B • One mark for substitution into formula for $\tan\theta$, and for correct simplifications 3
(g)	$\tan\theta \times (2 + \tan^2\alpha) = \tan\alpha$ $2\tan\theta + \tan\theta\tan^2\alpha = \tan\alpha$ $\tan\theta\tan^2\alpha - \tan\alpha + 2\tan\theta = 0$ $\tan\alpha = \frac{1 \pm \sqrt{1 - 8\tan^2\theta}}{2\tan\theta}$ For $\tan\alpha$ to be real, $\Delta \geq 0$. $\Delta = b^2 - 4ac$ $= 1 - 8\tan^2\theta \geq 0$ $8\tan^2\theta \leq 1$ $\tan^2\theta \leq \frac{1}{8}$ $0 \leq \tan\theta \leq \frac{1}{\sqrt{8}}$ (since θ is an acute angle) $\therefore 0° \leq \theta \leq 19°28'$ (correct to the nearest minute)	HE3 • One mark for forming the equation, and for showing $1 - 8\tan^2\theta \geq 0$ • One mark for deducing the range of values for θ 2

Marks

Question 1 **Begin a new page**

(a) Find the number of ways in which 6 books can be arranged in a row so that the shortest book and the longest book are at the ends. 1

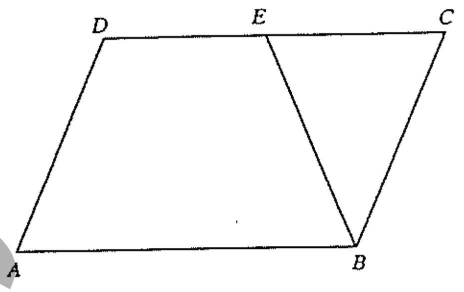
(b) The acute angle between the line $x-2y+3=0$ and the line $y=mx$ is $45°$. 3

(i) Show that $\left|\frac{2m-1}{m+2}\right|=1$

(ii) Find the possible values of m.

(c) Solve the equation $\ln\left(x^3+19\right)=3\ln(x+1)$. 3

(d) 5



$ABCD$ is a parallelogram. E is the point on CD such that $BE=BC$.

(i) Copy the diagram showing the above information.

(ii) Show that $ABED$ is a cyclic quadrilateral.

Marks

Question 2 Begin a new page

(a) Find $\lim_{x \to 0} \frac{\sin 2x}{x}$ **1**

(b) Solve the inequality $\frac{x^2+9}{x} \le 6$ **3**

(c) (i) Factorise $3x^3+3x^2-x-1$ **3**

(ii) Solve the equation $3\tan^3\theta+3\tan^2\theta-\tan\theta-1=0$ for $0 \le \theta \le \pi$

(d) $P(2t, t^2)$ is a point on the parabola $x^2=4y$ with focus F. The point M divides the interval FP externally in the ratio 3 : 1. **5**

(i) Show that as P moves on the parabola $x^2=4y$, then M moves on the parabola $x^2=6y+3$.

(ii) Find the coordinates of the focus and the equation of the directrix of the locus of M.

Question 3 Begin a new page

(a) Find the gradient of the tangent to the curve $y=\tan^{-1}\frac{1}{x}$ at the point on the curve where $x=1$. **2**

(b) A function is given by the rule $f(x)=\frac{x+1}{x+2}$. Find the rule for the inverse function $f^{-1}(x)$. **2**

(c) At any point on the curve $y=f(x)$ the gradient function is given by $\frac{dy}{dx} = 2\cos^2 x+1$. **4**
If $y=\pi$ when $x=\pi$, find the value of y when $x=2\pi$.

(d) Use the substitution $x=u^2$, $u>0$, to express the value of $\int_1^{100} \frac{1}{x+2\sqrt{x}}\,dx$ **4**
in the form $\ln a$ for some constant $a>0$.

Marks

Question 4 Begin a new page

(a) Find the exact value of $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(b) A particle is moving in a straight line. At time t seconds its displacement x metres from a fixed point O on the line is such that $t=x^2-3x+2$. **2**

(i) Find an expression for its velocity v in terms of x.

(ii) Find an expression for its acceleration a in terms of x.

(c) Consider the function $y=2\cos^{-1}(1-x)$. **4**

(i) Find the domain and range of the function.

(ii) Sketch the graph of the function.

domain 2 0

range 0 2π

(d) The radius r kilometres of a circular oil spill at time t hours after it was first observed is given by $r=\frac{1+3t}{1+t}$. Find the exact rate of increase of the area of the oil spill when the radius is 2 kilometres. **4**

Question 5 Begin a new page

(a) Consider the function $f(x)=\frac{\ln x}{x}$. **6**

(i) Find the coordinates and the nature of the stationary point on the curve $y=f(x)$.

(ii) Explain why $f(\pi)<f(e)$ and hence show that $\pi^e<e^\pi$.

(iii) $P(X,-2)$ is a point on the curve $y=f(x)$. Starting with an initial approximation of $X=0\cdot5$, use one application of Newton's method to find an improved approximation to the value of X, giving the answer correct to 2 decimal places.

Marks

Question 5 (Cont)

(b) A machine which initially costs \$49 000 loses value at a rate proportional to the difference between its current value \$$M$ and its final scrap value \$1000. After 2 years the value of the machine is \$25 000. **6**

(i) Explain why $\frac{dM}{dt} = -k(M-1000)$ for some constant $k>0$, and verify that $M = 1000 + Ae^{-kt}$, A constant, is a solution of this equation.

(ii) Find the exact values of A and k.

(iii) Find the value of the machine, and the time that has elapsed, when the machine is losing value at a rate equal to one quarter of the initial rate at which it loses value.

Question 6

Begin a new page

(a) Show that the coefficients of x^4, x^5, x^6 in the expansion of $(1+x)^{14}$ are consecutive terms in an arithmetic sequence. **2**

(b) A coin is biased so that in any one throw there is a constant probability p (where $p \neq 0 \cdot 5$) that the coin shows heads. In 6 throws of the coin the probability of 3 heads is twice the probability of 2 heads. Find the value of p. **4**

(c) A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds its displacement x metres from a fixed point O on the line is given by $x = 2\sin 3t - 2\sqrt{3}\cos 3t$. **6**

(i) Express x in the form $x = R\sin(3t-\alpha)$ for some constants $R>0$ and $0<\alpha<\frac{\pi}{2}$.

(ii) Describe the initial motion of the particle in terms of its initial position, velocity and acceleration.

(iii) Find the exact value of the first time that the particle is 2 metres to the left of O and moving towards O.

Marks

Question 7

Begin a new page

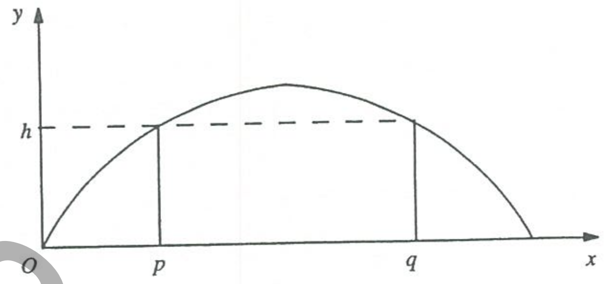
(a) $$S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} \quad , \quad n = 1, 2, 3, \ldots$$ **6**

(i) Use the method of mathematical induction to show that $S_n = 1 - \frac{1}{(n+1)!}$ for all positive integers $n \geq 1$.

(ii) Find the value of $\lim_{n\to\infty} S_n$.

(iii) Find the smallest positive integer n such that $|S_n - 1| < 10^{-6}$.

(b) **6**



A particle is projected with velocity V ms^{-1} from a point O at an angle of elevation α . Axes Ox and Oy are taken horizontally and vertically through O. The particle just clears two vertical chimneys of height h metres at horizontal distances of p metres and q metres from O. The acceleration due to gravity is taken as 10 ms^{-2} and air resistance is ignored.

(i) Write down expressions for the horizontal displacement x and the vertical displacement y of the particle after time t seconds.

(ii) Show that $V^2 = \frac{5p^2(1+\tan^2\alpha)}{p\tan\alpha - h}$.

(iii) Show that $\tan\alpha = \frac{h(p+q)}{pq}$.

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

CSSA MATHEMATICS 3 UNIT SOLUTIONS 1999

QUESTION 1

(a) $2!$ (shortest and longest) $\times 4!$ (others)

$= 2 \times 24 = \underline{48 \text{ ways}}$.

(b) $x - 2y + 3 = 0$, $2y = x + 3$, $y = {}^{x}/_{2} + {}^{3}/_{2}$ gradient $^{1}/_{2}$

$y = mx$ gradient m

(i) $\left|\frac{m - (^1/_2)}{1 + m(^1/_2)}\right| = \tan 45°$ $\therefore \left|\frac{(2m-1)/_2}{(2+m)/_2}\right| = 1$

$\therefore \left|\frac{2m-1}{m+2}\right| = 1$.

(ii) $\frac{2m-1}{m+2} = -1$, $2m - 1 = -m - 2$, $3m = -1$, $\underline{\underline{m = -^1/_3}}$

or $\frac{2m-1}{m+2} = 1$, $2m - 1 = m + 2$, $\underline{\underline{m = 3}}$

(c) $\ln(x^3 + 19) = 3\ln(x+1)$

$\therefore \ln(x^3 + 19) = \ln(x+1)^3$

$\therefore \ln(x^3 + 19) = \ln(x^3 + 3x^2 + 3x + 1)$

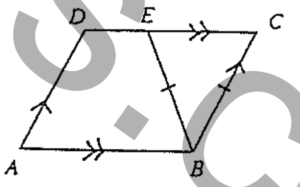
$\therefore x^3 + 19 = x^3 + 3x^2 + 3x + 1$

$\therefore 3x^2 + 3x - 18 = 0$ $\quad\therefore x^2 + x - 6 = 0$

$\therefore (x+3)(x-2) = 0$ $\quad\therefore$ ~~$x = -3$~~ or $\underline{\underline{x = 2}}$

($x \neq -3$ since $\ln((-3)^3 + 19)$, $\ln((-3)+1)$ are not defined)

(d) (i)



(ii) $\angle BEC = \angle BCE$ (in $\triangle BEC$ equal angles are opposite equal sides BC and BE)

$\angle BCE = \angle BAD$ (opposite angles are equal in parallelogram ABCD)

$\therefore \angle BEC = \angle BAD$

$\therefore$ ABED is a cyclic quadrilateral (exterior angle $\angle BEC$ is equal to interior opposite angle $\angle BAD$)

ESTION 2

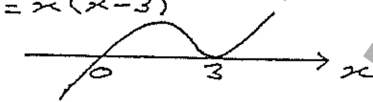
$\lim_{x \to 0} \frac{\sin 2x}{x} = 2 . \lim_{x \to 0} \frac{\sin 2x}{2x} = 2 \times 1 = \underline{\underline{2}}$.

$\frac{x^2+9}{x} \leq 6 \quad \therefore \frac{x^2+9}{x} \times x^2 \leq 6 \times x^2$

$\therefore x^3 + 9x \leq 6x^2 \quad \therefore x^3 - 6x^2 + 9x \leq 0$

$\therefore x(x^2 - 6x + 9) \leq 0 \quad \therefore x(x-3)^2 \leq 0$

$y = x(x-3)^2$



$\therefore \underline{\underline{x < 0 \text{ or } x = 3}}$

$(x \neq 0$ since $\{(0)^2+9\}/(0)$ is not defined$)$

i) $3x^3 + 3x^2 - x - 1 = (x+1)(3x^2-1)$

i) $3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0 \quad (0 \leq \theta \leq \pi)$

$\therefore (\tan\theta + 1)(3\tan^2\theta - 1) = 0$

$\therefore \tan\theta + 1 = 0, \tan\theta = -1, \underline{\underline{\theta = 3\pi/4}}$

r $3\tan^2\theta - 1 = 0, 3\tan^2\theta = 1, \tan^2\theta = 1/3,$

$\tan\theta = -1/\sqrt{3}, \underline{\underline{\theta = 5\pi/6}}$ or $\tan\theta = 1/\sqrt{3}, \underline{\underline{\theta = \pi/6}}$.

$^2 = 4y$. $F(0,1)$, $P(2t, t^2)$

$M(x,y)$ divides FP externally in the ratio $3:1$.

$\therefore x = \frac{3(2t)-1(0)}{3-1} = 3t, \; y = \frac{3(t^2)-1(1)}{3-1} = \frac{3t^2-1}{2}$

$x = 3t \quad \therefore t = x/3$

$y = (3t^2-1)/2 \quad \therefore 2y = 3t^2 - 1$

$\therefore 2y = 3(x/3)^2 - 1 \quad \therefore 2y = \cancel{3}(x^2/\cancel{9}_3) - 1$

$\therefore 6y = x^2 - 3 \quad \therefore x^2 = 6y + 3$.

M moves on the parabola $x^2 = 6y + 3$.

$x^2 = 6y + 3 \quad \therefore x^2 = 6(y + 1/2)$

$\therefore x^2 = 4(3/2)(y + 1/2) \quad \therefore (x-0)^2 = 4(3/2)(y - (-1/2))$

$\therefore$ vertex $V(0, -1/2)$, focal length $a = 3/2$

$\therefore$ focus $(0, -1/2 + 3/2)$ ie. $\underline{\underline{(0,1)}}$

$\therefore$ directrix $y = -1/2 - 3/2$ ie. $\underline{\underline{y = -2}}$

QUESTION 3

(a) $y = \tan^{-1} 1/x \quad \therefore \frac{dy}{dx} = \frac{1}{1+(1/x)^2} \cdot (-1/x^2)$

$\therefore$ when $x = 1$, $dy/dx = 1/(1+1) \cdot (-1) = -1/2$

$\therefore$ the gradient of the tangent is $\underline{\underline{-1/2}}$.

(b) $f(x) = \frac{x+1}{x+2} \quad \therefore y = \frac{x+1}{x+2}$

interchange x and y $\quad \therefore x = \frac{y+1}{y+2}$

$\therefore x(y+2) = y+1 \quad \therefore xy + 2x = y + 1$

$\therefore xy - y = 1 - 2x \quad \therefore y(x-1) = 1 - 2x$

$\therefore y = \frac{1-2x}{x-1} \quad \therefore \underline{\underline{f^{-1}(x) = \frac{1-2x}{x-1}}}$.

c) $\frac{dy}{dx} = 2\cos^2 x + 1 \quad \therefore y = \int (2\cos^2 x + 1)\,dx$

$\therefore y = \int (\cos 2x + 1 + 1)\,dx \quad \therefore y = \int (2 + \cos 2x)\,dx$

$\therefore y = 2x + 1/2 \sin 2x + c$

When $x = \pi, y = \pi \quad \therefore \pi = 2\pi + 1/2 \sin 2\pi + c$

$\therefore \pi = 2\pi + 0 + c \quad \therefore c = -\pi$

$\therefore y = 2x + 1/2 \sin 2x - \pi$

$\therefore$ when $x = 2\pi$, $y = 4\pi + 1/2 \sin 4\pi - \pi$

$\therefore y = 4\pi + 0 - \pi \quad \therefore \underline{\underline{y = 3\pi}}$.

) $\int_1^{100} \frac{1}{x + 2\sqrt{x}}\,dx$ $\qquad x = u^2 \, (u > 0)$

$\therefore dx/du = 2u, \; dx = 2u\,du$

when $x = 1, u = 1$

when $x = 100, u = 10$

$= \int_1^{10} \frac{1}{u^2 + 2u} \cdot 2u\,du$

$= \int_1^{10} \frac{1}{\cancel{u}(u+2)} \cdot 2\cancel{u}\,du = 2 \int_1^{10} \frac{1}{u+2}\,du$

$= 2\,[\ln(u+2)]_1^{10} = 2(\ln 12 - \ln 3)$

$= 2 \ln 12/3 = 2 \ln 4$

$= \ln 4^2 = \underline{\underline{\ln 16}}$

ESTION 4

) $\int_{\sqrt2}^{\sqrt3}\frac{1}{\sqrt{4-x^2}}dx = \int_{\sqrt2}^{\sqrt3}\frac{1}{\sqrt{2^2-x^2}}dx$

$= \left[\sin^{-1} x/2\right]_{\sqrt2}^{\sqrt3} = \sin^{-1}\sqrt3/2 - \sin^{-1}\sqrt2/2$

$= \pi/3 - \pi/4 = \underline{\underline{\pi/12}}$

(i) $t = x^2-3x+2 \quad \therefore dt/dx = 2x-3$

$v = dx/dt = 1/dt/dx \quad \therefore \underline{\underline{v = 1/(2x-3)}}$

(ii) $a = d/dx\,(1/2\,v^2) = d/dx\,(1/2 \cdot 1/(2x-3)^2)$

$\therefore a = 1/2 \cdot -2/(2x-3)^3 \cdot 2 \quad \therefore \underline{\underline{a = -2/(2x-3)^3}}$

$y = 2\cos^{-1}(1-x)$

i) domain : $-1 \le 1-x \le 1$

$\therefore -2 \le -x \le 0 \quad \therefore \underline{\underline{0 \le x \le 2}}$

range : $0 \le \cos^{-1}(1-x) \le \pi$

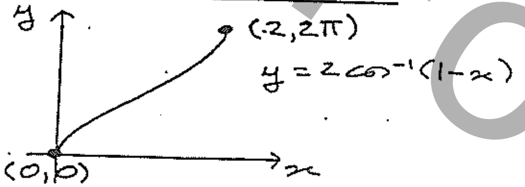
$\therefore 0 \le 2\cos^{-1}(1-x) \le 2\pi \quad \therefore \underline{\underline{0 \le y \le 2\pi}}$

) when $x = 0$

$y = 2\cos^{-1}(1) = 0$

when $x = 2$

$y = 2\cos^{-1}(-1) = 2\pi$



$r = \frac{1+3t}{1+t} \quad \therefore \frac{dr}{dt} = \frac{(1+t)(3)-(1+3t)(1)}{(1+t)^2} = \frac{2}{(1+t)^2}$

When $r = 2$, $\frac{1+3t}{1+t} = 2$

$\therefore 1+3t = 2+2t \quad \therefore t = 1.$

$A = \pi r^2 \quad \therefore \frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = 2\pi r \cdot \frac{2}{(1+t)^2}$

When $r = 2$, $t = 1$

$\frac{dA}{dt} = 2\pi(2)\cdot\frac{2}{(1+(1))^2} = 2\pi$

. the area of the oil spill is increasing at a rate of $\underline{\underline{2\pi \text{ kilometres}^2 / \text{hour}}}$.

QUESTION 5

) $f(x) = \ln x / x \quad (x > 0)$

(i) $f'(x) = \{(x)(1/x) - (\ln x)(1)\}/x^2 = (1-\ln x)/x^2$

When $f'(x) = 0$, $(1-\ln x)/x^2 = 0$, $1-\ln x = 0$, $\ln x = 1$, $x = e$

When $x = e$, $y = \ln e/e = 1/e$

When $0 < x < e$, $\ln x < 1$ and $f'(x) = (1-\ln x)/x^2 > 0$

When $x > e$, $\ln x > 1$ and $f'(x) = (1-\ln x)/x^2 < 0$

$\therefore$ $\underline{(e, 1/e) \text{ is a maximum turning point}}$.

) $f(e)$ is a local maximum and $f'(x) < 0$ for $x > e$ so that $f(x)$ is a decreasing function for $x > e$. $\therefore f(\pi) < f(e)$

$\therefore \ln\pi/\pi < \ln e/e \quad \therefore e\ln\pi < \pi\ln e$

$\therefore \ln\pi^e < \ln e^\pi \quad \therefore \underline{\underline{\pi^e < e^\pi}}$.

iii) $\ln x/x = -2 \quad \therefore \ln x = -2x \quad \therefore \ln x + 2x = 0$

$P(x) = \ln x + 2x$, $P'(x) = 1/x + 2$

$\therefore$ with initial approximation of $X = 0.5$, improved approximation

$= 0.5 - \frac{P(0.5)}{P'(0.5)} = 0.5 - \frac{\ln 0.5 + 2(0.5)}{1/(0.5) + 2} = \underline{\underline{0.42}}$ (to 2d.p.)

(i) $dM/dt < 0$, $dM/dt \propto (M-1000)$

$\therefore dM/dt = k(M-1000)\ (k<0)$ or $dM/dt = -k(M-1000)\ (k>0)$

If $M = 1000 + Ae^{-kt}$

$dM/dt = 0 + A(-ke^{-kt}) = -k(Ae^{-kt}) = -k(M-1000)$

$\therefore M = 1000 + Ae^{-kt}$ is a solution of $dM/dt = -k(M-1000)$.

(ii) When $t = 0$, $M = 49000$. $\therefore 49000 = 1000 + Ae^{-k(0)}$

$\therefore 49000 = 1000 + A \quad \therefore \underline{\underline{A = 48000}}$

When $t = 2$, $M = 25000 \quad \therefore 25000 = 1000 + 48000e^{-k(2)}$

$\therefore 24000 = 48000e^{-2k} \quad \therefore e^{2k} = 2 \quad \therefore \underline{\underline{k = 1/2\ln 2}}$

iii) $dM/dt = -k(M-1000)$; when $t = 0$, $M = 49000$

$\therefore$ initial rate of losing value $= -k(49000-1000) = -48000k$

When $dM/dt = 1/4(-48000k) = -12000k$,

$-k(M-1000) = -12000k$, $M - 1000 = 12000$, $\underline{\underline{M = \$13000}}$

When $M = 13000$, $13000 = 1000 + 48000e^{-kt}$

$\therefore 12000 = 48000e^{-kt} \quad \therefore e^{kt} = 4 \quad \therefore kt = \ln 4$

$\therefore t = \frac{\ln 4}{k} = \frac{\ln 4}{1/2\ln 2} = \underline{\underline{4 \text{ years}}}$

i) In the expansion of $(1+x)^{14}$ the coefficients of x^4, x^5, x^6 are ${}^{14}C_4 = 1001, {}^{14}C_5 = 2002, {}^{14}C_6 = 3003$ which are consecutive terms in an arithmetic sequence with common difference 1001.

) In any one throw $P(\text{heads}) = p$.

$P(3 \text{ heads in } 6 \text{ throws}) = 2P(2 \text{ heads in } 6 \text{ throws})$

$\therefore {}^6C_3\, p^3(1-p)^3 = 2\, {}^6C_2\, p^2(1-p)^4$

$\therefore 20p^3(1-p)^3 = 2.15p^2(1-p)^4$

$\therefore 20p^3(1-p)^3 = 30p^2(1-p)^4 \quad \therefore 2p = 3(1-p)$

$\therefore 2p = 3 - 3p \quad \therefore 5p = 3 \quad \therefore \underline{\underline{p = 3/5}}$

$x = 2\sin 3t - 2\sqrt{3}\cos 3t$

i) $2\sin 3t - 2\sqrt{3}\cos 3t \equiv R\sin(3t-\alpha)$

$\equiv R(\sin 3t\cos\alpha - \cos 3t\sin\alpha)$

$\equiv (R\cos\alpha)\sin 3t - (R\sin\alpha)\cos 3t$

$\therefore R\cos\alpha = 2 \quad (1) \qquad R\sin\alpha = 2\sqrt{3} \quad (2)$

$(1)^2 \quad R^2\cos^2\alpha = 4 \qquad R^2(\cos^2\alpha + \sin^2\alpha) = 16$

$(2)^2 \quad R^2\sin^2\alpha = 12 \quad + \qquad R^2 = 16, \quad R = 4$

$\frac{(2)}{(1)} \quad \frac{R\sin\alpha}{R\cos\alpha} = \frac{2\sqrt{3}}{2} \qquad \tan\alpha = \sqrt{3}, \quad \alpha = \pi/3$

$\therefore \underline{\underline{x = 4\sin(3t - \pi/3)}}$

) $x = 4\sin(3t - \pi/3)$; when $t = 0$, $x = -2\sqrt{3}$

$v = \dot{x} = 12\cos(3t - \pi/3)$; when $t = 0$, $v = 6$

$a = \ddot{x} = -36\sin(3t - \pi/3)$; when $t = 0$, $a = 18\sqrt{3}$

∴ initially the particle is $2\sqrt{3}$ metres to the left of O, moving to the right at a speed of $6\,ms^{-1}$ and speeding up at a rate of $18\sqrt{3}\ ms^{-2}$.

) When $x = -2$, $4\sin(3t - \pi/3) = -2$

$\therefore \sin(3t - \pi/3) = -1/2 \qquad \therefore 3t - \pi/3 = -\pi/6, \ldots$

$\therefore 3t = -\pi/6 + \pi/3, \ldots \qquad \therefore 3t = \pi/6, \ldots$

$\therefore t = \pi/18, \ldots \qquad$ When $t = \pi/18$, $v = 6\sqrt{3} > 0$

∴ the first time is after $\underline{\underline{\pi/18 \text{ seconds}}}$

QUESTION 7

(a) $S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} \quad (n = 1, 2, 3, \ldots)$

(i) $S(n): \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

When $n = 1$, $LHS = 1/2! = 1/2$

$RHS = 1 - 1/(1+1)! = 1 - 1/2! = 1 - 1/2 = 1/2$

∴ statement is true when $n = 1$.

If the statement is true when $n = k$ for some $k \geq 1$ then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{k}{(k+1)!} + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{\{(k+2) - (k+1)\}}{(k+2)!} = 1 - \frac{1}{(k+2)!}$

and the statement is also true when $n = k+1$.

Since $S(1)$ is true and if $S(k)$ is true for some $k \geq 1$ then $S(k+1)$ is also true it follows that $S(1+1) = S(2)$ is true, $S(2+1) = S(3)$ is true, ... and so $S(n)$ is true for all positive integers $n \geq 1$.

ii) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{(n+1)!} = 1 - 0 = \underline{\underline{1}}$

iii) $|S_n - 1| < 10^{-6}$

$\therefore \left| 1 - \frac{1}{(n+1)!} - 1 \right| < \frac{1}{10^6} \quad \therefore \left| \frac{-1}{(n+1)!} \right| < \frac{1}{10^6}$

$\therefore \frac{1}{(n+1)!} < \frac{1}{10^6} \quad \therefore (n+1)! > 10^6$

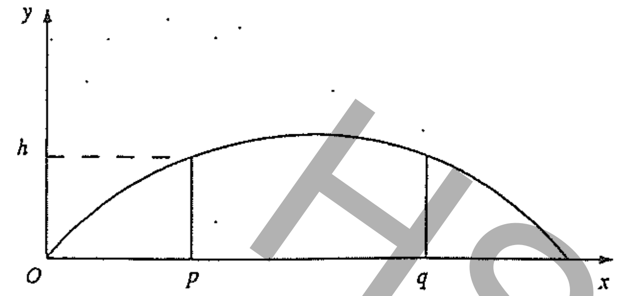
Now $9! = 362880$, $10! = 3628800$

$\therefore n+1 \geq 10 \quad \therefore n \geq 9$

∴ the smallest positive integer is $\underline{\underline{n = 9}}$.

JESTION 7

·)



i) $x = (V\cos\alpha)t$ (1), $y = (V\sin\alpha)t - \frac{1}{2}gt^2$

$y = (V\sin\alpha)t - 5t^2$ (2)

ii) When $x = p$, $y = h$

$\therefore$ in (1) $p = (V\cos\alpha)t$ $\therefore t = p/V\cos\alpha$

$\therefore$ in (2) $h = (V\sin\alpha)t - 5t^2$

$\therefore h = (V\sin\alpha)(p/V\cos\alpha) - 5(p/V\cos\alpha)^2$

$\therefore h = p\tan\alpha - 5p^2/V^2\cos^2\alpha$

$\therefore h = p\tan\alpha - 5p^2/V^2 \sec^2\alpha$

$\therefore h = p\tan\alpha - 5p^2/V^2 (1+\tan^2\alpha)$

$\therefore 5p^2/V^2 (1+\tan^2\alpha) = p\tan\alpha - h$

$$\therefore V^2 = \frac{5p^2(1+\tan^2\alpha)}{p\tan\alpha - h}$$

ii) When $x = q$, $y = h$

$\therefore$ similarly $V^2 = \frac{5q^2(1+\tan^2\alpha)}{q\tan\alpha - h}$

$$\therefore \frac{5p^2\cancel{(1+\tan^2\alpha)}}{p\tan\alpha - h} = \frac{5q^2\cancel{(1+\tan^2\alpha)}}{q\tan\alpha - h}$$

$\therefore p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$

$\therefore p^2q\tan\alpha - p^2h = pq^2\tan\alpha - q^2h$

$\therefore p^2h - q^2h = p^2q\tan\alpha - pq^2\tan\alpha$

$\therefore (p^2 - q^2)h = (p^2q - pq^2)\tan\alpha$

$\therefore (p+q)\cancel{(p-q)}h = pq\cancel{(p-q)}\tan\alpha$

$$\therefore \tan\alpha = \frac{h(p+q)}{pq}$$

NATIONAL NEAP

National Educational Advancement Programs

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

1999

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time Allowed: Three hours
(Plus 5 minutes' reading time)

This paper must be kept under strict security and may only be used on or after the afternoon of Friday 13 August, 1999, as specified in the NEAP Examination Timetable.

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 9.
- Approved calculators may be used.
- Each question is to be returned in a separate Writing Booklet clearly labelled, showing your Student Name or Number.
- You may ask for extra Writing Booklets if you need them.

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P.O. Box 214 St Leonards NSW 2065 (100 Christie St.) Tel: (02) 9438 1386 Fax: (02) 9438 1385

QUESTION 1. Use a *separate* Writing Booklet. **Marks**

(a) Solve $\frac{x+1}{x} \geq 2$. 3

(b) Find the acute angle between the lines $x + 3y = 4$ and $2x - 5y = 0$. 3

Give the answer to the nearest degree.

(c) Use the "t" results to solve the equation $2\cos\theta - \sin\theta = -1$ for $0 \leq \theta \leq 2\pi$. 4

Give any answers correct to three significant figures.

(d) Point A is $(2, -7)$ and B is $(-6, 9)$. Find the coordinates of point P which divides AB internally in the ratio 3 : 5. 2

QUESTION 2. Use a *separate* Writing Booklet. **Marks**

(a) Use the substitution $u = \ln x$ to find $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$. 2

(b) Consider the geometric series $1 - \tan^2 x + \tan^4 x - \ldots$, where $0 < x < \frac{\pi}{2}$. 3

(i) For what values of x does this series have a limiting sum?

(ii) Find the limiting sum in terms of $\cos x$.

(c) It is given that $x^2 + x - 2$ is a factor of $x^3 + rx^2 - 4x + s$, where r and s are constants. 3

(i) Show that $r + s = 3$.

(ii) Evaluate r and s.

(d) On a certain railway line, there are eleven stations at which a train can stop. The rail authority wishes to print tickets for travel between every possible pair of stations on the line. 1

How many different one-way tickets must be printed if the ticket specifies which direction the passenger is travelling?

(e) (i) Five men and five women are arranged in a straight line. How many arrangements are there in which men and women alternate? 3

(ii) Five men and five women are arranged in a circle. How many arrangements are there in which men and women alternate?

(iii) Find the probability that men and women alternate if five men and five women are arranged at random in a circle.

QUESTION 3. Use a *separate* Writing Booklet.

Marks

(a) (i) Sketch $y = 3\sin x$ and $y = x$, for $0 \le x \le 2\pi$, on the same set of axes. **4**

(ii) By substitution, show that a solution for $3\sin x - x = 0$ lies between $x = 2.2$ and $x = 2.4$.

(iii) Taking $x = 2.3$ as an approximation to a solution of $3\sin x - x = 0$, apply Newton's method once to find a better approximation correct to three decimal places.

(b) (i) Find $\frac{d}{dx}(x\tan^{-1}x)$. **4**

(ii) Hence find the exact value of $\int_0^1 \tan^{-1}x \, dx$.

(c) Use mathematical induction to show **4**

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1} \quad \text{for } n \ge 1.$$

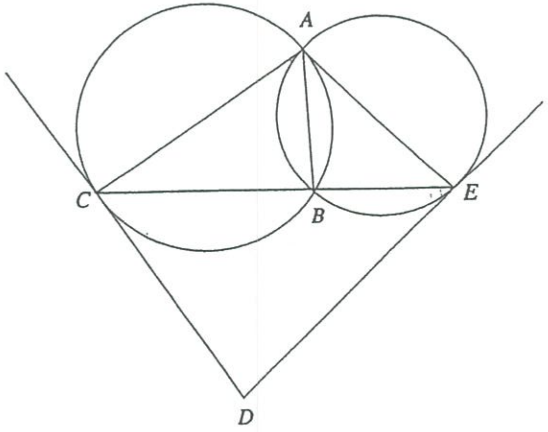
QUESTION 4. Use a *separate* Writing Booklet.

Marks

(a) Assume that Antarctica can be approximately represented as a circle, and that its area is decreasing at the rate of 3000 square kilometres per year. **2**

Find the rate at which the radius of the circle is decreasing when the length of the radius is 2100 kilometres.

(b) **4**



In the diagram, two circles intersect at A and B. Points C and E lie on the circles and C, B and E are collinear. Tangents at C and E meet at D.

Copy the diagram into your booklet and show that quadrilateral $AEDC$ is cyclic.

(c) A particle moves in a straight line and at time t seconds, its velocity, v metres per second, is related to its displacement, x metres, by **6**

$$v^2 = 108 + 36x - 9x^2.$$

(i) By deriving an expression for acceleration, show that the motion is simple harmonic.

(ii) Find the period of the motion.

(iii) Find the amplitude of the motion.

(iv) For what value of x does the particle have maximum speed?

QUESTION 5. Use a *separate* Writing Booklet.

Marks

(a) Consider the function $f(x) = \frac{1}{1+x^2}$ for $x \leq 0$. 4

(i) Sketch $y = f(x)$. (It is not necessary to show working.)

(ii) Find the inverse function $f^{-1}(x)$.

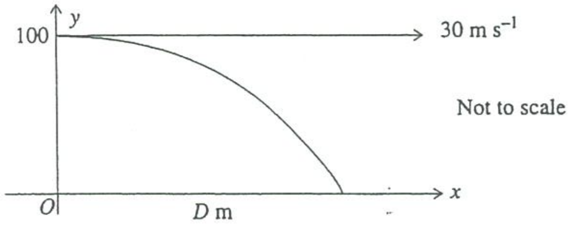
(iii) State the domain of $f^{-1}(x)$.

(b) Fish are taken at random from a large number of fish in a dam. For this particluar kind of fish, one quarter of the population is male. 4

(i) Find the probability that if ten fish are taken, then exactly four are male.

(ii) How many fish have to be taken from the dam for the probability of there being at least one male to be greater than 99%?

(c) A plane designed to drop water "bombs" flies in a horizontal line towards a spot fire. It maintains a steady speed of 30 metres per second, and an altitude of 100 metres. The plane releases its water when at a horizontal distance D metres from the fire. 4



Let the origin O be as shown in the diagram, and let x and y metres be the horizontal and vertical displacements of the centre of the water bomb at time t seconds after its release. Then $\dot{x}$ and $\dot{y}$ are the horizontal and vertical components of its velocity in metres per second. Let $g = 10$ metres per second per second, and assume that air resistance has negligible effect.

(i) Use calculus to derive expressions for $\dot{x}$, $\dot{y}$, x and y in terms of t.

(ii) Hence determine the value of D correct to the nearest metre.

QUESTION 6. Use a *separate* Writing Booklet.

Marks

(a) Solve $|2x-1| < 3x+2$. 3

(b) (i) On the same set of axes, sketch $y = \sin^{-1}x$ and $y = \cos^{-1}x$, showing all essential information. 4

(ii) Let $f(x) = \sin^{-1}x + \cos^{-1}x$. By referring to the graphs in part (i), or otherwise, explain why $f(x)$ is a constant function, and find its value.

(iii) Hence evaluate $\int_{-1}^{1} (\sin^{-1}x + \cos^{-1}x)\,dx$.

(c) At time t seconds after the parachute has opened, a parachutist falls with speed v metres per second and with acceleration given by 5

$$\frac{dv}{dt} = k(4-v)$$

where k is constant.

(i) Show that $v = 4 + Ae^{-kt}$, where A is constant, satisfies the above equation.

(ii) At the instant the parachute opens, the parachutist is falling at 25 metres per second. Find the value of A.

(iii) Two seconds after the parachute has opened, the parachutist's speed is 12 metres per second. Evaluate k correct to three significant figures.

(iv) As t increases indefinitely, v approaches a fixed value. Show that after 20 seconds v differs from this fixed value by less than 0.1%.

QUESTION 7.

Use a *separate* Writing Booklet. **Marks**

(a) An expression of the form $(1-ax)^n$, where a is a constant and n a positive integer, is expanded. The first three terms of the expansion are **4**

$$1-4x+\frac{20}{3}x^2.$$

Find the values of a and n.

(b) (i) Show that $\cos 3x = 4\cos^3 x - 3\cos x$ by using the expansion of $\cos(A+B)$. **8**

(ii) Show that the solution of $\cos 3x - \sin 2x = 0$ for $0 < x < \frac{\pi}{2}$ is given by

$$\sin x = \frac{\sqrt{5}-1}{4}.$$

(iii) Use a trigonometric identity to explain why $x = \frac{\pi}{10}$ is a solution to

$\cos 3x = \sin 2x$.

(iv) Hence, using the results referred to in parts (ii) and (iii), prove

$$\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \frac{\sqrt{5}}{4}.$$

The following list of standard integrals may be used:

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq 1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln(x+\sqrt{x^2-a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x+\sqrt{x^2+a^2}).$$

Note: $\ln x = \log_e x, \quad x > 0$

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National Educational Advancement Programs

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

1999

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

SOLUTIONS AND SUGGESTED MARKING SCHEME

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QUESTION 1.

(a) $\frac{x+1}{x} \geq 2 \quad x \neq 0$

$\therefore \frac{(x^2)(x+1)}{x} \geq 2(x^2)$ ✓

$\therefore x(x+1) \geq 2x^2$

$2x^2 - x^2 - x \leq 0$

$x^2 - x \leq 0$

$x(x-1) \leq 0$ ✓

but $x \neq 0$

$\therefore 0 < x \leq 1$ ✓

(b) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$x + 3y = 4$

$2x - 5y = 0$

$m_1 = -\frac{1}{3}$

$m_2 = \frac{2}{5}$

$\tan\theta = \left|\frac{\left(-\frac{1}{3}\right) - \left(\frac{2}{5}\right)}{1 + \left(-\frac{1}{3}\right)\left(\frac{2}{5}\right)}\right|$ ✓

$= \left|\frac{\frac{-5-6}{15}}{1 - \frac{2}{15}}\right|$

$= \left|\frac{-11}{13}\right|$

$= \frac{11}{13}$ ✓

$\therefore \theta = \tan^{-1}\left(\frac{11}{13}\right)$

$= 40.236\ldots°$ (calculator)

∴ the angle between the lines is 40° (to the nearest degree). ✓

(c) $2\cos\theta - \sin\theta = -1$

$2\left(\frac{1-t^2}{1+t^2}\right) - \left(\frac{2t}{1+t^2}\right) = -1$ ✓

$2 - 2t^2 - 2t = -1 - t^2$

$t^2 + 2t - 3 = 0$

$(t+3)(t-1) = 0$

$\therefore t = -3$ or $t = 1$ ✓

$\tan\frac{\theta}{2} = -3$ or $\tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = \pi - 1.2490\ldots \quad \frac{\theta}{2} = \frac{\pi}{4}$ ✓

$\therefore \theta = 3.79$ (3SF) or $\theta = 1.57$ (3SF) ✓

(d) $x = \frac{x_1 r_2 + x_2 r_1}{r_1 + r_2}$

$x_1 = 2 \quad x_2 = -6$

$y_1 = -7 \quad y_2 = 9$

$r_1 : r_2 = 3:5$

$= \frac{(2)(5) + (-6)(3)}{3+5}$

$= \frac{10-18}{8}$

$= -1$ ✓

$y = \frac{y_1 r_2 + y_2 r_1}{r_1 + r_2}$

$= \frac{(-7)(5) + (9)(3)}{3+5}$

$= \frac{-35+27}{8}$

$= -1$ ✓

∴ point P is $(-1, -1)$.

QUESTION 2.

(a) $\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-u^2}}$ ✓

let $u = \ln x$

$du = \frac{1}{x}dx$

$= \sin^{-1}u + c$

$= \sin^{-1}(\ln x) + c$ ✓

(b) $r = \frac{T_2}{T_1} = -\tan^2 x$

(i) $-1 < r < 1$

$\therefore -1 < -\tan^2 x < 1$

$\therefore -1 < \tan^2 x < 1$

But $\tan^2 x \geq 0$, and $x > 0$, given.

$\therefore 0 < \tan^2 x < 1$

$0 < \tan x < 1 \quad \left(\text{since } 0 < x < \frac{\pi}{2}\right)$

$0 < x < \frac{\pi}{4}$ ✓

(ii) $S_\infty = \frac{a}{1-r}$

$= \frac{1}{1-(-\tan^2 x)}$

$= \frac{1}{1+\tan^2 x}$ ✓

$= \frac{1}{\sec^2 x}$

$= \cos^2 x$ ✓

(c) Let $P(x) = x^3 + rx^2 - 4x + s$

and note $x^2 + x - 2 = (x+2)(x-1)$

If $x^2 + x - 2$ is a factor of $P(x)$

then $P(1) = 0$ and $P(-2) = 0$

(i) $P(1) = (1)^3 + r(1)^2 - 4(1) + s = 0$

$\therefore (1 + r - 4 + s) = 0$

$r + s = 3$ ① ✓

(ii) $P(-2) = (-2)^3 + (-2)^2 r - 4(-2) + s = 0$

$-8 + 4r + 8 + s = 0$

$\therefore 4r + s = 0$ ② ✓

from (i) $r + s = 3$ ①

② − ① $3r = -3$

$r = -1$

$s = 4$ ✓

(d) No. of tickets with direction $= 2 \times {}^{11}C_2$ (OR ${}^{11}P_2$)

$= 110$ ✓

(e) (i) No. of arrangements in a straight line $= 2! \times 5! \times 5!$ ✓

$= 28800$

(ii) No. of arrangements in a circle $= 4! \times 5!$ ✓

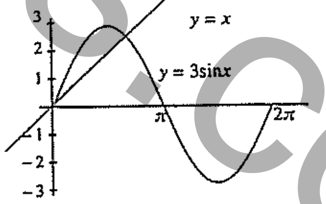
$= 2880$

(iii) P(m and w alternate) $= \frac{4! \times 5!}{9!}$

$= \frac{1}{126}$ ✓

QUESTION 3.

(a) (i)



✓

(ii) At $x = 2.2$ $3\sin(2.2) - (2.2) = 0.2254\ldots$

At $x = 2.4$ $3\sin(2.4) - (2.4) = -0.3736\ldots$

Since the sign of $3\sin x - x$ changes between $x = 2.2$ and $x = 2.4$ then a solution lies between 2.2 and 2.4. ✓

(iii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(x) = 3\sin x - x$

$f'(x) = 3\cos x - 1$

$x_2 = (2.3) - \frac{3\sin(2.3) - 2.3}{3\cos(2.3) - 1}$ ✓

$= 2.27903\ldots$(calculator)

$= 2.279$ (correct to 3 dec. places) ✓

(b) (i) $\frac{d}{dx}(x\tan^{-1}x) = (\tan^{-1}x)(1) + (x)\left(\frac{1}{1+x^2}\right)$

$= \tan^{-1}x + \frac{x}{1+x^2}$ ✓

(ii) $\int_0^1 \tan^{-1}x\,dx = \int_0^1 \left(\frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}\right)dx$ ✓

$= \left[x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2)\right]_0^1$ ✓

$= \left[(1)\tan^{-1}(1) - \frac{1}{2}\log_e(1+(1)^2)\right]$

$-\left[0 - \frac{1}{2}\log_e 1\right]$

$= \frac{\pi}{4} - \frac{1}{2}\log_e 2$ ✓

(c) To prove $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$, $n \geq 1$.

When $n = 1$

$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{1} \frac{1}{r(r+1)}$

$= \frac{1}{1(2)}$

$= \frac{1}{2}$

and $\frac{n}{n+1} = \frac{1}{1+1}$

$= \frac{1}{2}$

$\therefore \sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 1$. ✓

Assume $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k$.

i.e. assume $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$.

When $n = k + 1$

$$\sum_{r=1}^{n}\frac{1}{r(r+1)} = \sum_{r=1}^{k+1}\left(\frac{1}{r(r+1)}\right)$$

$$= \sum_{r=1}^{k}\frac{1}{r(r+1)} + \frac{1}{(k+1)(k+1+1)} \quad ✓$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{(from assumption)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{(k+1)}{(k+1)+1} \quad ✓$$

$\therefore$ if $\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k$

then $\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k+1$.

Conclusion:

Since $\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 1$

then $\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 2, 3, \ldots\ldots$

$\therefore \sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$. ✓

QUESTION 4.

(a) $A = \pi r^2$

$$\frac{dA}{dt} = \frac{dA}{dr}\times\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

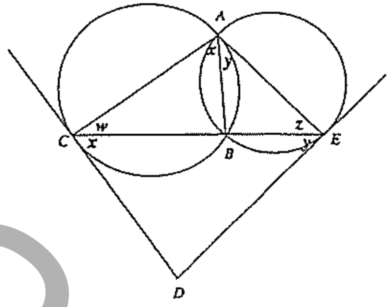
$$\frac{dr}{dt} = \frac{dA}{dt} \div 2\pi r$$

$$= -3000 \times \frac{1}{2\pi \times 2100} \quad ✓$$

$= -0.22736\ldots$(calculator)

$\therefore$ radius is decreasing at 227 metres per year (to the nearest metre) ✓

(b)



Let $x = \angle DCB$.

$\angle DCB = \angle CAB$ (Alternate segment theorem)

$\therefore \angle CAB = x$. ✓

Similarly, if $y = \angle DEB$,

then $\angle EAB = y$.

Let $\angle ACB = w$ and let $\angle AEB = z$.

In ΔAEC, $w + x + y + z = 180°$ (Angle sum of triangle) ✓

In quadrilateral $AEDC$,

$ACD + DEA = w + x + y + z$

$= 180$ as shown above ✓

$\therefore$ opposite angles are supplementary

$\therefore$ quadrilateral $AEDC$ is cyclic. ✓

OR

Proof:

$\angle BCD = \angle CAB$ (angle between tangent & chord = angle in alternate segement) ✓

$\angle BED = \angle EAB$ (same reason)

$\angle BCD + \angle BED + \angle D = 180°$ (angle sum of Δ) ✓

$\therefore \angle CAB + \angle EAB + \angle D = 180°$

$\therefore \angle CAE + \angle D = 180°$ ✓

$\therefore$ sum of opposite angles of $ACDE = 180°$ ✓

$\therefore$ $ACDE$ is cyclic.

(c) (i) $v^2 = 108 + 36x - 9x^2$

$$\frac{1}{2}v^2 = \frac{1}{2}(108 + 36x - 9x^2)$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 18 - 9x \quad ✓$$

$\therefore \ddot{x} = -9x + 18$

$\ddot{x} = -9(x-2)$

$\therefore$ motion is simple harmonic of the form

$\ddot{x} = -n^2(x-2)$ ✓

(ii) Period of motion $= \frac{2\pi}{n} = \frac{2\pi}{3}$ seconds ✓

(iii) To find the amplitude let $v = 0$ i.e. $v^2 = 0$.

$\therefore 9x^2 - 36x - 108 = 0$

$9(x^2 - 4x - 12) = 0$ ✓

$9(x-6)(x+2) = 0$

$\therefore x = 6$ or $x = -2$

$\therefore 2a = 6 - (-2)$

$= 8$

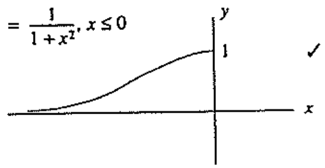
$a = 4$ ✓

$\therefore$ amplitude is 4 metres.

(iv) Maximum speed occurs at centre of motion, at $x = 2$. ✓

QUESTION 5.

(a) (i) $y = \frac{1}{1+x^2}, x \le 0$ ✓



(ii) Inverse $x = \frac{1}{1+y^2}$ $y \le 0$

$1 + y^2 = \frac{1}{x}$ ✓

$y = \pm\sqrt{\frac{1}{x} - 1}$, $y \le 0$

$\therefore f^{-1}(x) = -\sqrt{\frac{1}{x} - 1}$ ✓

(iii) Domain of $f^{-1}(x)$: $0 < x \le 1$. ✓

(b) Let p be the probability of selecting a male and q be the probability of selecting a female.

(i) Binomial expansion for ten fish:

$(p+q)^{10} = {}^{10}C_0p^{10} + {}^{10}C_1p^9q^1 + {}^{10}C_2p^8q^2\ldots$

$$P(4 \text{ males}) = {}^{10}C_6p^4q^6$$

$$= \frac{10\times9\times8\times7}{4\times3\times2\times1}\times\left(\frac{1}{4}\right)^4\times\left(\frac{3}{4}\right)^6$$

$$= \frac{10\times3\times7\times3^6}{4^{10}}$$

$$= \frac{76545}{524288}$$

$= 0.146$ (3 dp) ✓

(ii) $1 - P(\text{no males}) > \frac{99}{100}$ ✓

$$1 - \left(\frac{3}{4}\right)^n > \frac{99}{100}$$

$$\left(\frac{3}{4}\right)^n < 1 - \frac{99}{100}$$

$(0.75)^n < 0.01$

$n\log 0.75 < \log 0.01$

$n > \frac{\log 0.01}{\log 0.75}$ ✓

(N.B. $\log 0.75$ is negative)

$> 16.0078\ldots$ (calculator)

$n = 17$

$\therefore$ 17 fish have to be taken. ✓

(c) (i) $\ddot{x} = 0$

$\boxed{\dot{x} = 30}$

$x = \int 30\,dt$

$x = 30t + c$ (where c is a constant)

when $t = 0, x = 0 \quad \therefore c = 0$

$\boxed{x = 30t}$ ✓

$\ddot{y} = -10$

$\dot{y} = \int -10\,dt$

$= -10t + c$

when $t = 0, \dot{y} = 0 \quad \therefore c = 0$

$\boxed{\dot{y} = -10t}$

$y = \int -10t\,dt$

$= -5t^2 + c$

when $t = 0, y = 100 \quad \therefore c = 100$

$\boxed{y = -5t^2 + 100}$ ✓

(ii) Water bomb hits the ground when $y = 0$.

$\therefore 5t^2 = 100$

$t^2 = 20$

$t = 2\sqrt{5}$ sec. ✓

$\therefore x = 30(2\sqrt{5})$

$= 134.16\ldots$ (calculator) ✓

$\therefore D = 134$ metres (to the nearest metre)

QUESTION 6.

(a) $|2x-1| < 3x+2$

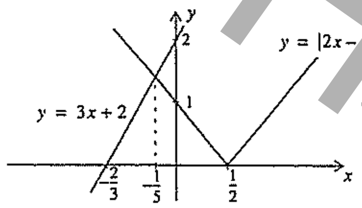
Solve $2x-1 = 3x+2$ and $2x-1 = -3x-2$

$x = -3$ $\qquad x = -\frac{1}{5}$

But if $x = -3$, $3x+2<0$ ✓

$\therefore x = -3$ is not a solution.

Now consider the graphs



✓

From graph $|2x-1| < 3x+2$ for $x > -\frac{1}{5}$. ✓

OR

$|2x-1| < 3x+2$

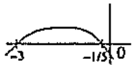
$(2x-1)^2 < (3x+2)^2$ (since $|2x-1| \geq 0$) ✓

$(2x-1)^2 - (3x+2)^2 < 0$

$(2x-1-(3x+2))(2x-1+(3x+2)) < 0$

$(-x-3)(5x+1) < 0$

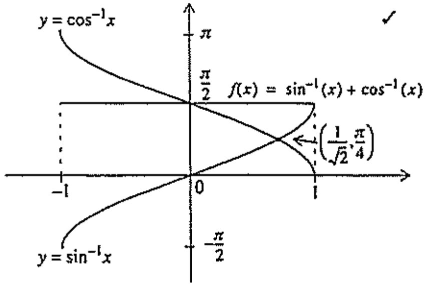
$\therefore x < -3$ or $x > -\frac{1}{5}$ ✓



But if $x < -3$, $3x+2<0$. ✓

$\therefore x > -\frac{1}{5}$

(b) (i)



✓

(ii) By adding ordinates at some key points on the graphs, and by noting the symmetry of the graphs, it can be seen that

$f(x) = \sin^{-1}x + \cos^{-1}x$ ✓

$= \text{constant}$

$= \frac{\pi}{2}$ ✓

OR

$f(x) = \sin^{-1}x + \cos^{-1}x$

$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$

$= 0$

But $f(0) = \frac{\pi}{2}$, and from graph $f(x)$ has one value only.

$\therefore f(x) = \text{constant function}$

$= \frac{\pi}{2}$

(iii) $\int_{-1}^{1} (\sin^{-1}x + \cos^{-1}x)\ dx$

$= \int_{-1}^{1} \left(\frac{\pi}{2}\right) dx$

$= \left[\frac{\pi}{2}x\right]_{-1}^{1}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

$= \pi$ ✓

OR

From graph, value of integral is given by area of rectangle width 2 and height $\frac{\pi}{2}$.

Area $= 2 \times \frac{\pi}{2}$

$= \pi$

$\therefore \int_{-1}^{1} (\sin^{-1}x + \cos^{-1}x)\ dx = \pi$

(c) (i) $v = 4 + Ae^{-kt}$ (given)

$\therefore \frac{dv}{dt} = -k \times Ae^{-kt}$.

Now $k(4-v) = k(4-(4+Ae^{-kt}))$

$= k \times (-Ae^{-kt})$

$= -k \times Ae^{-kt}$ ✓

$\therefore \frac{dv}{dt} = k(4-v)$.

$\therefore v = 4 + Ae^{-kt}$ satisfies $\frac{dv}{dt} = k(4-v)$

(ii) $v = 4 + Ae^{-kt}$

At $t = 0$, $v = 25$

$25 = 4 + Ae^{0}$

$\therefore A = 21$ ✓

(iii) $v = 4 + 21e^{-kt}$

At $t = 2$, $v = 12$

$12 = 4 + 21e^{-2k}$

$8 = 21e^{-2k}$

$\frac{8}{21} = e^{-2k}$

$-2k = \ln\left(\frac{8}{21}\right)$

$k = 0.4825\ldots$ ✓

$= 0.483$ (3 significant figures)

(iv) $v = 4 + 21e^{-0.483t}$

$= 4 + \frac{21}{e^{0.483t}}$

$\rightarrow 4$ as $t \rightarrow \infty$ ✓

At $t = 20$

$v = 4 + 21e^{-0.483 \times 20}$

$= 4.0013$ (4 decimal places)

% difference $= \frac{0.0013}{4} \times \frac{100}{1}\%$

$\doteqdot 0.03\% < 0.1\%$ ✓

$\therefore$ speed differs from 4 by less than 0.1%.

QUESTION 7.

(a) $(1-ax)^n = 1 - nax + \frac{n(n-1)}{2\times1}(ax)^2 - \ldots$

$= 1 - 4x + \frac{20}{3}x^2 - \ldots$(given)

$\therefore na = 4$ ①

and $\frac{n(n-1)}{2}a^2 = \frac{20}{3}$ ② ✓

From ① $a = \frac{4}{n}$ ③

Substituting in ② :

$\frac{n(n-1)}{2} \times \left(\frac{4}{n}\right)^2 = \frac{20}{3}$ ✓

$48n(n-1) = 40n^2$

$6n^2 - 6n = 5n^2$

$n^2 - 6n = 0$

$n(n-6) = 0$

$n = 0$ or 6.

But $n \neq 0$, since n is a positive integer.

$\therefore n = 6$ ✓

From ③ $a = \frac{4}{6}$

$= \frac{2}{3}$ ✓

$\therefore a = \frac{2}{3},\ n = 6$.

(b) (i) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\therefore \cos 3x = \cos(2x+x)$

$= \cos 2x\cos x - \sin 2x\sin x$ ✓

$= (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x$

$= (2\cos^2 x - 1)\cos x - 2(1-\cos^2 x)\cos x$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$ ✓

$= 4\cos^3 x - 3\cos x$

$\therefore \cos 3x = 4\cos^3 x - 3\cos x$.

(ii) $\cos 3x - \sin 2x = 0 \qquad 0 < x < \frac{\pi}{2}$

$4\cos^3 x - 3\cos x - 2\sin x\cos x = 0$

$\cos x(4\cos^2 x - 3 - 2\sin x) = 0$

$\cos x(4(1-\sin^2 x) - 3 - 2\sin x) = 0$

$\cos x(4 - 4\sin^2 x - 3 - 2\sin x) = 0$

$\cos x(4\sin^2 x + 2\sin x - 1) = 0$ ✓

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ which is not in required range.

$\therefore 4\sin^2 x + 2\sin x - 1 = 0$

$\sin x = \frac{-2 \pm \sqrt{4+16}}{8}$

$= \frac{\pm\sqrt{5}-1}{4}$

$\sin x = \frac{-\sqrt{5}-1}{4} \Rightarrow x$ is outside required range. ✓

$\therefore \sin x = \frac{\sqrt{5}-1}{4}$.

(iii) $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ ✓

Now if $\theta = 3 \times \frac{\pi}{10}$,

then $\frac{\pi}{2} - \theta = \frac{\pi}{2} - 3 \times \frac{\pi}{10} = 2 \times \frac{\pi}{10}$.

$\therefore x = \frac{\pi}{10}$ is a solution of $\cos 3x = \sin 2x$.

OR $3x + 2x = \frac{\pi}{2}$

$5x = \frac{\pi}{2}$

$\therefore x = \frac{\pi}{10}$.

(iv)
$$\begin{aligned}
\sin\frac{\pi}{5}\cos\frac{\pi}{10} &= \sin\left(2 \times \frac{\pi}{10}\right)\cos\frac{\pi}{10} \\
&= \left(2\sin\frac{\pi}{10}\cos\frac{\pi}{10}\right)\cos\frac{\pi}{10} \quad ✓ \\
&= 2\sin\frac{\pi}{10}\cos^2\frac{\pi}{10} \\
&= 2\sin\frac{\pi}{10}\left(1 - \sin^2\frac{\pi}{10}\right) \\
&= 2 \times \frac{\sqrt{5}-1}{4} \times \left(1 - \left(\frac{\sqrt{5}-1}{4}\right)^2\right) \quad ✓ \\
&= \frac{\sqrt{5}-1}{2} \times \left(\frac{16 - (5 - 2\sqrt{5} + 1)}{16}\right) \\
&= \frac{\sqrt{5}-1}{2} \times \frac{10 + 2\sqrt{5}}{16} \\
&= \frac{\sqrt{5}-1}{2} \times \frac{5 + \sqrt{5}}{8} \\
&= \frac{5\sqrt{5} + 5 - 5 - \sqrt{5}}{16} \quad ✓ \\
&= \frac{4\sqrt{5}}{16} \\
&= \frac{\sqrt{5}}{4}.
\end{aligned}$$

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES
- Attempt ALL questions.
- ALL questions are of equal value.
- Write your student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the seven questions are to be handed in separately clearly marked Question 1, Question 2, etc..
- *The question paper must be handed to the supervisor at the end of the examination.*

STUDENT NUMBER / NAME..

STUDENT NUMBER / NAME..

Question 1 (Start a new page) **Marks**

a. Two dice are rolled. If you know that at least one of the dice is a 5, what is the probability of getting a total of 8? **2**

b. At an election, 30% of the voters favoured candidate A. If 7 voters are selected at random, what is the probability that 4 of them favour A? **2**

c. The point C(–1, –4) divides the interval AB externally in the ratio 3:1. If the coordinates of A are (3, 2), find the coordinates of B. **2**

d. Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x \, dx$ using the substitution $u = \cos x$ **3**

e. Find the exact value of $\int_0^{\frac{\pi}{4}} \cos^2 \frac{1}{2}x \, dx$ **3**

Question 2 (Start a new page)

a. Solve $\frac{1}{x + 1} \geq 1 - x$ **3**

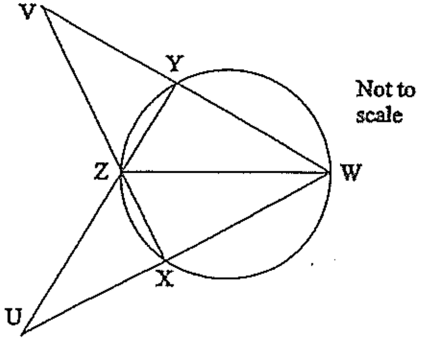
b. Find $\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{16 - 25x^2}}$ **3**

c. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x = 2at, y = at^2$. **3**

i. Find M, the midpoint of PQ.

ii. Show that, if the gradient of PQ is constant, the locus of M is a line parallel to the y–axis.

d. In the diagram, UZY, XZV, VYW and UXW are all straight lines. Given ZW bisects ∠XWY and ∠WUZ = ∠WVZ, prove that XW = YW. **3**



STUDENT NUMBER / NAME..

Question 3 (Start a new page) **Marks**

a. Show that $\frac{2x + 1}{x + 2} = 2 - \frac{3}{x + 2}$ **3**

Hence or otherwise, find the exact value of $\int_0^1 \frac{2x + 1}{x + 2} dx$

b. Solve $\cos x - \sqrt{3}\sin x + 1 = 0$ for $0 \leq x \leq 2\pi$ **3**

c. i. Show that the solution of $x \ln x - 1 = 0$ lies between $x = 1$ and $x = 2$. **3**

ii. Using $x = 2$ as a first approximation, apply Newton's method once to obtain a better approximation. Give your answer to one decimal place.

d. A mixed tennis team consisting of 2 men and 2 women is to be chosen from 5 men and 7 women. **3**

i. Find the probability that a particular woman is in the selected team.

ii. If one of the original 5 men is selected as the captain of the team, find the probability that his brother, who was one of the original 5 men, is also in the team.

Question 4 (Start a new page)

a. Two circles, C_1 and C_2, are members of the set of circles defined by the equation $x^2 + y^2 - 6x + 2ky + 3k = 0$, where k is real. **4**

The centre of C_1 lies on the line $x - 3y = 0$ and C_2 touches the x–axis.

Find the equations of C_1 and C_2.

b. The acceleration, a, of a particle is given in terms of its position, x, by the equation $a = 2x^3 + 2x$. **4**

i. If $v = 2$ when $x = 1$, show that $v^2 = (1 + x^2)^2$

ii. Show that, if $x = \frac{1}{\sqrt{3}}$ when $t = 0$, then $t = \frac{\pi}{6}$ when $x = \sqrt{3}$

c. Prove by Mathematical Induction that $5^{2n} - 1$ is divisible by 6 when n is a positive integer **4**

Question 5 (Start a new page) **Marks**

a. At 9 am, an ultralight aircraft flies directly over Daryl's head at 500 metres. It maintains a constant speed of 20 ms^{-1} and a constant altitude. 5

If x is the horizontal distance travelled by the plane and θ is the angle of elevation from Daryl to the plane,

i. show that $\frac{dx}{d\theta} = -500\operatorname{cosec}^2\theta$.

ii. Hence show that $\frac{d\theta}{dt} = -\frac{1}{25}\sin^2\theta$.

iii. Find the rate of change of the angle of elevation at 9:01 am.

b. Two groups of terrorists are 150 metres from their target. 7

The first group, Group A, is on the same horizontal level as the target and can fire their missiles in any direction at a speed of 50 ms^{-1}.

i. Show that Group A can hit the target and calculate the angle(s) at which their missiles are to be fired. [Use $g = 10$ ms^{-2}]

The second group, Group B, is positioned in a building 30 metres above the horizontal level of the target and can fire their missile only horizontally through a small window and at 55 ms^{-1}.

ii. Determine whether Group B can hit their target. [Use $g = 10$ ms^{-2}]

Question 6 (Start a new page) **Marks**

a. The displacement, x cm, of an object from the origin is given by 5

$$x = 2\sin t - 3\cos t, \quad t \geq 0$$

where time, t, is measured in seconds.

i. Show that the object is moving in Simple Harmonic Motion.

ii. Find the amplitude of the motion.

iii. At what time does the object first reach its maximum speed?

b. A cup of soup at temperature T°C loses heat when placed in the lounge room. It cools according to the law: 7

$$\frac{dT}{dt} = k(T - T_0)$$

where t is the elapsed time in minutes and T_0 is the temperature of the room in degrees centigrade.

i. Show that the equation $T = T_0 + Ae^{kt}$ satisfies the above law of cooling.

ii. A cup of soup at 95°C is placed in the freezer at −10°C for 5 minutes and cools to 65°C. Find the exact value of k

iii. The same cup, at 65°C, is then taken into the lounge room where the surrounding temperature is 26°C. Assuming k remains the same, find, to the nearest degree, the temperature of the soup after another 5 minutes.

STUDENT NUMBER / NAME..

Question 7 (Start a new page) Marks

a. Find the constant term in the expansion of $\left(3x - \dfrac{1}{x^2}\right)^6$ 3

b. i. Solve the equation $x^4 + x^2 - 1 = 0$, giving your answer(s) to two decimal places. 9

ii. On the same axes, draw the graphs of $y = \tan^{-1} x$ and $y = \cos^{-1} x$, showing all important features. Mark the point, P, where the curves intersect.

iii. Show that, if $\tan^{-1} x = \cos^{-1} x$, then $x^4 + x^2 - 1 = 0$. Hence find the coordinates of P.

iv. Find to two decimal places the area enclosed by the curves and the y-axis.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

HEMATICS
NIT TRIAL 1999

SUGGESTED ANSWERS

Possibilities are

1,5
2,5
(3,5)
4,5
5,2 (5,3) 5,4 5,5 5,6
6,5

Probability of total of 8 = $\frac{2}{11}$

et p = prob of supporting A = $3/10$
q = prob of supporting other = $7/10$
n = no. of A supporters

en $P(X=r) = {}^7C_r\left(\frac{3}{10}\right)^r\left(\frac{7}{10}\right)^{7-r}$

$P(X=4) = {}^7C_4\left(\frac{3}{10}\right)^4\left(\frac{7}{10}\right)^3$

$= 0\cdot0972405$

$\doteq 0\cdot1$

$x = \frac{kx_2+lx_1}{k+l}$ $y = \frac{ky_2+ly_1}{k+l}$

$-1 = \frac{-3\times x_2+1\times3}{-3+1}$ $-4 = \frac{-3\times y_2+1\times2}{-3+1}$

$2 = -3x_2+3$ $8 = -3y_2+2$

$x_2 = 1/3$ $y_2 = -2$

$\therefore B\left(\frac{1}{3}, -2\right)$

d. $u = \cos x$

$du = -\sin x \, dx$

if $x = \pi/2$, $u = 0$

if $x = \pi/3$, $u = \frac{1}{2}$

$\therefore I = \int_{.5}^{0} -u^3 \, du$

$= \left[\frac{u^4}{4}\right]_0^{.5}$

$= \frac{.5^4}{4} - 0 = \frac{1}{64}$

e. $\int_0^{\pi/4} \cos^2 \frac{1}{2}x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1+\cos x \, dx$

$= \frac{1}{2}\left[x+\sin x\right]_0^{\pi/4}$

$= \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)$

MATHS 3U ANSWERS - 1999

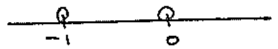
a. $\frac{1}{x+1} \geq 1-x$

Critical points at $x=-1$ and

$\frac{1}{x+1} = 1-x$

$1 = 1-x^2$

$\Rightarrow x = 0$



Test $x=-2$ False

Test $x=-\frac{1}{2} \Rightarrow 2 \geq 1\frac{1}{2}$ $\therefore$ True

$x = 1 \Rightarrow \frac{1}{2} \geq 0$ $\therefore$ True

Solution: $x > -1$

b. $\int_0^{2/5} \frac{dx}{\sqrt{16-25x^2}}$

$= \int_0^{2/5} \frac{dx}{5\sqrt{\frac{16}{25}-x^2}}$

$= \frac{1}{5}\left[\sin^{-1}\frac{x}{4/5}\right]_0^{2/5}$

$= \frac{1}{5}\left[\sin^{-1}\frac{5x}{4}\right]_0^{2/5}$

$= \frac{1}{5}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}0\right)$

$= \frac{1}{5}\cdot\frac{\pi}{6} = \frac{\pi}{30}$

c. (i) $M\left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

(ii) $M_{PQ} = \frac{p+q}{2} = k$, a constant

Then, for the point M,

$x = a(p+q)$

$= a.2k$

$x = 2ak$

Since a and k are constant, the locus of M is a line parallel to the y-axis

d. $\angle U = \angle V$ (given)

$\angle UZX = \angle VZY$ (vertically opposite)

Now $\angle ZXW = \angle UZX + \angle U$ (exterior angle of triangle)

and $\angle ZYW = \angle VZY + \angle V$ (ditto)

$\therefore$ $\angle ZXW = \angle ZYW$ (equal to sum of equal angles)

In ΔXZW & ΔYZW,

ZW is common

$\angle ZXW = \angle ZYW$ (above)

$\angle XWZ = \angle YWZ$ (given ZW bisects $\angle YWX$)

$\therefore \Delta XZW \equiv \Delta YZW$ (AAS)

and XW = YW

ITHS 3U ANSWERS - 1999

$2-\frac{3}{x+2} = \frac{2(x+2)-3}{x+2}$

$= \frac{2x+1}{x+2}$

$\int_0^1 \frac{2x+1}{x+2}\,dx$

$= \int_0^1 2-\frac{3}{x+2}\,dx$

$= \left[2x - 3\ln(x+2)\right]_0^1$

$= (2-3\ln 3)-(0-3\ln 2)$

$= 2+3\ln(2/3)$

Let $\cos x - \sqrt{3}\sin x = A\cos(x+\theta)$

$= A\cos x\cos\theta - A\sin x\sin\theta$

n $A\cos\theta = 1$

$A\sin\theta = \sqrt{3}$

⇒ $\tan\theta = \sqrt{3}$ and $\theta = \pi/3$

2nd $A = 2$

$\therefore 2\cos(x+\frac{\pi}{3}) + 1 = 0$

$\cos(x+\frac{\pi}{3}) = -1/2$

$x+\frac{\pi}{3} = \ldots.\ \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots\ldots$

$x = \pi, \frac{4\pi}{3}$ in given domain

i) Let $f(x) = x\ln x - 1$

$f(1) = 1.\ln 1 - 1 < 0$

$f(2) = 2\ln 2 - 1 > 0$

. a solution exists between $x=1$ &

$x=2$ (assuming $f(x)$ is continuous)

(ii) $f'(x) = x.\frac{1}{x} + \ln x = \ln x + 1$

By Newton's method,

$x_1 = x - \frac{f(x)}{f'(x)}$

$= x - \frac{x\ln x - 1}{\ln x + 1}$

If $x=2$, $x_1 = 2 - \frac{2\ln 2 - 1}{\ln 2 + 1}$

$= +1.77184832$

$= 1.8$

(d)(i) Total no. of possible teams

$= {}^7C_2 \times {}^5C_2 = 210$

Teams with a particular woman

$= {}^6C_1 \times {}^5C_2 = 60$

∴ Probability of a particular woman

$= \frac{60}{210} = \frac{2}{7}$

(ii) Captain is specified, so the number of possible teams is ${}^4C_1 \times {}^7C_2 = 84$

No. of teams with his brother is ${}^7C_2 =$

∴ Probability of captain and brother

$= \frac{21}{84} = \frac{1}{4}$

OR with the captain as specified, the probability that his brother is chosen from the remaining four men is $\frac{1}{4}$

MATHS 3U ANSWERS - 1999

4(a) $x^2+y^2-6x+2ky+3k=0$

Completing the squares:

$(x-3)^2+(y+k)^2 = k^2-3k+9$

If the centre $(3,-k)$ is on the line $x-3y=0$, then

$3-3x-k=0 \Rightarrow k=-1$

$\therefore C_1: (x-3)^2+(y-1)^2 = 13$

If C_2 touches the x-axis, the radius is k

$\therefore \sqrt{k^2-3k+9} = k$

$k^2-3k+9 = k^2$

$\Rightarrow k = 3$

$\therefore C_2: (x-3)^2+(y+3)^2 = 9$

(b)(i) $\frac{d}{dx}(\frac{1}{2}v^2) = 2x^3+2x$

$\therefore \frac{1}{2}v^2 = \frac{1}{2}x^4+x^2+C$

If $v=2$, $x=1$

$\therefore \frac{1}{2}.2^2 = \frac{1}{2}.1+1+C \Rightarrow C=\frac{1}{2}$

& $\frac{1}{2}v^2 = \frac{1}{2}x^4+x^2+\frac{1}{2}$

$v^2 = x^4+2x^2+1$

$v^2 = (x^2+1)^2$

(ii) so $v = \pm(x^2+1)$

but $v=2\ (>0)$ when $x=1$

$\therefore v = +(x^2+1)$

$\frac{dt}{dx} = \frac{1}{x^2+1}$

so $t = \tan^{-1}x + C$

Now $x=\frac{1}{\sqrt{3}}$ when $t=0$

$\therefore C = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$

so $t = \tan^{-1}x - \frac{\pi}{6}$

when $x=\sqrt{3}$, $t = \tan^{-1}\sqrt{3} - \frac{\pi}{6}$

$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

(c) Let $S(n)$: $5^{2n}-1 = 6I$, where I is an integer.

$S(1)$: LHS $= 5^2-1 = 24 = 6\times 4$

∴ $S(1)$ is true

Assume $S(k)$: $5^{2k}-1 = 6I$ (I, integer)

Consider $S(k+1)$:

LHS $= 5^{2k+2}-1$

$= 5^{2k}.5^2-1$

$= 25(5^{2k}-1)-1+25$

$= 25.6I+24$ by $S(k)$

$= 6[25I+4]$

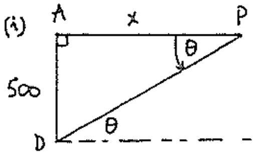
Now I is integer, ∴ $25I+4$ is integer.

Hence, if $S(k)$ is true, $S(k+1)$ is true

But $S(1)$ is true, so $S(2)$ is true, and then $S(3)$ is true and so on for all integer values of n.

MATHS 3U ANSWERS - 1999

)(i)



$\tan\theta = \frac{500}{x}$

$\therefore x = 500\cot\theta$

$\frac{dx}{d\theta} = -500\,\mathrm{cosec}^2\theta$

ii) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$= \frac{1}{-500\,\mathrm{cosec}^2\theta} \times 20$

$= -\frac{1}{25}\sin^2\theta$

iii) At 9:01, $t=60$, $x=1200$

Then $PD=1300$ (Pythagoras' Theorem)

so $\sin\theta = \frac{500}{1300} = \frac{5}{13}$

$\therefore \frac{d\theta}{dt} = -\frac{1}{25} \times \left(\frac{5}{13}\right)^2$

$= -\frac{1}{169}$ degrees/sec.

)(i) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = c_1$ $\quad \dot{y} = -10t + c_2$

initially $\dot{x} = 50\cos\alpha$ $\quad \therefore \dot{x} = 50\cos\alpha$

and $\dot{y} = 50\sin\alpha$ $\quad \therefore \dot{y} = -10t + 50\sin\alpha$

$x = 50t\cos\alpha + C_3$ $\quad y = -5t^2 + 50t\sin\alpha + C_4$

since $x=0$ when $t=0$, $\quad$ and $y=0$ when $t=0$

$C_3 = 0$ $\quad C_4 = 0$

$\therefore x = 50t\cos\alpha$ $\quad \therefore y = -5t^2 + 50t\sin\alpha$

when $x=150$, $150 = 50t\cos\alpha$

so $3 = t\cos\alpha$ ... (1)

when $y=0$, $0 = -5t^2 + 50t\sin\alpha$

$= -5t(t - 10\sin\alpha)$

$\Rightarrow t = 10\sin\alpha$ (2)

Solving (1) + (2):

$3 = 10\sin\alpha\cos\alpha$

$= 5\sin 2\alpha$

$\therefore \sin 2\alpha = 3/5$

$2\alpha = 36°52', 143°08'$

$\therefore \alpha = 18°26'$ or $71°29'$

(ii) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = C_1$ $\quad \dot{y} = -10t + C_2$

Initially, $\dot{x} = 55\cos\alpha$, $\dot{y} = 55\sin\alpha$

$\therefore \dot{x} = 55\cos\alpha$ $\quad \dot{y} = -10t + 55\sin\alpha$

$\dot{x} = 55$ $\quad \dot{y} = -10t$ since α

Then $x = 55t + C_3$ $\quad y = -5t^2 + C_4$

When $t=0$, $x=0$ and $y=30$

$\Rightarrow C_3 = 0$ $\quad C_4 = 30$

$\therefore x = 55t$ $\quad y = -5t^2 + 30$

Now when $y=0$, $-5t^2 + 30 = 0$

$\therefore t^2 = 6$

$\& \; t = \sqrt{6}$

At $t = \sqrt{6}$, $x = 55\sqrt{6}$

≈ 135 m

$\therefore$ Group B cannot reach the target

MATHS 3U ANSWERS - 1999

Q6(a)(i) $x = 2\sin t - 3\cos t$

$\dot{x} = 2\cos t + 3\sin t$

$\ddot{x} = -2\sin t + 3\cos t$

$= -(2\sin t + 3\cos t)$

$= -x$

$\therefore$ motion is simple harmonic.

(ii) Amplitude $= \sqrt{2^2 + 3^2}$

$= \sqrt{13}$ cm

(iii) $\dot{x} = 2\cos t + 3\sin t$

$\ddot{x} = -2\sin t + 3\cos t$

Max velocity when $\ddot{x} = 0$

$-2\sin t + 3\cos t = 0$

$3\cos t = 2\sin t$

$3/2 = \tan t$

$t = 0{\cdot}983, 4.1243....$ etc

$\therefore$ reaches maximum velocity when $t = 0{\cdot}983$

(b)(i) $T = T_0 + Ae^{kt}$

$\frac{dT}{dt} = k.Ae^{kt}$

$= k(T - T_0)$

(ii) When $t=0$, $T=95$, $T_0 = -10$

$\Rightarrow A = 105$

when $t=5$, $T=65$

$\therefore 65 = -10 + 105e^{5k}$

$e^{5k} = 75/105 = 5/7$

$\ln e^{5k} = \ln 5/7$

$\therefore k = \frac{1}{5}\ln\frac{5}{7}$

(iii) When $t=0$, $T_0 = 26$ & $T = 65$

$\therefore 65 = 26 + Be^{k.0}$

$\therefore B = 39$

Therefore, at $t=5$,

$T = 26 + 39e^{5k}$ with $k = \frac{1}{5}\ln\frac{5}{7}$

so $T = 53{\cdot}86°$

$= 54°$ (to the nearest degree)

ATHS 3U ANSWERS - 1999

$\left(3x-\frac{1}{x^2}\right)^6=\sum_{r=0}^{6}{}^6C_r\,(3x)^{6-r}\left(-\frac{1}{x^2}\right)^r$

ypical term, T_r, is

$T_r={}^6C_r\,3^{6-r}\cdot x^{6-r}\cdot(-1)^r\cdot(x^{-2})^r$

$={}^6C_r\,3^{6-r}(-1)^r\,x^{6-3r}$

onstant term when $6-3r=0$

$r=2$

then $T_2={}^6C_2\,3^4\,(-1)^2$

$=1215$

) $x^4+x^2-1=0$

$x^2=\frac{-1\pm\sqrt{1-4\times1\times-1}}{2}$

$=\frac{-1\pm\sqrt{5}}{2}$

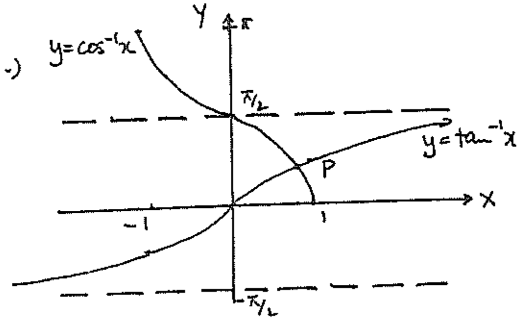
$\therefore x^2=\cancel{\frac{-1-\sqrt{5}}{2}}$ or $\frac{-1+\sqrt{5}}{2}$

$x^2=0.618033988$

$x=\pm0.786151377$

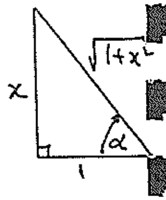
$=\pm0.79$

)



(iii) let $\tan^{-1}x=\alpha$

$\therefore x=\tan\alpha$



At P, $\cos^{-1}x=\tan^{-1}x=\alpha$

$\therefore$ at P $\cos^{-1}x=\alpha$ + $x=\cos\alpha$

But $\cos\alpha=\frac{1}{\sqrt{1+x^2}}$ (from diagram

$\therefore\ x=\frac{1}{\sqrt{1+x^2}}$

Squaring, $x^2=\frac{1}{1+x^2}$

+ $x^4+x^2=1$

$x^4+x^2-1=0$

$\therefore x=0.79$ (from (i))

and $y=\tan^{-1}0.79=0.6686$

so $P(0.79,\ 0.67)$

(iv) $A=\int_0^{0.67}\tan y\,dy+\int_{0.67}^{\pi/2}\cos y\,dy$

$=\left[-\ln|\cos y|\right]_0^{0.67}+\left[\sin y\right]_{0.67}^{\pi/2}$

$=-\ln|\cos0.67|+\sin\frac{\pi}{2}-\sin0.6$

$=0.62258$

$=0.62$ (to 2-decimal places)

WESTERN REGION

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

3 Unit (Additional)
and
3/4 Unit (Common)

Time allowed - TWO hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES
- Attempt ALL questions.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on the last page. These may be removed for your convenience.
- Board-approved calculators may be used.
- Each question should be started on a new page.

Question One. *Start a new page* Marks

(a) Find a general solution for x if $\tan x = \frac{1}{\sqrt{3}}$ (in terms of π). 2

(b) If α, β and χ are the roots of 2
$$3x^3 + 5x^2 - 7x + 4 = 0.$$
Find the values of $\alpha\beta + \alpha\chi + \beta\chi$ and $\alpha + \beta + \chi$.

(c) Solve $\sec^2 x + \tan x - 7 = 0$ for $0° \le x \le 360°$ (to the nearest minute). 3

(d) Given that 2
$$x = \cos\theta + 1 \quad \text{and} \quad y = \sin\theta - 2$$
By eliminating θ determine a relationship between x and y only.

(e) Solve $\frac{2x}{x-3} \le 1$ if $x \ne 3$. 3

Question Two. *Start a new page* **Marks**

(a) Given $\dfrac{x^3+3x^2+9x-1}{x^2+9} = Q(x) + \dfrac{R(x)}{x^2+9}$ **4**

(i) By performing the division or otherwise find $Q(x)$ and $R(x)$.

(ii) Hence find $\displaystyle\int \frac{x^3+3x^2+9x-1}{x^2+9}\,dx$

(b) Using Mathematical Induction prove that $7^n - 5^n$ is even for all positive integer n. **3**

(c) Eight people are seated about a round table. **3**

(i) How many different seating arrangements are possible ?

(ii) If the eight people consist of four couples find the probability that each person is seated adjacent to their partner.

(d) Given that the quadrilateral ABCD is cyclic show that the sum of the tangents of the angles is zero. **2**
That is, $\tan A + \tan B + \tan C + \tan D = 0$

Question Three. *Start a new page* **Marks**

(a) Evaluate $\displaystyle\int_0^1 \frac{2x}{(2x+1)^2}\,dx$ **3**

using the substitution $u = 2x + 1$.

(b) By making suitable substitutions for A and B in the expansion of $\cos(A+B)$ find the exact value of $\cos 75°$. **2**

(c) If $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$ determine the values of R and α. **3**

(d) Find the constant term in the expansion of **2**

$$\left(2x - \frac{1}{x^3}\right)^{20}$$

(Do not evaluate).

(e) If $f(x) = \sin^{-1}x + \cos^{-1}x$ and $1 \geq x \geq 0$, find **2**

(i) $f'(x)$

(ii) $\displaystyle\int_0^1 f(x)\,dx$

Question Four. *Start a new page* **Marks**

(a) A Horticulturalist knows that the probability of grafting a particular rose successfully is 0·4. **5**

(i) If ten grafts are made find the probability of six successful grafts. Give your answer correct to 3 significant figures.

(ii) How many grafts must be made to ensure that the probability of at least one success is at least 99·9 %.

(b) A spherical balloon is being inflated at the rate of 1000 $\text{cm}^3\ \text{s}^{-1}$ and given that **5**

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad A = 4\pi r^2$$

find :

(i) An expression for the instantaneous rate of change of the radius in terms of r. $\left(\text{Find } \frac{dr}{dt}\right)$.

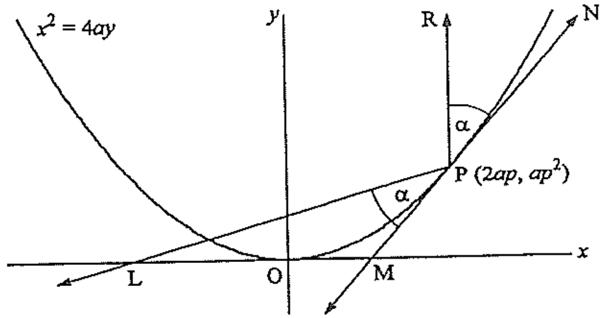
(ii) The rate of change of the surface area of the balloon when the radius is 10 cm.

(c) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \; dx$ **2**

Question Five. *Start a new page* **Marks**

(a) It is known that $f(x) = \frac{1}{2}x - \sin x$ has a root of $f(x) = 0$ near $x = 2$. **2**
By using one application of Newton's method find a better approximation to two decimal places.

(b) **7**



The parabola $x^2 = 4ay$ is shown in the sketch above.
The tangent at $P(2ap, ap^2)$ cuts the x axis at M and passes through the point N.
PR is parallel to the axis of the parabola and makes $\angle RPN = \alpha°$.
PL is such that it cuts the x axis at L and $\angle LPM = \alpha°$.

(i) Show that $\tan\alpha = \frac{1}{p}$.

(ii) Show that the gradient of LP is $\frac{p^2 - 1}{2p}$.

(iii) Show that the line LP passes through the focus of the parabola.

(c) A particle is projected such that at any time 't', the equation of the trajectory is given by **3**

$$x = 36t \quad \text{and} \quad y = 15t - \tfrac{1}{2}gt^2$$

find the angle of projection to the nearest minute.

Question Six. *Start a new page* **Marks**

(a) (i) State the domain and range of **3**

$$y = 2\sin^{-1}(3x).$$

(ii) Sketch $y = 2\sin^{-1}(3x)$.

(b) The rate at which a body cools is proportional to the difference between its temperature (T) and the constant temperature of the surrounding air (S). **7**
That is

$$\frac{dT}{dt} = k(T - S)$$

where t is the time elapsed and k is a constant.

(i) Show that $T = S + Be^{kt}$, where B is a constant, is a solution of the above differential equation.

(ii) A body cools from 150° C to 90° C in three hours. If the air temperature is 30° C find the value of B and hence the value of k to 3 decimal places.

(iii) Using the values of B and k found in (ii) determine the temperature of the body after a further three hours.

(c) Find the coordinates of the point P which divides the interval joining A (−1 , 2) and B (5 , −3) internally such that AP : PB = 3 : 2. **2**

Question Seven. *Start a new page* **Marks**

(a) Two circles touch internally at P. A line through P cuts the smaller circle at A and the larger circle at B. **4**
A second line through P cuts the smaller and larger circles at C and D respectively.

(i) Sketch this information.

(ii) Prove that the line joining A and C is parallel to the line joining B and D.

(b) You are given that x, v, a and t represent displacement, velocity, acceleration and time elapsed respectively. **2**
Show that

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

(c) A particle moves in a straight line such that its acceleration is given by **6**

$$a = -e^{-2x}$$

If $v = 1$ and $x = 0$ when $t = 0$

(i) Express v in terms of x. (Use the result proven in (b))

(ii) Express x in terms of t.

(iii) Express v in terms of t.

WESTERN REGION

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

3 Unit (Additional)
and
3/4 Unit (Common)

Solutions and Marking Scheme

PLEASE NOTE :

- These are suggested solutions. They are not intended to specify the amount of working required or the method to be applied.
- Teachers should accept any valid method of solution providing adequate working is shown.

Course : 3 UNIT

Solutions	Marks	Comments
Question 1		
(a) $\tan x = \frac{1}{\sqrt{3}}$		
$\therefore x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$ for $2\pi \geq x \geq 0$		
∴ General solution		
$x = \frac{\pi}{6} \pm 2n\pi, \pi + \frac{\pi}{6} \pm 2n\pi$	1	
$= \frac{\pi}{6} + n\pi$	1	Accept $\frac{\pi}{6} \pm n\pi$
for all integral n		
(b) $3x^3 + 5x^2 - 7x + 4 = 0$		
$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{7}{3}$	1	
$\alpha + \beta + \gamma = -\frac{5}{3}$	1	
(c) $\sec^2 x + \tan x - 7 = 0$		
$1 + \tan^2 x + \tan x - 7 = 0$	1	
$\tan^2 x + \tan x - 6 = 0$		
$(\tan x + 3)(\tan x - 2) = 0$		
$\therefore \tan x = -3, 2$	1	
$\therefore x = 108°26', 288°26',$	1	
$63°26', 243°26'$		

Course : 3 UNIT

Solutions	Marks	Comments
Question 1 (continued)		
(d) $x = \cos\theta + 1$		
$\therefore x - 1 = \cos\theta$ — ①		
$y = \sin\theta - 2$		
$y + 2 = \sin\theta$ — ②	1	
①^2 + ②^2		
$(x-1)^2 + (y+2)^2 = \sin^2\theta + \cos^2\theta$		
$\therefore (x-1)^2 + (y+2)^2 = 1$ is the	1	
required equation		
(e) $\frac{2x}{x-3} \leq 1 \quad x \neq 3$		Some may use another method. If solution is correct give full marks
If $x - 3 > 0$		
$x > 3$		
$2x \leq x - 3$		
$x \leq -3$	1	
No solution		
If $x - 3 < 0$		
$x < 3$		
$\therefore 2x \geq x - 3$	1	
$x \geq -3$		
$\therefore$ Solution is $3 > x \geq -3$	1	

Solutions	Marks	Comments
Question 2 (a) (i) $x^2+9\,\overline{)\,x^3+3x^2+9x-1\,}$ quotient $x+3$ $x^3 \qquad +9x$ $3x^2 \quad -1$ $3x^2 \quad +27$ -28 $\therefore Q(x)=x+3 \quad R(x)=-28$	2	
(ii) $\int \frac{x^3+3x^2+9x-1}{x^2+9}\,dx$ $=\int\left(x+3\cdot-\frac{28}{x^2+9}\right)dx$ $=\frac{1}{2}x^2+3x-\frac{28}{3}\tan^{-1}\frac{x}{3}+C$	2	Do not worry abo C
(b) Let $f(n)=7^n-5^n$ $\therefore f(1)=7-5$ $=2$ $\therefore$ True for $n=1$ Assume true for $n=k$ $\therefore f(k)=7^k-5^k=2A$ where A is an integer	1	
when $n=k+1$ $f(k+1)=7^{k+1}-5^{k+1}$ $=7\times7^k-5\times5^k$ $=7(7^k-5^k)+2\times5^k$ $=7\times2A+2\times5^k$ $=2(7A+5^k)$	1	Some may test $f(k+1)-f(k$ even
$\therefore$ True as $7A+5^k$ is an integer $\therefore$ if true for $n=1$ the true $n=2$ " " " $n=2$ " " $n=3$ etc. $\therefore$ True for all $n\geq0$ integers	1	

Solutions	Marks	Comments
Question 2 (continued) (c) (i) Number of arrangements $=(8-1)!$ $=5040$	1	
(ii) Number of couple arrangements $=2^4\times(4-1)!$ $=96$	1	
$\therefore$ Probability of pairs together $=\frac{96}{5040}$ $=\frac{2}{105}$	1	
(d) $\because$ Quadrilateral is cyclic $\therefore$ opposite angles are supplementary $\therefore \angle C=180^\circ-\angle A$ $\angle D=180^\circ-\angle B$	1	
$\therefore \tan A+\tan B+\tan C+\tan D$ $=\tan A+\tan B+\tan(180-A)+\tan(180-B)$ $=\tan A+\tan B-\tan A-\tan B$ $=0$	1	

Solutions	Marks	Comments
Question 3		
(a) $\int_0^1 \frac{2x\,dx}{(2x+1)^2}$ $\quad u = 2x+1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $x=1$ $u=3$, $x=0$ $u=1$		
$\therefore \int_1^3 \frac{u-1}{u^2}\frac{du}{2}$	1	
$= \frac{1}{2}\int_1^3 \left(\frac{1}{u} - \frac{1}{u^2}\right) du$		
$= \frac{1}{2}\left[\ln u + \frac{1}{u}\right]_1^3$	1	
$= \frac{1}{2}\left(\ln 3 + \frac{1}{3}\right) - \frac{1}{2}(\ln 1 + 1)$		
$= \frac{1}{2}\left(\ln 3 - \frac{2}{3}\right)$	1	
(b) $\cos(A+B) = \cos A\cos B - \sin A \sin B$ $\cos 75^\circ = \cos(45^\circ + 30^\circ)$	1	
$\therefore \cos 75^\circ = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$ $= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$		
$= \frac{\sqrt{3}-1}{2\sqrt{2}}$	1	
(c) $\sqrt{3}\cos x - \sin x$ $= R\cos x\cos\alpha - R\sin x\sin\alpha$	1	
$\therefore$ $R\cos\alpha = \sqrt{3}$ $\quad R\sin\alpha = 1$ $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$ $\quad R^2\sin^2\alpha + R^2\cos^2\alpha = 1+3$ $\quad R^2 = 4$ $\tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = 30^\circ$ $\quad \therefore R = 2$	2	1each

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Solutions	Marks	Comments
Question 3 (continued)		
(d) General Term $= {}^{20}C_r (2x)^{20-r}\left(\frac{-1}{x^3}\right)^r$		
Ignoring coefficient to find r $x^{20-r}\cdot(x^{-3})^r = x^0$ $x^{20-4r} = x^0$	1	
$\therefore 4r = 20$ $r = 5$		
$\therefore$ Term is ${}^{20}C_5 \cdot 2^{15}\cdot(-1)^5$		
$= -{}^{20}C_5 \cdot 2^{15}$ -508035072	1	
(e) $f(x) = \sin^{-1}x + \cos^{-1}x$ $= \frac{\pi}{2}$		
(i) $f'(x) = 0$	1	
(ii) $\int_0^1 f(x)\,dx = \int_0^1 \frac{\pi}{2}\,dx$ $= \left[\frac{\pi}{2}x\right]_0^1$ $= \frac{\pi}{2} - 0$		
$= \frac{\pi}{2}$	1	

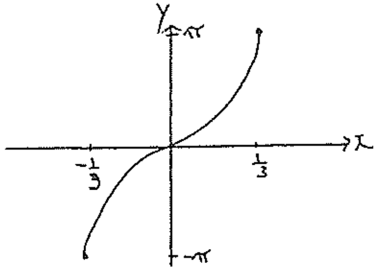
Solutions	Marks	Comments
Question 4 (a)(i) Let $P(S)=0.4$ and $P(F)=0.6$ $\therefore P(S=6) = {}^{10}C_6(0.6)^4(0.4)^6$ $=0.111$ (3sig figs)	2	
(ii) Let there be n grafts $\therefore P(F=n)=(0.6)^n$ $\therefore P(S\geqslant 1) = 1-(0.6)^n \geqslant 0.999$	1	
$\therefore$ $0.001 \geqslant (0.6)^n$ $\frac{\ln(0.001)}{\ln(0.6)} \leqslant n$	1	
$13.523 \leqslant n$ $\therefore$ 14 or more grafts are required.	1	
(b)(i) Given $V=\frac{4}{3}\pi r^3$ $\frac{dV}{dt}=1000$ $\frac{dV}{dr}=4\pi r^2$	1	
$\therefore \frac{dr}{dt} = \frac{dV}{dt} \div \frac{dV}{dr}$ $= \frac{1000}{4\pi r^2}$ $= \frac{250}{\pi r^2}$	1	
Radius is changing at $\frac{250}{\pi r^2}$ cm s^{-1}		

Solutions	Marks	Comments
Question 4 (continued) (b) (ii) Given $A=4\pi r^2$ $\frac{dA}{dr}=8\pi r$ $\therefore \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$	1	
$= 8\pi r \times \frac{250}{\pi r^2}$ $= \frac{2000}{r}$	1	
when $r=10$ $\frac{dA}{dt} = \frac{2000}{10} = 200$ Area is changing at 200 $cm^2 s^{-1}$	1	
(c) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2x)\,dx$ $=\frac{1}{2}\left[x+\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$	1	
$=\frac{1}{2}\left[(\frac{\pi}{2}+0)-0\right]$ $=\frac{\pi}{4}$	1	

Solutions	Marks	Comments
Question 5		
(a) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$	1	
$f(x) = \frac{1}{2}x - \sin x$ $f'(x) = \frac{1}{2} - \cos x$		
$x_1 = 2 \quad \therefore x_2 = 2 - \frac{\frac{1}{2}\cdot 2 - \sin 2}{\frac{1}{2} - \cos 2}$	1	
$= 1\cdot 90$		
b)(i) Gradient of tangent = p		
$\angle PMX = 90 - \alpha$	1	
$\therefore$ Gradient of MN = $\tan(90-\alpha)$ $\therefore \quad p = \tan(90-\alpha)$ $= \cot\alpha$		
$\frac{1}{p} = \tan\alpha$	1	
(ii) $\angle PLM + \angle LPM = \angle PMX$ (exterior $\angle$ $\triangle PLM$)	1	
$\therefore \angle PLM + \alpha = 90 - \alpha$		
$\angle PLM = 90 - 2\alpha$ ✓	1	
$\therefore$ Gradient $LP = \tan(90-2\alpha)$ $= \cot 2x$	1	
$= \frac{1-\tan^2 x}{2\tan x}$ $= \frac{1-\frac{1}{p^2}}{2\cdot\frac{1}{p}}$		
$= \frac{p^2-1}{2p}$ ✓	1	

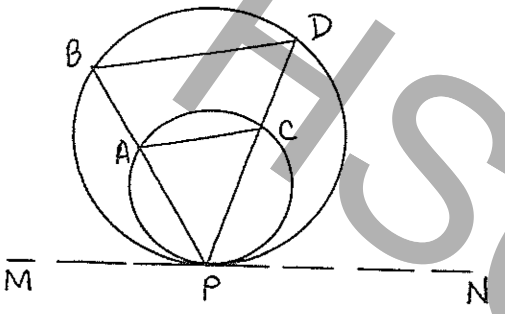
Solutions	Marks	Comments
Question 5 (continued)		
(b)(iii) Coords. of S are (0, a)		Other ways are o.k.
Gradient $PS = \frac{ap^2 - a}{2ap}$ $= \frac{p^2-1}{2p}$	1	
Gradient PS = Gradient LP $\therefore$ LP passes through S		
(c) $x = 36t \quad y = 15t - \frac{1}{2}gt^2$ $\frac{dx}{dt} = 36 \quad \frac{dy}{dt} = 15 - gt$ $\therefore \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$ $= \frac{15-gt}{36}$	2	Give marks for correct method
If θ is the angle of projection $\tan\theta = \frac{15-gt}{36}$ when $t=0$ $= \frac{15}{36}$		
$\therefore \theta = \tan^{-1}\frac{15}{36}$ $= 22°37'$	1	

Course : 3 Unit

Solutions	Marks	Comments
Question 6		
(a)(i) $y = 2\sin^{-1}(3x)$.		
$1 \geqslant 3x \geqslant -1$		
$\therefore$ Domain $\frac{1}{3} \geqslant x \geqslant -\frac{1}{3}$	1	
Range $\pi \geqslant y \geqslant -\pi$	1	
(ii) 	1	
(b)(i) $T = S + Be^{kt}$ ___ ①		
$\frac{dT}{dt} = Bke^{kt}$	1	
Now $Be^{kt} = T - S$ from ①		
$\therefore Bke^{kt} = k(T-S)$		
$\therefore$ It is a solution.	1	

Course : 3 UNIT

Solutions	Marks	Comments
Question 6 (continued)		
(ii) $T = S + Be^{kt}$		
$S = 30°$, $T = 150°$ when $t = 0$		
$150° = 30° + Be^{0}$		
$120° = B$	2	
when $t = 3$ $T = 90°$		
$\therefore 90 = 30 + 120e^{3k}$		
$60 = 120e^{3k}$		
$0.5 = e^{3k}$		
$\therefore k = \frac{1}{3}\ln 0.5$		
$= -0.231$ (3 d.p.)	2	
(iii) when $t = 6$		
$T = 30° + 120 \times e^{-0.231 \times 6}$		
$= 60°$		
Temperature is 60°	1	
(c) $x = \frac{kx_2 + lx_1}{k+l}$ $y = \frac{ky_2 + ly_1}{k+l}$		
$= \frac{3 \times 5 + 2 \times -1}{3+2}$ $= \frac{3 \times -3 + 2 \times 2}{3+2}$		
$= 2\frac{3}{5}$ $= -1$		
$\therefore$ Coordinates of P are		
$(2\frac{3}{5}, -1)$		

Solutions	Marks	Comments
Question 7		
(a)(i)		
	1	
(ii) Draw tangent MN through P as shown		
$\angle ACP = \angle APM$ (Angle in alternate segment.)	1	
$\angle BDP = \angle BPM$ (" " ")		
$\angle BPM = \angle APM$ (Same angle)		
$\therefore \angle ACP = \angle BDP$ ($= \angle APM$)	1	
$\therefore$ BD $\parallel$ CA corresponding angles equal	1	
b) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$		
$= v \cdot \frac{dv}{dx}$	1	
$= \frac{dx}{dt} \times \frac{dv}{dx}$		
$= \frac{dv}{dt}$	1	
$= a$		

Solutions	Marks	Comments
Question 7 (Continued)		
(c)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-2x}$		
$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + C$		
Given $v=1$ when $x=0$	1	
$\frac{1}{2} = \frac{1}{2}e^0 + C$		
$\therefore C = 0$		
$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x}$		
$v^2 = e^{-2x}$		
$v = e^{-x}$	1	strictly speaking $v = \pm e^{-x}$
(ii) $\frac{dx}{dt} = e^{-x}$		
$\therefore \frac{dt}{dx} = e^{x}$	1	
$t = e^{x} + D$		
When $t=0$ $x=0$		
$\therefore D = -1$	1	
$\therefore t = e^{x} - 1$		
$t+1 = e^{x}$		
$\therefore x = \ln(t+1)$	1	
(iii) $v = \frac{dx}{dt} = \frac{1}{t+1}$	1	

Student Number: ....

1999
TRIAL EXAMINATION
MATHEMATICS
3 Unit (Additional)
and
3/4 Unit (Common)

Time Allowed - Two (2) hours
(plus 5 minutes reading time)

Directions to Candidates

- Attempt ALL questions
- Show all necessary working, marks may be deducted for careless or untidy work
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- Additional Answer Booklets are available

Directions to School or College

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All care has been taken to ensure that this examination paper is error free and that it follows the style, format and material content of previous Higher School Certificate Examination. Candidates are advised that authors of this examination paper cannot in any way guarantee that the 1999 HSC Examination will have a similar content or format.

Question One

(a) For the function $f(x)=e^{x+1}$ find the inverse function $f^{-1}(x)$ and hence show that $f[f^{-1}(x)]=f^{-1}[f(x)]=x$ *(3 marks)*

(b) Solve the inequality $\frac{1}{x+2}\geq\frac{2}{x-3}$ and represent the solution on a number line. *(3 marks)*

(c) If $\sum_{r=1}^{n}\frac{3}{2}(2)^{r-1}=766\frac{1}{2}$, find n *(3 marks)*

(d) The word **EQUATION** contains all five vowels. How many 3-letter 'words' consisting of at least 1 vowel and 1 consonant can be made from the letters of **EQUATION**? *(3 marks)*

Question Two

(a) Prove that $\frac{2\cos A}{\operatorname{cosec} A-2\sin A}=\tan 2A$ *(3 marks)*

(b) Show that $\tan^{-1}x=\sin^{-1}\frac{x}{\sqrt{1+x^2}}$ *(3 marks)*

(c) Write down the expansion for $\sin(\alpha-\beta)$ and hence prove that $\sin(-\theta)=-\sin\theta$ *(3 marks)*

(d) Find $\frac{d}{dx}\left[\frac{\ln x}{x}\right]$ and hence find the primitive function of $\frac{2-\ln x}{x^2}$ *(4 marks)*

Question Three

(a) The sides of a square sheet of cardboard are each 12m long. At each corner a square of $x^2\,\text{m}^2$ is cut away. The sides of the sheet are then turned up to form a box. Calculate:

(i) the values of x so that the box has a volume of 108m^3

(ii) the value of x so that the box has a maximum volume *(5 marks)*

(c) Use mathematical induction to show that for all positive integers n, $\sum_{r=1}^{n} a^{-r} = \frac{a^n - 1}{(a-1)a^n}$

(4 marks)

(d) A polynomial $P(x) = ax^3 + bx^2 + cx + d$ has zeroes at -2, 2 and $\frac{3}{2}$.
It leaves a remainder of 12 when divided by x - 1.
Find the values of a, b, c and d.

(3 marks)

Question Four

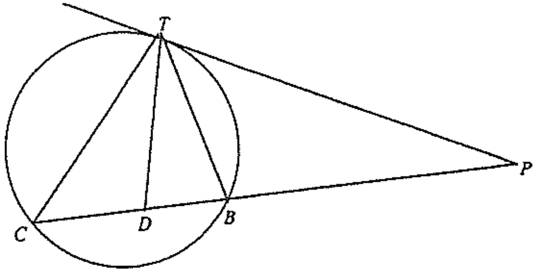
(a) The tangent at the point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ cuts the y-axis at T. The line through the focus S parallel to this tangent cuts the directrix at V. M is the mid-point of TV. Find the locus of M as P moves on the parabola.

(5 marks)

(b) If $3^x = 2^y = 6^z$, prove that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

(3 marks)

(c)

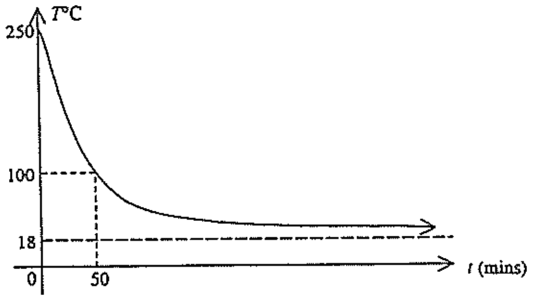


PT is a tangent to the circle and PBC is a secant. D is a point on PBC such that $TD = TB$.
Prove that $\angle CTD = \angle P$.

(4 marks)

Question Five

(a) The graph shown below shows the cooling curve for a container of paraffin oil which as been heated to a temperature of 250° then allowed to cool in air whose temperature is 18°C.



It is known that the rate at which the temperature T of the oil is changing is given by $\frac{dT}{dt} = k(T - M)$
where M is the temperature of the surrounding air and t is the time elapsed after cooling begins, in minutes.

(i) Show that $T = M + Ae^{kt}$ is a solution to the given equation.

(ii) Use the graph to write down the values of M and A.

(iii) Find the value of k to one significant figure if the temperature of the oil drops to 103.3°C in 50 minutes of cooling.

(6 marks)

(b) Write the equation $2\tan\theta - 3\sec\theta = -2\sqrt{3}$ in the form $a\sin\theta + b\cos\theta = c$, and then by expressing the left hand side as sine of a compound angle, solve the equation for $0 \le \theta \le 360°$

(6 marks)

Question Six

(a) A particle moving in simple harmonic motion, passes through the centre of oscillation O with a velocity of 5cm/s. If it has a period of π seconds, find

(i) the value of n

(ii) the amplitude of the motion

(iii) the time taken for the particle to first reach $x = 1.5$cm *(5 marks)*

(b) Express $\sec(\sin^{-1} x)$ in terms of x and hence write down $\int \sec(\sin^{-1} x)dx$ $(-1<x<1)$ *(2 marks)*

(c) The acceleration of a body moving in a straight line is given by

$$\frac{d^2x}{dt^2} = -\frac{24}{x^2}$$

where x is the displacement from the origin after t seconds. When $t = 0$ the body is 3 metres to the right of the origin with a velocity of 4m/s.

(i) Show that the velocity v of the body in terms of x, is

$$v = \frac{4\sqrt{3}}{\sqrt{x}}$$

(ii) Find an expression for t in terms of x

(iii) How long does it take for the body to reach a point 10m to the right of the origin? *(5 marks)*

Question Seven

(a) (i) Show that the range of flight of a projectile fired at an angle of α to the horizontal and at a velocity v is

$$\frac{v^2 \sin 2\alpha}{g}$$

where g is the acceleration due to gravity

(ii) A cannon fires a shell at an angle of 45° to the horizontal and strikes a point 50m beyond its target. When fired with the same velocity at an angle of 30° it hits a point 20m in front of the target. Calculate

(I) the distance of the target from the cannon

(II) the correct angles required to hit the target *(7 marks)*

(b) By considering the value of $(1+x)^{2n}$ when $x = 1$, prove that

$$\sum_{k=0}^{n} \binom{2n}{k} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$$

(5 marks)

SUGGESTED SOLUTIONS for 3 UNIT MATHS - 1999

Question One

(a)

$f(x) = y = e^{x+1}$

$x = e^{y+1}$

$y + 1 = \ln x$

$y = \ln x - 1$ the inverse function $f^{-1}(x)$

$$\begin{aligned} f[f^{-1}(x)] &= e^{\ln x - 1 + 1} \\ &= e^{\ln x} \\ &= x \end{aligned}$$

$$\begin{aligned} f^{-1}[f(x)] &= \ln[e^{x+1}] - 1 \\ &= (x+1)\ln e - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

(b)

$$\frac{1}{x+2} \geq \frac{2}{x-3}$$

$(x-3)^2(x+2) \geq 2(x-3)(x+2)^2$

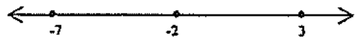
$(x-3)^2(x+2) - 2(x-3)(x+2)^2 \geq 0$

$(x-3)(x+2)[x-3-2(x+2)] \geq 0$

$(x-3)(x+2)[x-3-2x-4] \geq 0$

$(x-3)(x+2)(-x-7) \geq 0$ ....(1)

Consider $(x-3)(x+2)(-x-7) = 0$



Test $x = -10$, inequality (1) holds true

Test $x = -5$, inequality (1) does not hold true

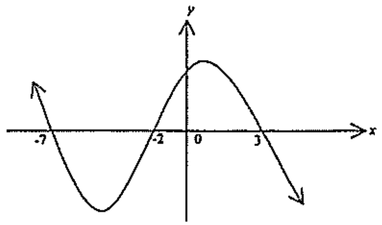
Test $x = 0$, inequality (1) holds true

Test $x = 4$, inequality (1) does not hold true

$\therefore$ Solution is $x \leq -7$ or $-2 < x < 3$

$(x \neq -2), (x \neq 3)$

From the graph of $y = (x-3)(x+2)(-x-7)$



Solution is $x \leq -7$ or $-2 < x < 3$

(c)

$$\sum_{r=1}^{n} \frac{3}{2}(2)^{r-1} = \frac{3}{2} + \frac{3}{2}(2) + \frac{3}{2}(2)^2 + \ldots + \frac{3}{2}(2)^n$$

$a = \frac{3}{2}; r = 2$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\begin{aligned} \therefore \quad \frac{\frac{3}{2}(2^n - 1)}{2 - 1} &= 766\tfrac{1}{2} \\ \tfrac{3}{2}(2^n - 1) &= \frac{1533}{2} \\ 3(2^n - 1) &= 1533 \\ 2^n - 1 &= 511 \\ 2^n &= 512 \\ n &= 9 \end{aligned}$$

(d)

There are 5 vowels and 3 consonants. For words with 2 vowels and 1 consonant the number of selections is ${}^5C_2 \times {}^3C_1 = 30$. For words with 1 vowel and 2 consonants the number of selections is ${}^5C_1 \times {}^3C_2 = 15$.

$\therefore$ The total number of selections = 45

But each selection may be arranged 3! ways.

$\therefore$ The total number of words $= 45 \times 3!$

$= 270$

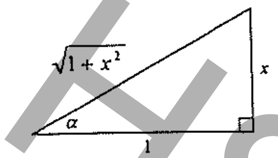
Question Two

(a)

$$\begin{aligned} \text{LHS} &= \frac{2\text{cosA}}{\text{cosecA} - 2\text{sinA}} \\ &= \frac{2\cos A}{\frac{1}{\sin A} - 2\sin A} \\ &= \frac{2\cos A}{\frac{1 - 2\sin^2 A}{\sin A}} \\ &= \frac{2\sin A\cos A}{1 - 2\sin^2 A} \\ &= \frac{\sin 2A}{\cos 2A} \\ &= \tan 2A \\ &= \text{RHS} \end{aligned}$$

(b) Let $\tan^{-1} x = \alpha$

$\therefore \tan\alpha = x$



From the triangle $\sin\alpha = \dfrac{x}{\sqrt{1+x^2}}$

$$\therefore \alpha = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\therefore \tan^{-1} x = \frac{\sin^{-1} x}{\sqrt{1+x^2}}$$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Put $\alpha = 0$ and $\beta = \theta$

$\therefore$

$$\begin{aligned}\sin(-\theta) &= \sin 0\cos\theta - \cos 0\sin\theta\\ &= 0-(1)\sin\theta\\ &= -\sin\theta\end{aligned}$$

(d)

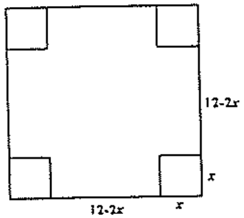
$$\begin{aligned}{}^{d}\!/_{dx}\left(\frac{\ln x}{x}\right) &= \frac{x\cdot\frac{1}{x} - \ln x(1)}{x^2}\\ &= \frac{1-\ln x}{x^2}\end{aligned}$$

$\therefore$

$$\begin{aligned}\int\left(\frac{2-\ln x}{x^2}\right)dx &= \int\left(\frac{1+1-\ln x}{x^2}\right)dx\\ &= \int\left(\frac{1}{x^2}+\frac{1-\ln x}{x^2}\right)dx\\ &= \int x^{-2}dx + \int\left(\frac{1-\ln x}{x^2}\right)dx\\ &= \frac{x^{-1}}{-1}+\frac{\ln x}{x}+c\\ &= \frac{\ln x}{x}-\frac{1}{x}+c\end{aligned}$$

Question Three

(a) (i)



$$\begin{aligned}V &= x(12-2x)^2\\ &= 144x - 48x^2 + 4x^3\end{aligned}$$

$\therefore 4x^3 - 48x^2 + 144x = 108$

$x^3 - 12x^2 + 36x - 27 = 0$

Let $f(x) = x^3 - 12x^2 + 36x - 27$

$$\begin{aligned}f(3) &= 27-108+108-27\\ &= 0\end{aligned}$$

$\therefore x-3$ is a factor

$$\begin{array}{r l}
 & x^2 - 9x + 9\\
x-3 & \overline{)\,x^3 - 12x^2 + 36x - 27}\\
 & \underline{x^3 - 3x^2}\\
 & -9x^2 + 36x\\
 & \underline{-9x^2 + 27x}\\
 & 9x - 27\\
 & \underline{9x - 27}\\
 & 0
\end{array}$$

$\therefore (x-3)(x^2-9x+9) = 0$

$x=3$ or

$$\begin{aligned}x &= \frac{9 \pm \sqrt{81-4(1)(9)}}{2}\\ &= \frac{9\pm\sqrt{81-36}}{2}\\ &= \frac{9\pm\sqrt{45}}{2}\\ &= \frac{9\pm3\sqrt{5}}{2}\end{aligned}$$

$x = \dfrac{9+3\sqrt{5}}{2}$ is rejected since it is >12

$\therefore$ Values of x so that box has a volume of 108m^3 are

3m and $\frac{9-3\sqrt{5}}{2}$m ($\approx$1.15m)

(ii) $V = 4x^3 - 48x^2 + 144x$

$dV/dx = 12x^2 - 96x + 144 = 0$
for maximum/minimum

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

reject $x = 6$; $x = 2$

$d^2V/dx^2 = 24x - 96$

When $x = 2$, $\quad d^2V/dx^2 = 48 - 96$

$= -48 < 0$

$\therefore$ maximum

$\therefore$ the box has a maximum volume when $x = 2$m

(b) $\sum_{r=1}^{n} a^{-r} = \frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^n}$

We are required to prove that

$\frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^n} = \frac{a^n - 1}{(a-1)a^n}$ for all positive integers n

Test $n = 1$:

$LHS = \frac{1}{a}$

$RHS = \frac{a-1}{(a-1)a}$

$= \frac{1}{a}$

$\therefore$ The result is true for $n = 1$

Assume that it is true for $n = k$

$\frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^k} = \frac{a^k - 1}{(a-1)a^k}$

Put $n = k + 1$

$LHS = \frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^k} + \frac{1}{a^{k+1}}$

$= \frac{a^k - 1}{(a-1)a^k} + \frac{1}{a^{k+1}}$

$= \frac{a(a^k - 1) + (a-1)}{(a-1)a^{k+1}}$

$= \frac{a^{k-1} - a + a - 1}{(a-1)a^{k+1}}$

$= \frac{a^{k+1} - 1}{(a-1)a^{k+1}}$

Hence if the result is true for $n = k$, it is true for $n = k + 1$. But it is true for $n = 1$. Therefore it is true for $n = 2$ and so, by mathematical induction, for all positive integers n.

(c) Since $P(x)$ has zeros at -2, 2 and $\frac{3}{2}$, $P(x)$ may be expressed in the form

$P(x) = k(x+2)(x-2)(2x-3)$ where k is a real number.

Now $P(1) = k(3)(-1)(-1) = 12$

$\therefore k = 4$

$\therefore P(x) = 4(x+2)(x-2)(2x-3)$

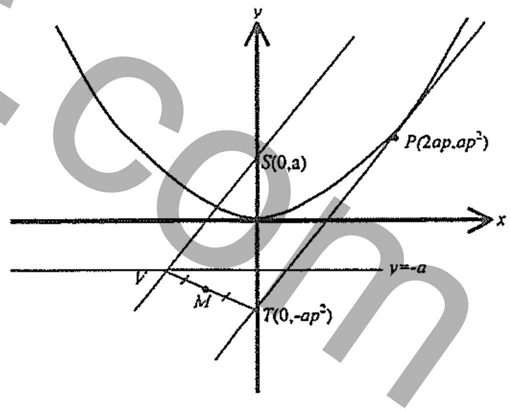
$= (x^2 - 4)(8x - 12)$

$= 8x^3 - 12x^2 - 32x + 48$

$\therefore a = 8, b = -12, c = -32, d = 48$

Question Four

(a)



Gradient of tangent at $P = p$

Equation of tangent at P: $\quad y = px - ap^2$

$\therefore$ It cuts the y-axis at $-ap^2$

$\therefore$ $T(0,-ap^2)$

Equation of VS: $y = px + a$

y-coordinate of $V = -a$

$\therefore$ $-a = px_v + a$

$x_v = \frac{-2a}{p}$

$\therefore$ $V\left(\frac{-2a}{p},-a\right)$

$\therefore$ $x_m = \frac{\frac{-2a}{p}+0}{2}$ $y_m = \frac{-ap^2-a}{2}$

$M\left(\frac{-a}{p},\frac{-ap^2-a}{2}\right)$

$x = \frac{-a}{p}$ $\therefore p = \frac{-a}{x}$

$y = \frac{-ap^2-a}{2} = \frac{-a\left(\frac{-a}{x}\right)^2-a}{2}$

$\therefore$ $y = \frac{-a^3}{2x^2} - \frac{a}{2}$ which is the locus of M.

(b) $3^x = 6^z$

$\therefore x\log 3 = z\log 6$

Similarly $y\log 2 = z\log 6$

so that

$x = \frac{z\log 6}{\log 3}$

$y = \frac{z\log 6}{\log 2}$

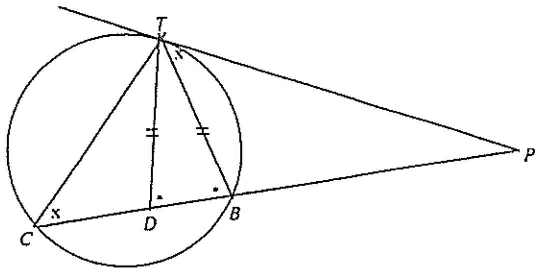
$\therefore \frac{1}{x} + \frac{1}{y} = \frac{\log 3}{z\log 6} + \frac{\log 2}{z\log 6}$

$= \frac{\log 3 + \log 2}{z\log 6}$

$= \frac{\log 6}{z\log 6}$

$= \frac{1}{z}$

(c)



$\angle TDB = \angle C + \angle CTD$ (exterior $\angle$ of ΔCTD)

and $\angle TBD = \angle P + \angle PTB$ (exterior $\angle$ of ΔPTB)

But $\angle TDB = \angle TBD$ (base $\angle$s of isosceles ΔTBD)

$\therefore$ $\angle C + \angle CTD = \angle P + \angle PTB$

But $\angle PTB = \angle C$ ($\angle$ between tangent and chord)

$\therefore$ $\angle CTD = \angle P$

Question Five

(a) (i) $T = M + Ae^{kt}$

$\frac{dT}{dt} = k \cdot Ae^{kt}$

$= k(T - M)$

$\therefore$ $T = M + Ae^{kt}$ is a solution to the equation

(ii) $M = 18°C$

$A = 250 - 18 = 232°C$

(iii) $T = 18 + 232e^{kt}$

When $t = 50$ mins and $T = 103.3°C$

$103.3 = 18 + 232e^{50k}$

$\therefore 232e^{50k} = 85.3$

$e^{50k} = \frac{85.3}{232}$

$50k = \ln\frac{85.3}{232}$

$k = \frac{\ln\frac{85.3}{232}}{50}$

≈ -0.002 (one significant figure)

(b)

$2\tan\theta - 3\sec\theta = -2\sqrt{3}$

$\frac{2\sin\theta}{\cos\theta} - \frac{3}{\cos\theta} = -2\sqrt{3}$

$2\sin\theta - 3 = -2\sqrt{3}\cos\theta$

ie. $2\sin\theta + 2\sqrt{3}\cos\theta = 3$

Let $2\sin\theta + 2\sqrt{3}\cos\theta = r\sin(\theta+\alpha)$

where $r = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$

and $\tan\alpha = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$\therefore 4\sin(\theta + \frac{\pi}{3}) = 3$

$\sin(\theta + \frac{\pi}{3}) = \frac{3}{4}$

$\therefore\ \theta + 60° = \underset{reject}{48°35'}, 180° - 48°35', 360° + 48°35'$

$\theta = 71°25'$ or $348°35'$

Question Six

(a) (i)

$T = \frac{2\pi}{n}$

$\pi = \frac{2\pi}{n}$

$n = 2$

(ii) When $t = 0, x = 0$ and $v = 5$cm/s

$v^2 = n^2(a^2 - x^2)$

$25 = 4(a^2 - 0)$

$a^2 = \frac{25}{4}$

$a = \frac{5}{2}$cm

(iii) Let $x = a\sin(nt + \alpha)$ describe the SHM

When $t = 0, x = 0$ so that $\alpha = 0$

$\therefore x = a\sin nt$

$x = \frac{5}{2}\sin 2t$

When $x = 1.5$cm

$\frac{3}{2} = \frac{5}{2}\sin 2t$

$\sin 2t = 0.6$

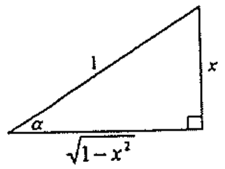
$2t = 0.64$

$t = 0.32$s.

(b) Let $\sec(\sin^{-1} x) = y$

Put $\sin^{-1} x = \alpha$

then $\sin\alpha = x$ $(-1 < x < 1)$



$\therefore y = \sec\alpha$

$= \frac{1}{\sqrt{1-x^2}}$

$\therefore \int \sec(\sin^{-1} x)dx = \int \frac{1}{1-x^2}dx$

$= \sin^{-1} x + c$

(c) (i)

$v^2 = 2\int \ddot{x}dx$

$= 2\int \frac{-24}{x^2}dx$

$= -48\int x^{-2}dx$

$= 48x^{-1} + c$

$v^2 = \frac{48}{x} + c$

When $t = 0$, $v = 4$ and $x = 3$

$\therefore\ 16 = \frac{48}{3} + c$

$\therefore\ c = 0$

$\therefore\ v^2 = \frac{48}{x}$

$v = \sqrt{\frac{48}{x}}$

$= \frac{4\sqrt{3}}{\sqrt{x}}$

(ii)
$$v = \frac{dx}{dt}$$
$$= \frac{4\sqrt{3}}{\sqrt{x}}$$
$$dt/_{dx} = \frac{\sqrt{x}}{4\sqrt{3}}$$
$$t = \int \frac{\sqrt{x}}{4\sqrt{3}} dx$$
$$= \frac{1}{4\sqrt{3}} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \frac{1}{6\sqrt{3}} x^{\frac{3}{2}} + c$$
When $t = 0, x = 3$m
$$\therefore 0 = \frac{3\sqrt{3}}{6\sqrt{3}} + c$$
$$\therefore c = -\tfrac{1}{2}$$
$$\therefore t = \frac{x^{\frac{3}{2}}}{6\sqrt{3}} - \frac{1}{2}$$

(iii) When $x = 10$
$$t = \frac{10^{\frac{3}{2}}}{6\sqrt{3}} - \frac{1}{2}$$
$$\approx 2.5\text{sec}$$

Question Seven

(a) (i) Starting with
$$y = vt\sin\alpha - \tfrac{1}{2}gt^2$$
$$x = vt\cos\alpha$$
When $y = 0$,
$$vt\sin\alpha - \tfrac{1}{2}gt^2 = 0$$
$$t(v\sin\alpha - \tfrac{1}{2}gt) = 0$$
$$t = 0 \text{ or } t = \frac{2v\sin\alpha}{g}$$

$$\therefore \quad x = \frac{v(2v\sin\alpha)\cos\alpha}{g}$$
$$= \frac{2v^2\sin\alpha\cos\alpha}{g}$$
I.e. $Range = \frac{v^2\sin 2\alpha}{g}$

(ii) (I) Let T = distance from cannon to target:

When $\alpha = 45°$, $T + 50 = \frac{v^2\sin 90°}{g}$
$$\therefore T + 50 = \frac{v^2}{g} \text{..............(1)}$$
When $\alpha = 30°$, $T - 20 = \frac{v^2\sin 60°}{g}$
$$T - 20 = \frac{v^2\sqrt{3}}{2g} \text{...........(2)}$$

(1) ÷ (2):
$$\frac{T+50}{T-20} = \frac{2}{\sqrt{3}}$$
$$\sqrt{3}(T+50) = 2(T-20)$$
$$\sqrt{3}T + 50\sqrt{3} = 2T - 40$$
$$T(2-\sqrt{3}) = 40 + 50\sqrt{3}$$
$$T = \frac{40+50\sqrt{3}}{2-\sqrt{3}}$$
$$= 472.5\text{m}$$

(II)
$$T + 50 = \frac{v^2}{g} \text{.........(1)}$$
$$T = \frac{v^2\sin 2\alpha}{g} \text{........(2)}$$
Substitute $T = 472.5$,
$$522.5 = \frac{v^2}{g}$$
$$472.5 = \frac{v^2\sin 2\alpha}{g}$$
$$\frac{472.5}{522.5} = \sin 2\alpha$$
$\therefore \quad 2\alpha = 64°44'$ or $115°16'$

$\therefore \quad \alpha = 32°22'$ or $57°38'$

(b)

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \dots + \binom{2n}{n}x^n + \dots + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}$$

Putting $x = 1$,

$$(2)^{2n} = \binom{2n}{0} + \binom{2n}{1} + \dots + \binom{2n}{n} + \dots + \binom{2n}{2n-1} + \binom{2n}{2n}$$

$$\begin{aligned}(2)^{2n} &= \binom{2n}{0} + \binom{2n}{2n} + \binom{2n}{1} + \binom{2n}{2n-1} \dots + \binom{2n}{n} + \binom{2n}{n} - \binom{2n}{n} \\ &= 2\left[\binom{2n}{0} + \binom{2n}{1} + \dots + \binom{2n}{n}\right] - \binom{2n}{n} \\ &= 2\sum_{k=0}^{n}\binom{2n}{k} - \binom{2n}{n}\end{aligned}$$

$$\therefore \ \frac{2^{2n}}{2} = \sum_{k=0}^{n}\binom{2n}{k} - \tfrac{1}{2}\binom{2n}{n}$$

$$2^{2n-1} = \sum_{k=0}^{n}\binom{2n}{k} - \frac{1}{2}\frac{(2n)!}{n!(2n-n)!}$$

$$\therefore \ \ 2^{2n-1} = \sum_{k=0}^{n}\binom{2n}{k} - \frac{(2n)!}{2(n!)^2}$$

ie. $\displaystyle\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$