Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Afternoon Session
Thursday, 20 August 2009

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value

6300 - 1

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\lim_{x \to 0} \frac{\sin 3x}{x}$. **1**

(b) Use the substitution $u = 3x - 1$ to evaluate $\int_1^2 \frac{x}{3x-1}\, dx$. **3**

(c) Find all real numbers such that $\ln(2x+3) + \ln(x-2) = 2\ln(x+4)$. **4**

(d) (i) From a group of 7 girls and 6 boys, 3 girls and 2 boys are chosen. **2**

How many different groups of 5 are possible?

(ii) If the group of 5 stands in a line what is the probability that the boys stand together? **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Solve the inequality $\frac{x^2-4}{x+3} < x-4$ for x. **3**

(b) Prove by Mathematical Induction that $3^{3n} + 2^{n+2}$ is divisible by 5, for all positive integers n. **4**

(c) A particle, P, moves on the x-axis for time $t \geq 0$, in seconds, with velocity $v = \frac{2}{1+3x}$ cms^{-1}, where x, in centimetres, is the displacement from the origin $x = 0$.

(i) Find an expression for the acceleration, a cms^{-2}, and show that a varies directly with v^3. **3**

(ii) If the particle was initially at the origin, describe the motion both initially and as $t \to \infty$. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) The function $f(x) = e^x - x - 2$ has a zero near $x = 1.2$. **2**

Use one application of Newton's method to find a second approximation to the zero. Write your answer correct to three significant figures.

(b) A function is defined by $f(x) = e^{3x} - 1$ for all real x.

(i) Draw the graph of $y = f(x)$ and state the range of the function. **2**

(ii) Find the inverse function, $f^{-1}(x)$, clearly indicating any restrictions. **3**

(c) A particle moves in a straight line so that its displacement x cm from the origin at time $t \geq 0$, in seconds, is given by $x = \sqrt{3}\cos 3t - \sin 3t$.

(i) Show that the particle moves in simple harmonic motion. **2**

(ii) Find the velocity when the particle is 1 cm from the origin on its first oscillation. **3**

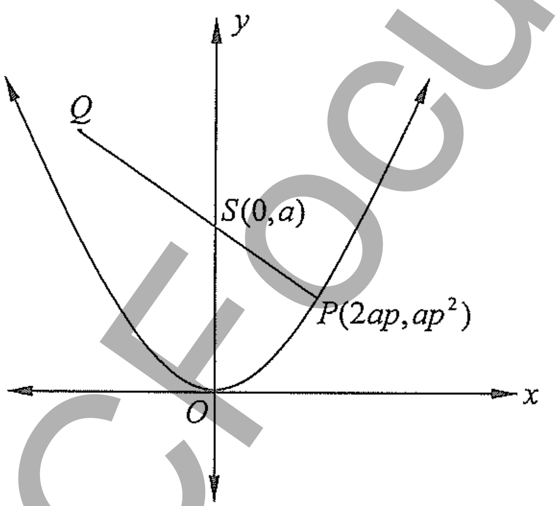
Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) If the roots of $x^3 - 6x^2 + 3x + k = 0$ are consecutive terms of an arithmetic series show that one of the roots is 2. **2**

(ii) Hence find the value of k and the other two roots. **3**

(b) Show that $\dfrac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} = \tan^2\theta$. **3**

(c) $P(2ap,\ ap^2)$ is a point on the parabola $x^2 = 4ay$ with focus $S(0,\ a)$. The point Q lies on PS produced and Q divides PS so that $PQ:QS = -4:3$.



NOT TO SCALE

(i) Show that Q has coordinates $\left(-6ap,\ a(4-3p^2)\right)$. **2**

(ii) Show that as P varies, the locus of Q is a parabola. **2**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Simplify $\dfrac{2^{4n} \times 3^{2n}}{8^n \times 6^n} + 3^n$. **3**

(b) A balloon in the shape of a cylinder, with height h and radius r, expands so that h is always proportional to r, that is $h = kr$ for some constant k.
When $r = 4\,\text{cm}$, the volume is expanding at the rate of $0.2\,\text{cm}^3\,\text{s}^{-1}$.

(i) Show that when $r = 4\,\text{cm}$ the rate of change of the radius is given by $\dfrac{dr}{dt} = \dfrac{1}{240\pi k}$. **2**

(ii) If the surface area of the balloon is expanding at the rate of $0.1\,\text{cm}^2\,\text{s}^{-1}$ when $r = 4\,\text{cm}$, find the constant of proportionality, k. **3**

(c) (i) Differentiate both sides of the expansion $(1+x)^{2n} = \sum_{k=0}^{2n} {}^{2n}C_k\, x^k$. **2**

(ii) Hence show that $\sum_{k=1}^{2n} k\, {}^{2n}C_k = n \times 4^n$. **2**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

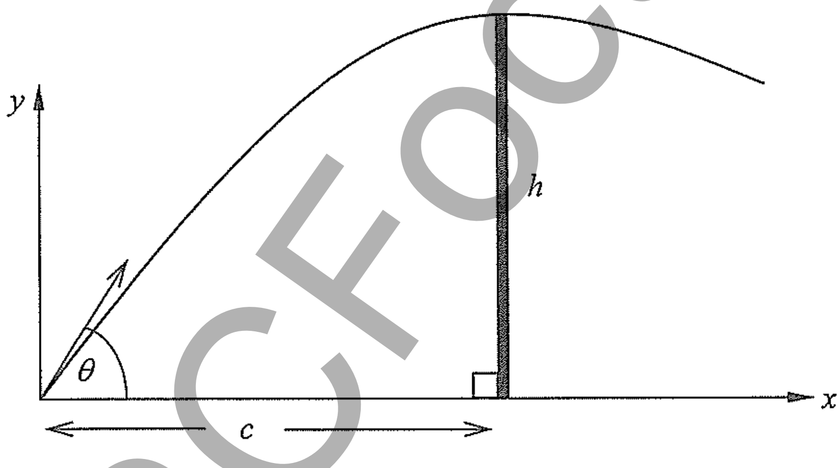
(a) A student is taking a test with 50 multiple-choice questions and guesses the answer to each one. The probability of guessing a question correctly is 0.3.

(i) What is the probability that the student answers 25 questions correctly? **2**

(ii) What is the most likely number of questions answered correctly? **3**

(b) A vertical wall, height h metres, stands on horizontal ground. When a projectile is fired, in a vertical plane which is at right angles to the wall, from a point on the ground c metres from the wall, it just clears the wall at the highest point of its path. The equations of motion for the projectile with angle of projection, θ, are:

$x = Vt\cos\theta \qquad y = Vt\sin\theta - \frac{1}{2}gt^2$ (Do <u>not</u> prove these.)



NOT TO SCALE

(i) Show that the particle reaches the highest point on its path when $t = \dfrac{V\sin\theta}{g}$. **2**

(ii) Show that the speed of projection is given by $V^2 = \dfrac{g}{2h}\left(4h^2 + c^2\right)$. **3**

(iii) Find the angle of projection, θ, in terms of h and c. **2**

End of paper

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Examiners

Carolyn Gavel (convenor)	Kambala, Rose Bay
Cynthia Athayde	St John Bosco College, Engadine
Joe Grabowski	Freeman Catholic College, Bonnyrigg
Anne Hastings	Kambala, Rose Bay
Br Domenic Xuereb fsp	Patrician Brothers' College, Fairfield

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = \frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2009 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 1

Question 1 (12 marks)
(a) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• applies the Remainder Theorem or equivalent progress towards solution	1
• finds correct remainder	1

Sample Answer:

$P(x) = x^3 - 3x^2 + 3x - 5$

By the Remainder Theorem $P(2) =$ remainder

$\therefore$ remainder $= 8 - 12 + 6 - 5$

$= -3$

OR

Correct division of polynomial.

(b) (2 marks)
Outcomes assessed: HE6, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• correct trigonometric substitution in integral	1
• finds a correct primitive (+C not necessary)	1

Sample Answer:

$$\int \sin^2 6x \, dx = \frac{1}{2}\int (1 - \cos 12x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{12}\sin 12x\right) + C$$

$$= \frac{x}{2} - \frac{\sin 12x}{24} + C$$

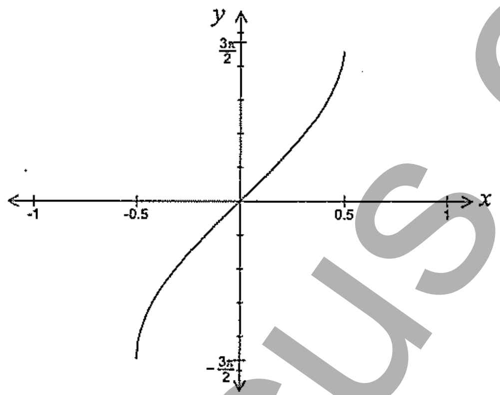
(c) (3 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• draws correctly shaped graph	1
• identifies correct domain	1
• identifies correct range	1

Sample Answer:

$$y = 3\sin^{-1}(2x)$$

domain: $\frac{-1}{2} \le x \le \frac{1}{2}$ range: $\frac{-3\pi}{2} \le y \le \frac{3\pi}{2}$



(d) (i) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses correct trigonometric identity	1
• substitutes correctly and determines correct equation	1

Sample Answer:

$x = \cos t$

$y = 3 + \sin t \quad \Rightarrow \quad \sin t = y - 3$

substitute into $\cos^2 t + \sin^2 t = 1$

$x^2 + (y-3)^2 = 1$

(d) (ii) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• correctly describes locus	1

Sample Answer:

Geometrically the locus is a circle with centre $(0,\ 3)$ and radius 1.

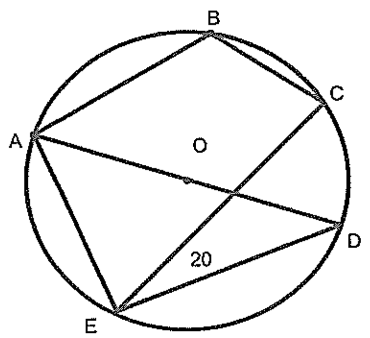
(e) (2 marks)

Outcomes assessed: PE2, PE3

Targeted Performance Bands: E2-E3

Criteria	Marks
• finds $\angle AED$, giving correct reason	1
• finds $\angle ABC$, giving correct reason	1

Sample Answer:



$\angle AED = 90^0$ (angle in a semicircle, AD is a diameter)

$\therefore \angle AEC = 70^0$

$\angle ABC = 110^0$ (opposite angles of cyclic quadrilateral $ABCE$ are supplementary)

Question 2 (12 marks)

(a) (1 mark)

Outcomes assessed: PE2

Targeted Performance Bands: E2-E3

Criteria	Marks
• gives correct result	1

Sample Answer:

$$\lim_{x\to 0}\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x}$$

$$= 3\times 1 \qquad \text{using } \lim_{x\to 0}\tfrac{\sin x}{x} = 1$$

$$= 3$$

(b) (3 marks)

Outcomes assessed: HE6

Targeted Performance Bands: E2-E3

Criteria	Mark
• rewrites the integral using the substitution	1
• finds the new limits	1
• evaluates the integral correctly (correct numerical equivalence)	1

Sample Answer:

$$\int_1^2 \frac{x}{3x-1}dx = \frac{1}{9}\int_1^2 \frac{3x}{3x-1}\times 3dx$$
$$= \frac{1}{9}\int_2^5 \frac{u+1}{u}\,du$$
$$= \frac{1}{9}\int_2^5 \left(1+\frac{1}{u}\right)du$$
$$= \frac{1}{9}\left[u+\ln u\right]_2^5$$
$$= \frac{1}{9}\left[5+\ln 5-(2+\ln 2)\right]$$
$$= \frac{1}{9}\left(3+\ln\tfrac{5}{2}\right)$$
$$= \tfrac{1}{3}+\tfrac{1}{9}\ln\tfrac{5}{2}$$

$u = 3x-1 \qquad 3x = u+1$

$\frac{du}{dx} = 3$

Limits $\quad x = 2 \Rightarrow u = 5$

$\qquad x = 1 \Rightarrow u = 2$

(c) (4 marks)

Outcomes assessed: HE7

Targeted Performance Bands: E2-E3

Criteria	Mark
• uses logarithmic laws	1
• establishes the quadratic equation	1
• solves the quadratic equation	1
• gives correct solution	1

Sample Answer

$$\ln(2x+3)+\ln(x-2) = 2\ln(x+4)$$
$$\ln(2x+3)(x-2) = \ln(x+4)^2$$
$$2x^2-x-6 = x^2+8x+16$$
$$x^2-9x-22 = 0$$
$$(x+2)(x-11) = 0$$
$$\therefore x = -2 \text{ or } x = 11$$

for valid solutions $x > 2$

but $x = -2$ is not valid $\therefore x = 11$ is the only solution

(d) (i) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Mark
• uses combinations correctly or significant progress towards answer	1
• gives correct answer	1

Sample Answer:

Girls can be selected in ${}^7C_3 = 35$ ways

Boys can be selected in ${}^6C_2 = 15$ ways

There are ${}^7C_3 \times {}^6C_2 = 525$ groups of 5.

(d) (ii) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• calculates the number of ways that the boys can stand together	1
• finds the correct probability	1

Sample Answer:

If the boys stand together then there are $2! = 2$ ways to arrange themselves.
In the line there are 3 girls and the group of boys to be arranged $\Rightarrow 4! = 24$ arrangements.
$\therefore 2! \times 4! = 48$ ways of the boys standing together in the line.

If no restrictions the 5 can be arranged in $5! = 120$ ways in a line.

$$\text{P(boys stand together)} = \frac{48}{120} = \frac{2}{5}.$$

Question 3 (12 marks)
(a) (3 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct quadratic or other correct significant step towards solution	1
• further significant step towards solution	1
• finds solution	1

Sample Answer:

$$\frac{x^2-4}{x+3} < x-4 \qquad \times (x+3)^2 \qquad x \neq -3$$

$$(x+3)(x^2-4) < (x-4)(x+3)^2$$

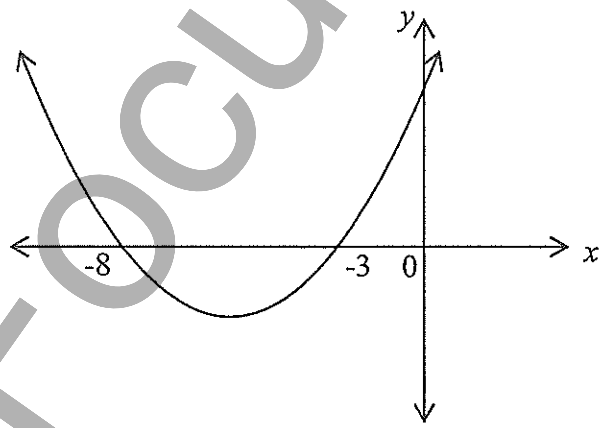
$$(x+3)(x^2-4) - (x-4)(x+3)^2 < 0$$

$$(x+3)(x^2-4-(x-4)(x+3)) < 0$$

$$(x+3)(x^2-4-(x^2-x-12)) < 0$$

$$(x+3)(x+8) < 0$$

$$-8 < x < -3$$



(b) (4 marks)
Outcomes assessed: HE2
Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes the truth of $S(1)$	1
• establishes the result for $S(k)$	1
• substitutes result in $S(k+1)$	1
• deduces the required result	1

Sample Answer:

Let $S(n)$ be the statement $3^{3n}+2^{n+2}$ is divisible by 5

Consider $S(1)$: $3^3+2^3=35$ which is divisible by 5.
Hence $S(1)$ is true

If $S(k)$ is true: $3^{3k}+2^{k+2}=5M$ where M is an integer *

RTP $S(k+1)$ is true i.e. prove $3^{3(k+1)}+2^{(k+1)+2}=5Q$ where Q is an integer

$$\begin{aligned}
LHS &= 3^{3k+3}+2^{k+3}\\
&= 3^3\times 3^{3k}+2\times 2^{k+2}\\
&= 27(5M-2^{k+2})+2\times 2^{k+2} \qquad \text{if } S(k) \text{ is true using } *\\
&= 27\times 5M-27\times 2^{k+2}+2\times 2^{k+2}\\
&= 5\times 27M-25\times 2^{k+2}\\
&= 5(27M-5\times 2^{k+2})\\
&= 5Q \text{ where } Q \text{ is an integer since } M \text{ and } k \text{ are integers}
\end{aligned}$$

Hence if $S(k)$ then $S(k+1)$ is true. Thus since $S(1)$ is true it follows by induction that $S(n)$ is true for positive integral n.

OR

$$\begin{aligned}
LHS &= 3^{3k+3}+2^{k+3}\\
&= 3^3\times 3^{3k}+2\times 2^{k+2}\\
&= 25\times 3^{3k}+2\times 3^{3k}+2\times 2^{k+2}\\
&= 25\times 3^{3k}+2(3^{3k}+2^{k+2})\\
&= 25\times 3^{3k}+2\times 5M \qquad \text{if } S(k) \text{ is true using } *\\
&= 5(5\times 3^{3k}+2M)\\
&= 5Q \text{ where } Q \text{ is an integer since } M \text{ and } k \text{ are integers}
\end{aligned}$$

Conclusion as above

(c) (i) (3 marks)

Outcomes assessed: HE5

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards correct differentiation	1
• finds a correct expression for acceleration	1
• shows correct relationship	1

Sample Answer:

$$v = \frac{2}{1+3x}$$

$$\frac{1}{2}v^2 = \frac{1}{2}\frac{4}{(1+3x)^2}$$

$$= 2(1+3x)^{-2}$$

Now

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= 2\times -2(1+3x)^{-3}\times 3$$

$$= \frac{-12}{(1+3x)^3}$$

$$= -12\times\frac{8}{(1+3x)^3}\times\frac{1}{8}$$

$$= -\frac{12}{8}v^3$$

$$= -\frac{3}{2}v^3$$

$\therefore a$ varies directly as v^3

(c) (ii) (2 marks)

Outcomes assessed: HE7

Targeted Performance Bands: E2-E3

Criteria	Marks
• describes initial motion	1
• describes motion as $t \to \infty$	1

Sample Answer:

Initially $v = 2$ cms^{-1} $\therefore$ the particle moves in a positive direction from the origin.

As t increases, x increases and v decreases.
As $t \to \infty$, the particle continues in a positive direction with $v \to 0$.

Question 4 (12 marks)
(a) (2 marks)
Outcomes assessed: PE3, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• progress towards solution	1
• finds correct approximation (correct numerical equivalence)	1

Sample Answer:

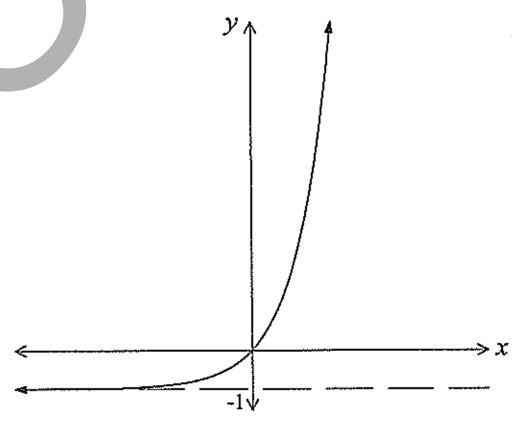
$$f(x) = e^x - x - 2$$
$$\therefore f'(x) = e^x - 1$$
Let $x_1 = 1.2$
$$f(x_1) = e^{1.2} - 1.2 - 2 = 0.1201169...$$
$$f'(x_1) = e^{1.2} - 1 = 2.3201169...$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 1.2 - \frac{0.1201169...}{2.3201169...}$$
$$= 1.14822...$$
$$= 1.15$$

(b) (i) (2 marks)
Outcomes assessed: PE6
Targeted Performance Bands: E2-E3

Criteria	Marks
• draws correct graph	1
• states correct range	1

Sample Answer:

$y = e^{3x} - 1$
Range: $y > -1$



(b) (ii) (3 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• interchanges variables or progress towards solution	1
• changes subject of equation or further progress towards solution	1
• states inverse function with correct restriction	1

Sample Answer:

$y = e^{3x} - 1$

Swap x and y

$x = e^{3y} - 1$

$e^{3y} = x + 1$

$3y = \ln(x+1)$

$y = \frac{1}{3}\ln(x+1)$

$f^{-1}(x) = \frac{1}{3}\ln(x+1), \ \ x > -1$

(c) (i) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• differentiates correctly	1
• shows motion is simple harmonic	1

Sample Answer:

$$\begin{aligned}
x &= \sqrt{3}\cos 3t - \sin 3t \\
v &= \frac{dx}{dt} \\
&= -3\sqrt{3}\sin 3t - 3\cos 3t \\
a &= \frac{dv}{dt} \\
&= -9\sqrt{3}\cos 3t + 9\sin 3t \\
&= -9(\sqrt{3}\cos 3t - \sin 3t) \\
&= -9x
\end{aligned}$$

which is of the form $a = -n^2 x$ where $n = 3$

$\therefore$ motion is simple harmonic

(c) (ii) (3 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes result using auxiliary angle or other progress toward solution	1
• solves correctly for time	1
• finds correct velocity (correct numerical equivalence)	1

Sample Answer:

when $x = 1$, $\sqrt{3}\cos 3t - \sin 3t = 1$

Let $\sqrt{3}\cos 3t - \sin 3t = R\cos(3t + \alpha)$

$R\cos(3t + \alpha) = R\cos 3t \cos\alpha - R\sin 3t \sin\alpha$
$\therefore R\cos\alpha = \sqrt{3}$
$R\sin\alpha = 1$
i.e. $\tan\alpha = \dfrac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha = \dfrac{\pi}{6}$
$R^2 = 1 + 3 \quad \Rightarrow \quad R = 2$
$\sqrt{3}\cos 3t - \sin 3t = 2\cos\left(3t + \dfrac{\pi}{6}\right)$

i.e. solve $2\cos\left(3t + \dfrac{\pi}{6}\right) = 1$

$\cos\left(3t + \dfrac{\pi}{6}\right) = \dfrac{1}{2}$

$3t + \dfrac{\pi}{6} = \dfrac{\pi}{3}$ (first oscillation)

$t = \dfrac{\pi}{18}$ seconds

When $t = \dfrac{\pi}{18}$ $\quad v = -3\sqrt{3}\sin\dfrac{\pi}{6} - 3\cos\dfrac{\pi}{6}$
$= -3\sqrt{3} \times \dfrac{1}{2} - 3 \times \dfrac{\sqrt{3}}{2}$
$= -3\sqrt{3}$ cms^{-1}

Question 5 (12 marks)
(a) (i) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• defines roots in arithmetic series	1
• uses sum of roots to show result	1

Sample Answer:

Let the roots be $\alpha - d,\ \alpha$ and $\alpha + d$

$x^3 - 6x^2 + 3x + k = 0$

sum of roots $= \dfrac{-b}{a} = 6$

Also sum of roots $= \alpha - d + \alpha + \alpha + d = 3\alpha$

$\therefore 3\alpha = 6$

$\alpha = 2$

i.e. one of the roots is 2

(a) (ii) (3 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Mark
• finds correct value for k	1
• progress toward solution	1
• finds correct roots	1

Sample Answer:

Since one root is 2 substitute into equation to find k.

$2^3 - 6 \times 2^2 + 3 \times 2 + k = 0$

$\therefore k = 10$

i.e. equation is $x^3 - 6x^2 + 3x + 10 = 0$

product of roots $= \dfrac{-d}{a} = -10$

product of roots $= \alpha(\alpha - d)(\alpha + d)$ from (i)

$= \alpha(\alpha^2 - d^2)$

$\therefore -10 = 2 \times (2^2 - d^2)$

$-5 = 4 - d^2$

$d^2 = 9$

$d = \pm 3$

$\therefore$ roots are $-1, 2, 5$

(b) (3 marks)

Outcomes assessed: PE2

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct t-formula or other progress towards result	1
• significant progress toward the result	1
• completes the proof	1

***Sample Answer*:**

$$\text{Let } t = \tan\theta,\ \therefore \tan 2\theta = \frac{2t}{1-t^2}$$

$$\begin{aligned}
\text{LHS} &= \frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} \\
&= \left(\frac{2t}{1-t^2} - t\right) \div \left(\frac{2t}{1-t^2} + \frac{1}{t}\right) \\
&= \frac{2t - t + t^3}{1-t^2} \div \left(\frac{2t^2 + 1 - t^2}{t(1-t^2)}\right) \\
&= \frac{t(1+t^2)}{1-t^2} \times \frac{t(1-t^2)}{t^2+1} \\
&= t^2 \\
&= \tan^2\theta \\
&= \text{RHS}
\end{aligned}$$

OR

$$\begin{aligned}
\text{LHS} &= \frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} \\
&= \left(\frac{2\tan\theta}{1-\tan^2\theta} - \tan\theta\right) \div \left(\frac{2\tan\theta}{1-\tan^2\theta} + \frac{1}{\tan\theta}\right) \\
&= \left(\frac{2\tan\theta - \tan\theta + \tan^3\theta}{1-\tan^2\theta}\right) \times \left(\frac{\tan\theta(1-\tan^2\theta)}{2\tan^2\theta + 1 - \tan^2\theta}\right) \\
&= \tan\theta(1+\tan^2\theta) \times \frac{\tan\theta}{\tan^2\theta + 1} \\
&= \tan^2\theta \\
&= \text{RHS}
\end{aligned}$$

(c) (i) (2 marks)

Outcomes assessed: PE4

Targeted Performance Bands: E3-E4

Criteria	Mark
• uses correct formula for division of interval or progress using other correct method	1
• finds correct coordinates from working	1

Sample Answer:

$P(2ap,\ ap^2)$, $S(0,\ a)$ and $PQ:QS=-4:3$

Let Q have coordinates $(x_q,\ y_q)$

$$x_q=\frac{3\times 2ap-4\times 0}{-4+3} \qquad y_q=\frac{3\times ap^2-4\times a}{-4+3}$$

$$=\frac{6ap}{-1} \qquad =\frac{3ap^2-4a}{-1}$$

$$=-6ap \qquad =a\left(4-3p^2\right)$$

$\therefore\ Q$ has coordinates $\left(-6ap,\ a(4-3p^2)\right)$

(c) (ii) (2 marks)

Outcomes assessed: PE4

Targeted Performance Bands: E3-E4

Criteria	Marks
• makes progress to finding the locus	1
• shows locus is a parabola	1

Sample Answer:

From (i) $x=-6ap$

$$\therefore p=\frac{-x}{6a} \text{ and } p^2=\frac{x^2}{36a^2}$$

$$\therefore y=a\left(4-3p^2\right)$$

$$=a\left(4-\frac{3x^2}{36a^2}\right)$$

$$=4a-\frac{x^2}{12a}$$

$$\frac{x^2}{12a}=4a-y$$

$$x^2=48a^2-12ay$$

$$=-12a(y-4a)$$

which is the form of a parabola [with vertex $(0,\ 4a)$]

Question 6 (12 marks)
(a) (3 marks)
Outcomes assessed: PE2
Targeted Performance Bands: E2-E3

Criteria	Marks
• simplifies some indices	1
• further progress with simplifying indices	1
• gives correct expression	1

Sample Answer:

$$\begin{aligned}\frac{2^{4n}\times 3^{2n}}{8^n\times 6^n}+3^n &= \frac{2^{4n}\times 3^{2n}}{2^{3n}\times 2^n\times 3^n}+3^n\\ &= \frac{2^{4n}\times 3^n}{2^{4n}}+3^n\\ &= 3^n+3^n\\ &= 2\times 3^n\end{aligned}$$

(b) (i) (2 marks)
Outcomes assessed: PE5, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes correct derivative	1
• shows the result	1

Sample Answer:

$$\begin{aligned}V &= \pi r^2 h\\ &= \pi r^3 k \qquad \text{since } h = kr\end{aligned}$$

$$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$$

$$\frac{dV}{dt}=3\pi r^2 k\times\frac{dr}{dt}$$

$$\frac{dV}{dt}=0.2 \text{ when } r=4$$

$$\therefore 0.2 = 3\pi\times 4^2 k\times\frac{dr}{dt}$$

$$\begin{aligned}\frac{dr}{dt} &= \frac{0.2}{48\pi k}\\ &= \frac{1}{240\pi k}\end{aligned}$$

(b) (ii) (3 marks)

Outcomes assessed: PE5, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• finds expression for $\frac{dr}{dt}$ using surface area or progress toward result	1
• equates expressions using (i) or significant progress toward result	1
• finds correct value of k	1

Sample Answer:

$$\begin{aligned}S &= 2\pi rh + 2\pi r^2 \\ &= 2\pi r^2 k + 2\pi r^2 \qquad \text{since } h = kr \\ &= 2\pi r^2(k+1)\end{aligned}$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$\frac{dS}{dt} = 4\pi r(k+1) \times \frac{dr}{dt}$$

$$\frac{dS}{dt} = 0.1 \text{ when } r = 4$$

$$\therefore 0.1 = 4\pi \times 4(k+1) \times \frac{dr}{dt}$$

$$\begin{aligned}\frac{dr}{dt} &= \frac{0.1}{16\pi(k+1)} \\ &= \frac{1}{160\pi(k+1)}\end{aligned}$$

$$\therefore \frac{1}{160\pi(k+1)} = \frac{1}{240\pi k} \qquad \text{from (i)}$$

$$\begin{aligned}240k &= 160k + 160 \\ 80k &= 160 \\ k &= 2\end{aligned}$$

(c) (i) (2 marks)

Outcomes assessed: HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• differentiate LHS correctly	1
• differentiate RHS correctly	1

Sample Answer:

$$(1+x)^{2n} = \sum_{k=0}^{2n} {}^{2n}C_k\, x^k = {}^{2n}C_0 + {}^{2n}C_1\, x + {}^{2n}C_2\, x^2 + \ldots + {}^{2n}C_k\, x^k + \ldots + {}^{2n}C_{2n}\, x^{2n}$$

Differentiate both sides with respect to x.

$$\text{LHS} = 2n(1+x)^{2n-1}$$

$$\begin{aligned} \text{RHS} &= {}^{2n}C_1 + {}^{2n}C_2\, 2x + \ldots + {}^{2n}C_k\, kx^{k-1} + \ldots + {}^{2n}C_{2n}\, 2nx^{2n-1} \\ &= \sum_{k=1}^{2n} {}^{2n}C_k\, kx^{k-1} \end{aligned}$$

$$\left[\therefore 2n(1+x)^{2n-1} = \sum_{k=1}^{2n} k\, {}^{2n}C_k\, x^{k-1}\right]$$

(c) (ii) (2 marks)

Outcomes assessed: HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• correct substitution into equation	1
• gives correct conclusion	1

Sample Answer:

Let $x = 1$ in the expansion of $2n(1+x)^{2n-1} = \sum_{k=1}^{2n} k\, {}^{2n}C_k\, x^{k-1}$.

$$\begin{aligned} \text{LHS} &= 2n \times 2^{2n-1} \\ &= n \times 2^{2n} \\ &= n \times 4^n \end{aligned}$$

$$\text{RHS} = \sum_{k=1}^{2n} k\, {}^{2n}C_k$$

$$\therefore \sum_{k=1}^{2n} k\, {}^{2n}C_k = n \times 4^n$$

Question 7 (12 marks)

(a) (i) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes correct binomial probability	1
• gives correct answer (correct numerical equivalence)	1

Sample Answer:

Let probability of correct guess, $p = 0.3$ and incorrect guess, $q = 0.7$

Binomial probability; $(0.7 + 0.3)^{50}$

$P(25\text{ correct}) = {}^{50}C_{25}(0.7)^{25}(0.3)^{25}$

$[= 0.0014]$

(a) (ii) (3 marks)

Outcomes assessed: H5

Targeted Performance Bands: E3-E4

Criteria	Marks
• applies greatest coefficient method or some progress towards solution	1
• further progress towards solution (e.g. solution of inequality)	1
• gives correct answer	1

Sample Answer:

Most likely number correct $\Rightarrow$ find the greatest term in $(0.7 + 0.3)^{50}$

Find k such that $\dfrac{T_{k+1}}{T_k} \geq 1$

$$\frac{T_{k+1}}{T_k} = \frac{50 - k + 1}{k} \times \frac{0.3}{0.7}$$

$$\text{i.e. } \frac{153 - 3k}{7k} \geq 1$$

$$153 - 3k \geq 7k$$

$$10k \leq 153$$

$$k \leq 15.3$$

$$\therefore k = 15$$

Most likely number correct is 15. $\left[T_{16} = {}^{50}C_{15}(0.3)^{15}(0.7)^{35} = 0.122\right]$

(b)(i) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E2-E3

Criteria	Marks
• differentiates and equates to zero	1
• shows correct result	1

Sample Answer:

Particle reaches maximum height when $y' = 0$

$$y = Vt\sin\theta - \frac{1}{2}gt^2 \qquad \Rightarrow \qquad y' = V\sin\theta - gt$$

when $y' = 0$, $gt = V\sin\theta$ i.e. $t = \dfrac{V\sin\theta}{g}$

(b) (ii) (3 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• some progress toward solution	1
• further progress toward solution	1
• substitutes and simplifies to obtain desired result	1

Sample Answer:

At maximum height $t = \dfrac{V\sin\theta}{g}$, $x = c$ and $y = h$

$$h = \frac{V^2\sin^2\theta}{g} - \frac{1}{2}g\frac{V^2\sin^2\theta}{g^2} \qquad \text{and} \qquad c = \frac{V^2\cos\theta\sin\theta}{g}$$

$$h = \frac{V^2\sin^2\theta}{2g} \qquad\qquad c^2 = \frac{V^4\cos^2\theta\sin^2\theta}{g^2}$$

$$\therefore \sin^2\theta = \frac{2gh}{V^2} \quad (1) \qquad\qquad = \frac{V^4\sin^2\theta(1-\sin^2\theta)}{g^2}$$

$$= \frac{V^4\dfrac{2gh}{V^2}\left(1-\dfrac{2gh}{V^2}\right)}{g^2} \qquad \text{substituting for } \sin^2\theta \text{ from (1)}$$

$$= \frac{2h(V^2-2gh)}{g}$$

$$\therefore V^2 = 2gh + \frac{c^2g}{2h}$$

$$= \frac{4gh^2 + c^2g}{2h}$$

$$= \frac{g}{2h}\left(4h^2 + c^2\right)$$

(b) (iii) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• significant progress towards solutions	1
• finds a correct expression for θ	1

Sample Answer:

$$c=\frac{V^2\cos\theta\sin\theta}{g}\qquad h=\frac{V^2\sin^2\theta}{2g}$$

$$\frac{h}{c}=\frac{V^2\sin^2\theta}{2g}\times\frac{g}{V^2\cos\theta\sin\theta}$$

$$\frac{h}{c}=\frac{\sin\theta}{2\cos\theta}$$

$$\frac{2h}{c}=\tan\theta$$

$$\therefore\theta=\tan^{-1}\left(\frac{2h}{c}\right)$$

OR

$$V^2=\frac{g}{2h}\left(4h^2+c^2\right)\qquad h=\frac{V^2\sin^2\theta}{2g}\ \text{ i.e. }\sin^2\theta=\frac{2gh}{V^2}$$

$$\sin^2\theta=\frac{2gh}{\frac{g}{2h}\left(4h^2+c^2\right)}$$

$$\sin^2\theta=\frac{4h^2}{\left(4h^2+c^2\right)}$$

$$\sin\theta=\frac{2h}{\sqrt{4h^2+c^2}}\quad(\theta\text{ acute})$$

$$\theta=\sin^{-1}\left(\frac{2h}{\sqrt{4h^2+c^2}}\right)$$

Neap:

HSC Trial Examination 2009

Mathematics Extension 1

This paper must be kept under strict security and may only be used on or after the afternoon of Thursday 13 August, 2009 as specified in the Neap Examination Timetable.

General Instructions

Reading time – 5 minutes

Working time – 2 hours

Write using black or blue pen

Board-approved calculators may be used

A table of standard integrals is provided at the back of this paper

All necessary working should be shown in every question

Total marks – 84

Attempt questions 1–7

All questions are of equal value

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2009 HSC Mathematics Extension 1 Examination.

Total marks 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 Marks) Use a SEPARATE writing booklet.

(a) Solve the inequality $\frac{x}{x-3} \geq 2$. **2**

(b) Determine the coordinates of the point that divides the interval joining $(6, 2)$ to $(4, 9)$ externally in the ratio of $2 : 1$. **2**

(c) Determine the general solution to the equation $2\sin 2x = 1$. **2**

(d) (i) Sketch the graph of $y = 4\sin^{-1}3x$. **1**

(ii) What is the domain and range of $y = 4\sin^{-1}3x$? **2**

(e) $(x+2)$ is a factor of the polynomial $P(x) = 3x^3 + kx^2 - 5x + 10$.

(i) Determine the value of k. **1**

(ii) The roots of $P(x) = 0$ are -2, α and β. Determine the value of $\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{2}$. **2**

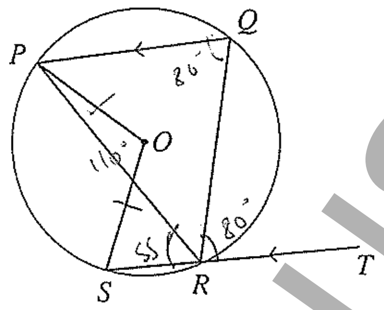
Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) The line $y = 12 - x$ crosses the parabola $y = x^2$ at the point $(3, 9)$. Calculate the size of the acute angle between the line and the parabola at $(3, 9)$. **2**

(b) Use the substitution $u = \sqrt{x}$ to evaluate $\displaystyle\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. **3**

(c) In the diagram below, PQ is parallel to ST, O is the centre of the circle, $\angle PQR = 80°$ and $\angle POS = 110°$. **3**



Find the value of $\angle PRQ$ giving reasons.

(d) Use mathematical induction to prove that, when n is a positive integer, **4**

$$\frac{1}{4} + \frac{1}{28} + \frac{1}{77} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$$

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

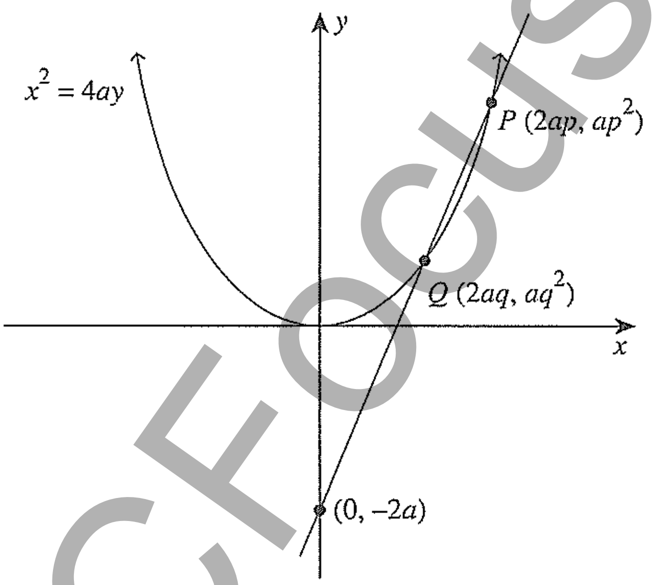
(a) Differentiate $y = x\tan^{-1}x$. **2**

(b) Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$. **2**

(c) (i) Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **2**

(ii) Hence find $\displaystyle\int 2\sin^3\theta \, d\theta$. **1**

(d) The diagram below shows the parabola $x^2 = 4ay$ and the points $P\ (2ap, ap^2)$ and $Q\ (2aq, aq^2)$.



(i) Show that the equation of the chord PQ is $2y = (p+q)x - 2apq$. **2**

(ii) The line joining P and Q passes through the point $(0, -2a)$. Show that $pq = 2$. **1**

(iii) The normals to the parabola $x^2 = 4ay$ at points $P\ (2ap, ap^2)$ and $Q\ (2aq, aq^2)$ intersect at K. The coordinates of K are $\left(-apq(p+q),\ a(p^2+q^2+pq+2)\right)$. Do *not* prove this. **2**

Prove that the locus of K is the parabola $x^2 = 4ay$.

Marks

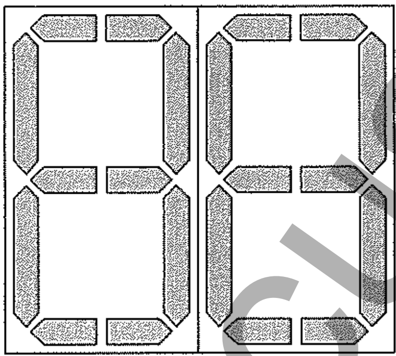
Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Find the coordinates of the stationary points on the curve $y = \dfrac{5x^2 + 5}{x}$ and determine their nature. **2**

(ii) Sketch the graph of $y = \dfrac{5x^2 + 5}{x}$. **1**

(iii) Hence determine the range of the function $y = \dfrac{5x^2 + 5}{x}$. **1**

(b) The 10 segments on each digital display panel below are lights that can be switched on or off. The panels are connected to a computer chip that randomly switches on 14 segments.



(i) How many sets of 14 lights can be selected from the 20 lights on the two display panels? **1**

(ii) Determine the total number of combinations possible if 4 vertical and 10 horizontal segments are to be lit. **1**

(iii) How many of the different 14-light displays show exactly 5 vertical segments lit up? **2**

(c) (i) Sketch the graph of $y = x^3 - 5x^2$. **1**

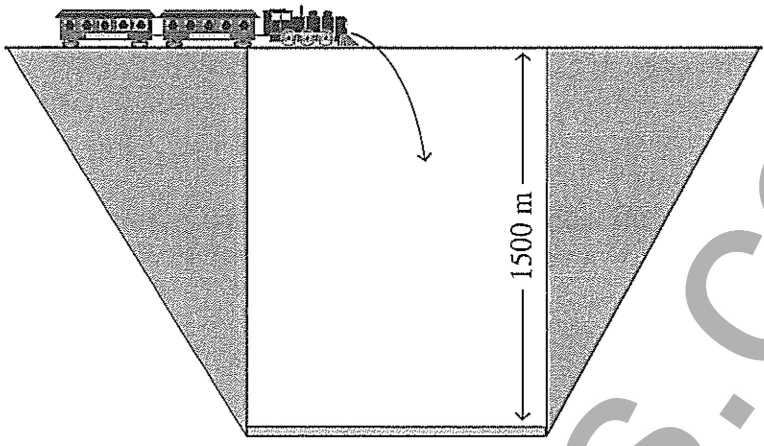
(ii) Using a diagram, or otherwise, explain how you know that $y = x^3 - 5x^2 - 1$ has only one root. **1**

(iii) Use one iteration of Newton's method to approximate the root of $y = x^3 - 5x^2 - 1$ starting with $x_1 = 5$. Express the root correct to two decimal places. **2**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) At the exact time a train travelling at 24 m/s begins to cross a 600 m–long bridge, a weight falls off the front of the train and becomes a projectile as it falls into the ravine as shown on the diagram below. The ravine is 1 500 m deep. Use $g \approx 10$ m/s^2.



(i) Determine the equations of motion for the flight of the weight. **2**

(ii) Will the weight crash into the side of the ravine or onto the floor of the ravine? Use calculations to justify your answer. **2**

(b) The acceleration of a particle moving in simple harmonic motion is given by $\ddot{x} = -8x$. Initially the particle is stationary 2 m to the right of the origin.

(i) Show that $v^2 = 8(4 - x^2)$. **2**

(ii) Calculate the time taken for the particle to first pass the origin. **2**

(c) The rate of change of a cool item placed in a hot environment is proportional to the difference between the temperature of the cool item (T) and the temperature of the hot environment (S). That is, $\frac{dT}{dt} \propto (S - T)$.

(i) Prove that $T = S - Ae^{kt}$ is a solution to the differential equation $\frac{dT}{dt} \propto (S - T)$. **1**

(ii) Jamie is cooking a large roast in an oven set to 160 °C. The roast will be cooked when the thermometer shows that the temperature of the centre of the meat is 150 °C. **3**

When Jamie started cooking, the temperature of the centre of the meat was 4 °C and 30 minutes later the temperature was 60 °C.

How long will it take for the roast to be cooked?

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) What is the value of the term in x^4 in the expansion of $(1 + 2x)^{15}$? **1**

(b) In a large pile of chips, the ratio of red to black chips is 3 : 5.
Gemma is going to select 12 chips at random from the pile.

(i) What is the probability that one quarter of the chips Gemma selects will be black? Express your answer correct to 4 decimal places. **2**

(ii) What is the most likely number of red and black chips Gemma will select? Use a calculation to justify your answer. **4**

(c) At the beginning of each year Cassandra invests in superannuation. Her account pays 5.6% p.a. annually compounding interest.

Cassandra's first investment was \$1000 and each following year her investment was 20% more than the previous year's. That is, her first three investments were \$1000, \$1200 and \$1440.

(i) Show that the total value, \$$V$, of Cassandra's investment at the end of three years is given by $V = 1000M(1.2^2 + 1.2M + M^2)$, where $M = 1.056$. **1**

(ii) Show that the total value of Cassandra's investment after n years is given by $V = 8800 \times (1.2)^{n-1} \times (1 - 0.88^n)$. **2**

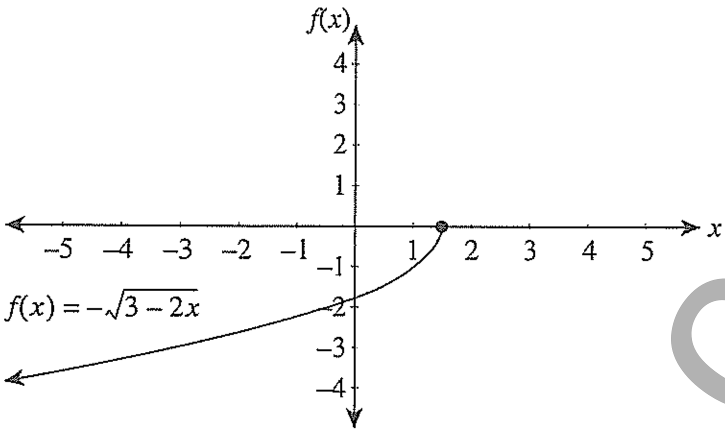
(iii) Cassandra continued her investment for 30 years. That is, until the end of the year following her 30th deposit. **2**

What percentage of the total value of the account at the end of 30 years was interest? Express your answer correct to the nearest whole percentage.

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $f(x) = -\sqrt{3-2x}$.



(i) Copy the diagram onto your answer page and show the graph of $y = f^{-1}(x)$, the inverse function of $y = f(x)$, on it. **1**

(ii) Write down the equation for the inverse function of $f(x) = -\sqrt{3-2x}$. Include the domain of the inverse function in your answer. **1**

(b) (i) By determining the coefficient of x^2 on both sides of the equation $(1+x)^n \times (1+x)^{2n} = (1+x)^{3n}$, or otherwise, show that **2**

$$\binom{n}{0}\binom{2n}{2} + \binom{n}{1}\binom{2n}{1} + \binom{n}{2}\binom{2n}{0} = \binom{3n}{2}.$$

(ii) Hence, or otherwise, show $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{n-k} = \binom{3n}{n}$. **2**

Question 7 continues on page 9

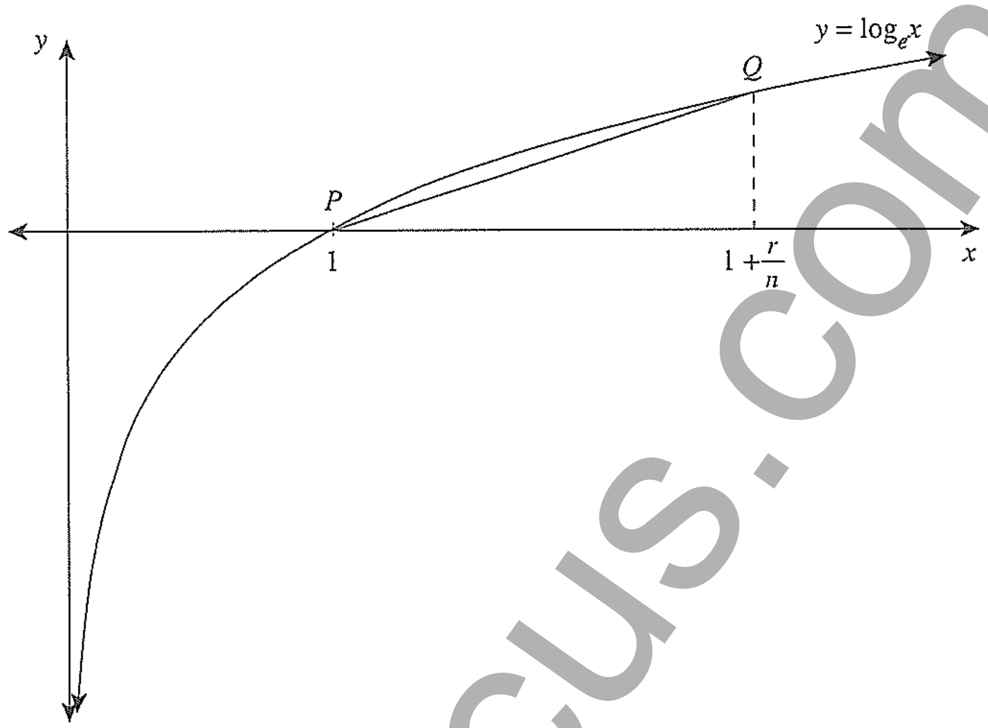
 TEHME1_QA_09.FM

Marks

Question 7 (continued)

(c) (i) Use the compound interest formula to show that a rate of 6% p.a. is effectively 6.168% p.a. when the investment is compounded monthly. **1**

(ii)



The diagram above shows the graph of $y = \log_e x$ and the secant joining points P and Q on the curve. P is at $x = 1$ and Q is at $x = 1 + \frac{r}{n}$.

(α) Show that the gradient of the secant PQ is $\frac{1}{r}\log_e\left(1 + \frac{r}{n}\right)^n$. **1**

(β) Use $\frac{d}{dx}\log_e x = \frac{1}{x}$ to show that $\lim_{n \to \infty}\left(1 + \frac{r}{n}\right)^n = e^r$. **3**

(γ) Hence or otherwise determine an expression for the effective annual rate of interest when an annual rate of 6% p.a. is compounded continually, that is, compounded an infinite number of times per year. **1**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

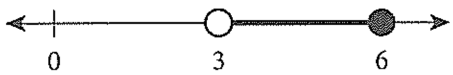
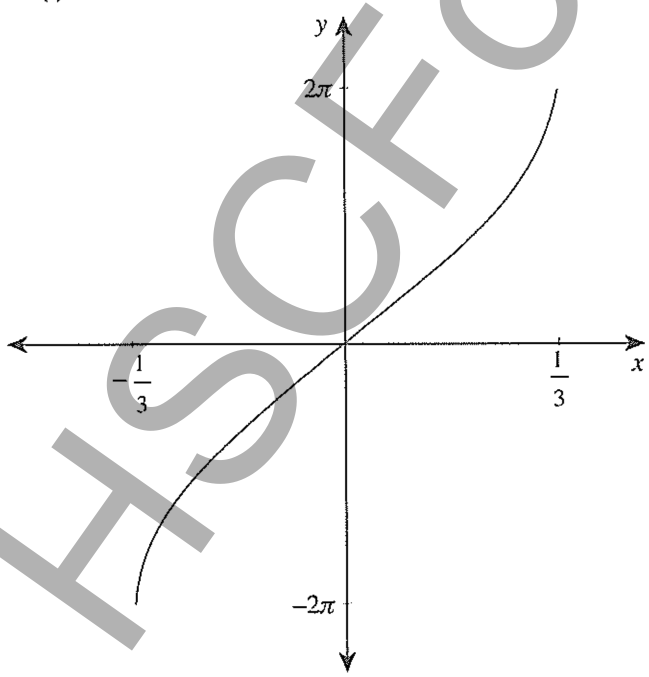
Neap:

HSC Trial Examination 2009

Mathematics Extension 1

Solutions and marking guidelines

Question 1

	Syllabus outcomes and marking guide
(a) $(x-3)^2 \times \frac{x}{x-3} \geq 2(x-3)^2,\ x \neq 3$ $(x-3)(-x+6) \geq 0$  $3 < x \leq 6$	PE3 • Gives the correct answer .... 2 • Gives $x \neq 3$. OR • $3 \leq x \leq 6$ 1
(b) $\left(\frac{6 \times -1 + 2 \times 4}{2-1}, \frac{2 \times -1 + 9 \times 2}{2-1}\right)$ $= (2, 16)$	PE2 • Gives the correct answer .... 2 • Internal division, i.e. $\left(\frac{14}{3}, \frac{20}{3}\right)$. OR • Minor numerical error in otherwise correct answer 1
(c) $\sin 2x = \frac{1}{2}$ $2x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{13\pi}{6}$ or $\frac{17\pi}{6}$ $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$ General solution: $\frac{\pi}{12} + \pi k, \frac{5\pi}{12} + \pi k$, k integer	HE3 • Gives correct answer .... 2 *(Note: accept answers in degrees and in the form $30° + 360k$, $150° + 360k$ etc., k integer)* • Gives the correct general solution from incorrect values in $0 \leq x \leq 2\pi$ 1
(d) (i)  Graph of function is $y = 4\sin^{-1} 3x$.	P4 • Draws the correct graph.............. 1

Question 1

	Syllabus outcomes and marking guide
(ii) Domain: $-\frac{1}{3} \le x \le \frac{1}{3}$ Range: $-2\pi \le x \le 2\pi$	PE6, HE4 • Gives the correct domain and range. . . . . 2 • Gives either the correct domain or the correct range 1
(e) (i) $P(x) = 3x^3 + kx^2 - 5x + 10$ $P(-2) = -24 + 4k + 10 + 10 = 0$ $4k = 4$ $k = 1$	PE3 • Gives the correct answer 1
(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\lambda}$ $= \frac{-\frac{5}{3}}{-\frac{10}{3}}$ $= \frac{1}{2}$	PE3 • Gives the correct answer . . . . 2 • Uses $\frac{\beta\lambda + \alpha\lambda + \alpha\beta}{\alpha\beta\lambda}$ or equivalent merit. 1

Question 2	Syllabus outcomes and marking guide
(a) $\tan\theta = \left\lvert\dfrac{m_1 - m_2}{1 + m_1 m_2}\right\rvert$ $m_1 = -1,\ m_2 = 2 \times 3 = 6$ $\tan\theta = \left\lvert\dfrac{6+1}{1 + 6 \times -1}\right\rvert$ $= \dfrac{7}{5}$ $\theta = 54°28'$	PE3 • Gives the correct answer .... 2 *(Note: accept 54° or similar rounding.)*
	• Use of correct formula with one correct gradient. OR • Gradient of curve = 6. OR • Similar merit 1
(b) $u = x^{\frac{1}{2}}$ $\dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}}$ $= \dfrac{1}{2\sqrt{x}}$ $x = 1,\ u = 1$ $x = 9,\ u = 3$ $\dfrac{1}{2}\displaystyle\int_1^3 e^u\,du = \dfrac{1}{2}[e^u]_1^3$ $= \dfrac{1}{2}(e^3 - e)$	HE6 • Correct answer, MUST show change of limits 3
	• Correct expression for integral with change of limits 2
	• Makes some progress, e.g. changes limits, or determines $du = \dfrac{1}{2\sqrt{x}}dx$ or similar merit............................ 1
(c) $\angle QRT = 80°$ (alternate angles, $PQ \parallel ST$) $\angle PRS = 55°$ (angle at the circumference is half the angle at the centre, on arc PS) $\angle PRQ + 55° + 80° = 180°$ (angles making a straight line add to 180°) $\therefore \angle PRQ = 45°$	PE2 • Gives correct answer with supporting reasons 3
	• Uses two relevant, correct geometrical facts with reason...................... 2
	• Uses one relevant, correct geometrical fact with reason. OR • $\angle PRQ = 45°$ without giving adequate reasons 1

Question 2 (Continued)

	Syllabus outcomes and marking guide
(d) To prove $\frac{1}{4}+\frac{1}{28}+\frac{1}{77}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$ Test for $n=1$ $LHS=\frac{1}{(3\times 1-2)(3\times 1+1)}$ $=\frac{1}{4}$ $RHS=\frac{1}{3\times 1+1}$ $=\frac{1}{4}$ $\therefore$ true for $n=1$ Assume $\frac{1}{4}+\frac{1}{28}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$, where k is a positive integer. Hence, test for $n=k+1$. $\frac{1}{4}+\frac{1}{28}+\ldots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{[3(k+1)-2][3(k+1)+1]}$ $=\frac{k}{3k+1}+\frac{1}{[3k+1][3k+4]}$ $=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$ $=\frac{(k+1)(3k+1)}{(3k+4)(3k+1)}$ $\frac{k+1}{3(k+1)+1}$ $\therefore$ if the formula is true for a value of n, then it is true for the following value of n. It is true for $n=1$, $\therefore$ it will be true for $n=2$, $n=3$, etc. $\therefore$ true for all n.	HE2 • Gives the correct, complete answer . . . . . 4 • Gives an essentially correct solution with a minor omission or flaw . . . . 3 • Makes significant progress, e.g. tests for $n=1$. AND • Attempts to show true for $n=k+1$ assuming $n=k$ is true . . . . 2 • Makes some progress, e.g. tests for $n=1$. OR • Attempts to show true for $n=k+1$ assuming $n=k$ is true, or similar merit. 1

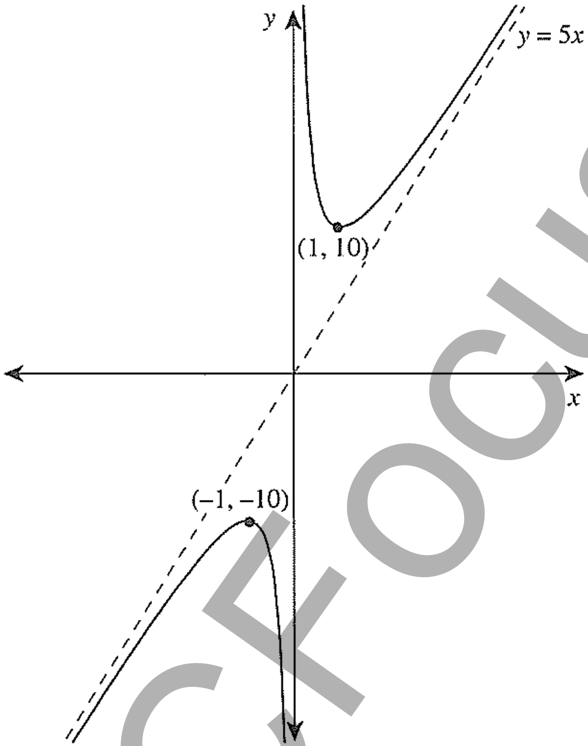
Question 3

	Sample answer	Syllabus outcomes and marking guide
(a)	$y = x\tan^{-1}x$ let $u = x$ $\quad v = \tan^{-1}x$ $\therefore u' = 1$ $\quad v' = \dfrac{1}{1+x^2}$ $\therefore \dfrac{d}{dx}(x\tan^{-1}x)$ $= uv' + vu'$ $= \dfrac{x}{1+x^2} + \tan^{-1}x$ $= \dfrac{x + (1+x^2)\tan^{-1}x}{1+x^2}$	HE4 • Obtains the correct answer in any form . . 2 • Shows $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$ 1
(b)	$\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \int_0^1 \frac{dx}{\sqrt{2^2-x^2}}$ $= \left[\sin^{-1}\dfrac{x}{2}\right]_0^1$ $= \left[\sin^{-1}\left(\dfrac{1}{2}\right) - \sin^{-1}0\right]$ $= \left[\dfrac{\pi}{6} - 0\right]$ $= \dfrac{\pi}{6}$	HE4 • Gives the correct answer in any form . . . 2 • Shows $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}$. 1
(c) (i)	$\sin 3\theta = \sin(2\theta + \theta)$ $= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$ $= 2\sin\theta\cos\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$ $= 3\sin\theta\cos^2\theta - \sin^3\theta$ $= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$ $= 3\sin\theta - 4\sin^3\theta$	HE3, HE6 • Correct proof . . . . 2 • Demonstrates the correct use of at least one multiple angle formula e.g. $\sin(A+B)$ or $\sin 2\theta$ or $\cos 2\theta$ 1
(ii)	$\displaystyle\int 2\sin^3\theta\, d\theta$ $4\sin^3\theta = 3\sin\theta - \sin 3\theta$ $2\sin^3\theta = \dfrac{1}{2}(3\sin\theta - \sin 3\theta)$ $\displaystyle\int 2\sin^3\theta\, d\theta = \frac{1}{2}\int(3\sin\theta - \sin 3\theta)\,d\theta$ $= \dfrac{1}{2}\left(-3\cos\theta + \dfrac{1}{3}\cos 3\theta\right) + C$ $= \dfrac{1}{6}\cos 3\theta - \dfrac{3}{2}\cos\theta + C$	HE6 • Gives the correct answer, ignore $+C$. . . . 1

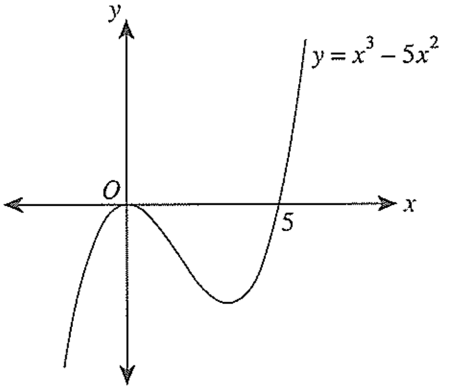
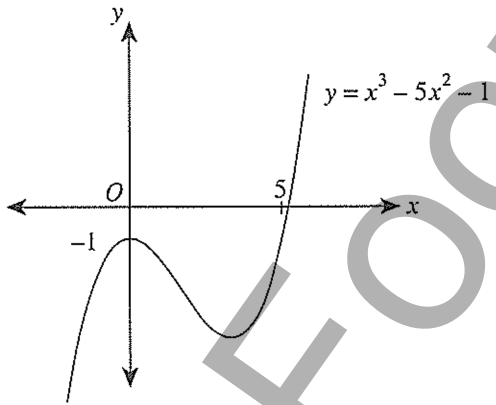
Question 3 (Continued)

	Sample answer	Syllabus outcomes and marking guide
(d) (i)	$m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$ $= \dfrac{a(p^2 - q^2)}{2a(p-q)}$ $= \dfrac{a(p-q)(p+q)}{2a(p-q)}$ $= \dfrac{p+q}{2}$ Use point P (or Q) $y - ap^2 = \dfrac{p+q}{2}(x - 2ap)$ $2y - 2ap^2 = (p+q)x - 2ap(p+q)$ $2y - 2ap^2 = (p+q)x - 2ap^2 - 2apq$ $2y = (p+q)x - 2apq$	PE4 • Gives the correct answer .... 2 • Shows $m_{pq} = \dfrac{p+q}{2}$ or equivalent merit. . 1
(ii)	Substitute point $(0, -2a)$ into: $2y = (p+q)x - 2apq$. When $x = 0$ and $y = -2a$ $2(-2a) = (p+q) \times 0 - 2apq$ $-4a = -2apq$ $\dfrac{-4a}{-2a} = pq$ $pq = 2$	PE4 • Gives the correct answer 1
(iii)	$x = -apq(p+q)$ and $y = a(p^2 + q^2 + pq + 2)$ $pq = 2$ $\therefore x = -2a(p+q)$ and $y = a(p^2 + q^2 + 2pq)$ $\dfrac{x}{-2a} = p + q,\ y = a(p+q)^2$ $y = a\left(\dfrac{x}{-2a}\right)^2$ $= a \times \dfrac{x^2}{4a^2}$ $4ay = x^2$ therefore, K is on the original parabola $x^2 = 4ay$.	PE4 • Gives the correct answer .... 2 • Makes significant progress, e.g. obtains $x = -2a(p+q)$ and $y = a(p^2 + q^2 + 2pq)$ or similar merit. 1

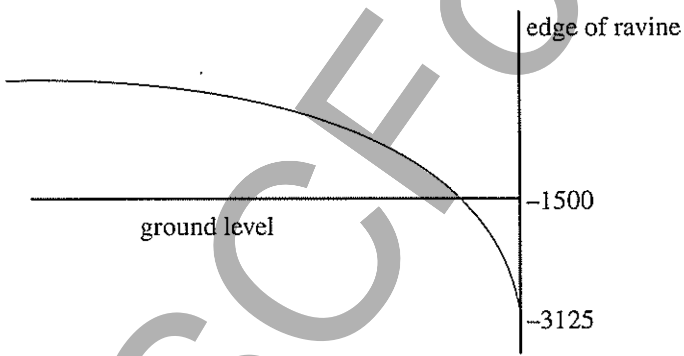
Question 4

Sample answer	Syllabus outcomes and marking guide
(a) (i) $y = 5x + 5x^{-1}, x \neq 0$ $y' = 5 - 5x^{-2}$ $= 0$ $x^{-2} = 1$ $x = \pm 1, (1, 10)(-1, -10)$ $y'' = 10x^{-3}$ $f''(1) = 10 > 0, \therefore \min\ (1, 10)$ $f''(-1) = -10 < 0 \therefore \max\ (-1, -10)$	H6 • Gives the correct answer .... 2 • Gives the correct stationary points. OR • Gives one stationary point with correct nature 1
(ii) As $x \to \infty, y \to 5x$ and $x \neq 0$  	H6 • Correct graph showing $(1, 10), (-1, -10)$ and asymptotes at $x = 0$ and $y = 5x$ ..... 1 *Note: accept equation of asymptotes shown anywhere in Question 4(a)(i), (ii) or (iii).*
(iii) Range: $y \geq 10$ and $y \leq -10$	H5 • Gives the correct answer 1
(b) (i) ${}^{20}C_{14} = 38\ 760$	PE3 • Gives the correct answer (accept correct numerical expression) 1
(ii) ${}^{8}C_{4} \times {}^{12}C_{10} = 4620$	PE3 • Gives the correct answer (accept correct numerical expression) 1
(iii) ${}^{8}C_{5} \times {}^{12}C_{9} \times 14! = 1.074 \times 10^{15}$	PE3 • Gives the correct answer in any form (accept correct numerical expression) .... 2 • Multiplying combinations by 14! 1

Question 4 **(Continued)**

Sample answer	Syllabus outcomes and marking guide
(c) (i) 	PE3 HE3 • Sketches correct graph 1
(ii) $y = x^3 - 5x^2 - 1$ has only one root as it is the graph of $y = x^3 - 5x^2$ shifted down one unit. Since $y = x^3 - 5x^2$ has a maximum turning point as a repeated root at $x = 0$, a downwards shift of the graph results in this root disappearing, leaving only one root near $x = 5$. 	PE3 HE3 • Correct translation of graph in part (i). OR • Correct explanation without graph 1
(iii) Let $f(x) = x^3 - 5x^2 - 1$ $f'(x) = 3x^2 - 10x$ Using Newton's method: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ $= 5 - \dfrac{(5)^3 - 5(5)^2 - 1}{3(5)^2 - 10(5)}$ $= 5 - \dfrac{-1}{25}$ $= 5 + \dfrac{1}{25}$ $= 5.04$	HE7, HE3 • Gives answer correct to two decimal places . . . . 2 • Correctly finds $f(5)$ and $f'(x)$ or similar merit 1

Question 5

	Syllabus outcomes and marking guide
(a) (i) Position the origin where the weight falls off the train. $\ddot{x} = 0$ $\dot{x} = v$ $\dot{x} = 24 \text{ m s}^{-1}$ $x = 24t + C_1$ $C_1 = 0$ $x = 24t$ $\ddot{y} = -g = -10$ $\dot{y} = -10t + C_2$ $C_2 = 0$ (starts from vertical rest) $\dot{y} = -10t$ $y = -5t^2 + C_3$ $C_3 = 0$ (starts at origin) $\therefore y = -5t^2$	HE3 • Gives the correct answer, allow origin in any position. . . . . 2 • Makes progress, e.g. obtains $\dot{x} = 24$ or $\dot{y} = -10t$ or similar merit . . . 1
(ii) The value of y when $x = 600$ is required. $24t = 600$ $t = 25$ $y = -\frac{1}{2}g \times 25^2$ $y = -5 \times 25^2$ $= -3125 \text{ m}$  The weight will hit the ground, not the vertical side of the ravine.	HE3 • Correct answer with justification . . . . 2 • Makes progress towards a solution, e.g. obtains $t = 25$ and attempts to find y when $t = 25$, or similar merit 1

Question 5 (Continued)	**Syllabus outcomes and marking guide**
(b) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -8x$ $\frac{1}{2}v^2 = -\int 8x\,dx$ $v^2 = -8x^2 + C$ or $v^2 = C - 8x$ From initial conditions: $0 = C - 8(2)^2$ $C = 32$ $\therefore v^2 = 32 - 8x^2$ $\therefore v^2 = 8(4 - x^2)$	HE3, HE5 • Obtains the correct solution . . . . 2 • Uses $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ 1
(ii) From (i), $v^2 = 8(4 - x^2)$ $= n^2(a^2 - x^2)$ since the particle is exerting simple harmonic motion $\therefore n^2 = 8$ and $a^2 = 4$ $\therefore n = 2\sqrt{2}$ and $a = 2$(where $a > 0$) $\therefore x = a\cos(nt + \alpha)$ $\therefore x = 2\cos(2\sqrt{2}t + \alpha)$. $v = -4\sqrt{2}\sin(2\sqrt{2}t + \alpha)$ when $t = 2$, $v = 0$ $0 = -4\sqrt{2}\sin\alpha \quad \therefore \alpha = 0$ $x = 2\cos(2\sqrt{2}t)$ when $x = 0$ $2\sqrt{2}t = \frac{\pi}{2}$ $t = \frac{\sqrt{2}\pi}{8}$	HE3, HE5 • Obtains the correct answer. . . . . 2 • Shows $x = 2\cos(2\sqrt{2}t)$ or similar merit. 1

Question 5 (Continued)	**Syllabus outcomes and marking guide**
(c) (i) $T = S - Ae^{kt} \Rightarrow Ae^{kt} = S - T$ $\frac{dT}{dt} = -Ae^{kt} \times k$ $= -(S-T) \times k$ That is, the rate is proportional to $S - T$. $\therefore T = S - Ae^{kt}$ is a solution to $\frac{dT}{dt} \propto (S-T)$.	HE3 • Correct demonstration 1
(ii) $S = 160$ °C $t = 0, T = 4$ °C $t = 30, T = 60$ °C $t = ?, T = 150$ °C $T = 160 - Ae^{kt}$ $4 = 160 - Ae^{0}$ $\therefore A = 156$ $t = 30$ and $T = 60$ $60 = 160 - 156e^{30k}$ $e^{30k} = \frac{100}{156}$ $30k = \log_e\left(\frac{100}{156}\right)$ $k = \frac{1}{30}\log_e\left(\frac{100}{156}\right)$ $t = ?$ and $T = 150$ $150 = 160 - 156e^{kt}$ $e^{kt} = \frac{10}{156}$ $t = \frac{\log_e\left(\frac{10}{156}\right)}{k}$ $t = 185.34$ $t \approx 186$ min	HE3 • Gives the correct answer . . . . 3 • Makes significant progress, e.g. determines the value of S, A and k, or makes a minor error . . . . 2 • Makes some progress, e.g. determines two of the values for S, A and k 1

Question 6

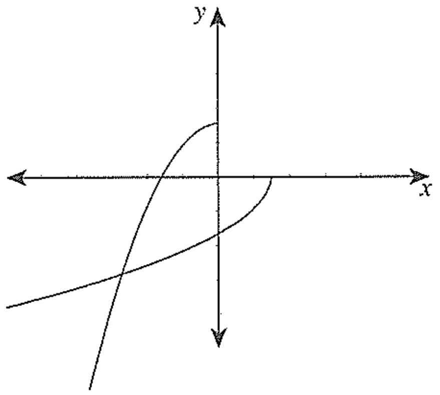
	Sample answer	Syllabus outcomes and marking guide
(a)	$T_5 = {}^{15}C_4(1)^{11}(2x)^4$ $= 1365 \times 2^4x^4$ $= 21\ 840x^4$	HE3 • Gives the correct answer 1
(b) (i)	$P(\text{red}) = \frac{3}{8}$ and $P(\text{black}) = \frac{5}{8}$ Terms in $\left(\frac{3}{8}+\frac{5}{8}\right)^{12}$ are required. A quarter black means 3 black, i.e. 9 red. ${}^{12}C_3\left(\frac{3}{8}\right)^9\left(\frac{5}{8}\right)^3 = 0.007877$	HE3 • Gives the correct answer, ignore rounding .... 2 • Uses $\left(\frac{3}{8}+\frac{5}{8}\right)^{12}$ 1
(ii)	The most likely number $\rightarrow$ the term with the largest probability. Need $C_{k+1} > C_k$ $\dfrac{{}^{12}C_{k+1}\left(\frac{3}{8}\right)^{11-k}\left(\frac{5}{8}\right)^{k+1}}{{}^{12}C_k\left(\frac{3}{8}\right)^{12-k}\left(\frac{5}{8}\right)^{k}} > 1$ $\dfrac{\frac{12!}{(k+1)!(11-k)!}\times\frac{5}{8}}{\frac{12!}{(12-k)!k!}\times\frac{3}{8}} > 1$ $\frac{5}{k+1}\times\frac{12-k}{3} > 1$ $8k < 57$ $k < 7\frac{1}{8}$ Need $k = 7$, that is the term involving ${}^{12}C_8$. $T_9 = {}^{12}C_8\left(\frac{3}{8}\right)^4\left(\frac{5}{8}\right)^8$ The most likely combination is 4 red and 8 black.	HE3 • Gives the correct answer, accept any method .... 4 • Obtains $k < 7\frac{1}{8}$, or similar value through a different approach .... 3 • Makes some progress in the solution to $C_{k+1} > C_k$ e.g. correctly uses the ${}^nC_r$ formula, or similar standard with a different approach.......................... 2 • Attempts to use: term with ${}^{12}C_{k+1}$ > term with ${}^{12}C_k$ or $\frac{T_{k+1}}{T_k} > 1$ 1
(c) (i)	Let A_n = value of investment at the end of n years. $A_1 = 1000(1 + 0.056)^1$ $A_2 = 1000(1.056)^2 + 1.2 \times 1000(1.056)^1$ Let $M = 1.056$ $A_3 = 1000M^3 + 1.2 \times 1000M^2 + 1.2^2 \times 1000M$ $A_3 = 1000M(1.2^2 + 1.2M + M^2)$	H5 • Correct demonstration 1

Question 6 (Continued) Sample answer	Syllabus outcomes and marking guide
(ii) $A_n = 1000M(M^{n-1} + 1.2M^{n-2} + 1.2^2M^{n-3} \ldots 1.2^{n-1})$ $= 1000M \times \dfrac{a(r^n - 1)}{r - 1}$ $= 1000M \times 1.2^{n-1} \times \dfrac{\left[\left(\frac{M}{1.2}\right)^n - 1\right]}{\frac{M}{1.2} - 1}$ $= 1000 \times 1.056 \times 1.2^{n-1} \times \dfrac{\left(\frac{1.056}{1.2}\right)^n - 1}{\frac{1.056}{1.2} - 1}$ $= 1056 \times (1.2)^{n-1} \times \dfrac{(0.88)^n - 1}{-0.12}$ $= 8800(1.2)^{n-1}[1 - (0.88)^n]$	H5 • Correct demonstration . . . . 2 • Makes substantial progress, e.g. attempts to sum the correct sequence or finds a correct sequence for A_n 1
(iii) $A_{30} = \$1\ 703\ 156.94$ The amount Cassie invested $= 1000 + 1000(1.2) + 1000(1.2)^2 \ldots 1000(1.2)^{29}$ $= 1000 \times \dfrac{[(1.2)^{30} - 1]}{1.2 - 1}$ $= \$1\ 181\ 881.57$ interest $= \$521\ 275.37$ % interest $= \dfrac{521\ 275.37}{1\ 703\ 156.94} \times 100$ $= 30.6\%$	H5 • Gives correct answer . . . . 2 • Determines that Cassie invested \$1 181 881.57 (ignore rounding) or equivalent merit 1

Question 7

Sample answer | **Syllabus outcomes and marking guide**

(a) (i)



HE4
- Draws a graph that is the mirror image of $y = f(x)$ in the line $y = x$ 1

(ii) $x = -\sqrt{3-2y}$ $\quad x \le 1\frac{1}{2}, y \le 0$

$x^2 = 3 - 2y$

$y = \frac{1}{2}(3 - x^2)$ $\quad x \le 0$

HE4
- Gives the correct answer 1

(b) (i) $(1+x)^n(1+x)^{2n} = (1+x)^{3n}$

Terms in x^2

$RHS = {}^{3n}C_2x^2$

On LHS

$({}^nC_0 + {}^nC_1x + {}^nC_2x^2 \ldots)({}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 \ldots$

The product of the paired terms give x^2

$\therefore$LHS term in x^2

$= {}^nC_0{}^{2n}C_2x^2 + {}^nC_1{}^{2n}C_1x^2 + {}^nC_2{}^{2n}C_0x^2$

Equation coefficients

${}^nC_0{}^{2n}C_2 + {}^nC_1{}^{2n}C_1 + {}^nC_2{}^{2n}C_0 = {}^{3n}C_2$

HE4
- Correct demonstration 2
- Determines the expression for x^2 on either side of the equation 1

(ii) Consider terms in x^n

on the RHS ${}^{3n}C_nx^n$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{n-k} = {}^nC_0{}^{2n}C_n + {}^nC_1{}^{2n}C_{n-1} \ldots + {}^nC_n{}^{2n}C_0$$

Terms in x^n on LHS

$= {}^nC_0{}^{2n}C_nx^n + {}^nC_1x{}^{2n}C_{n-1}x^{n-1} \ldots {}^nC_nx^n{}^{2n}C_0$

$= [{}^nC_0{}^{2n}C_n + {}^nC_1{}^{2n}C_{n-1} \ldots + {}^nC_n{}^{2n}C_0]x^n$

Coefficients of x^n on both sides of the equation are equal.

$\therefore {}^nC_0{}^{2n}C_n + {}^nC_1{}^{2n}C_{n-1} \ldots + {}^nC_n{}^{2n}C_0 = {}^{3n}C_n$

i.e. $\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{n-k} = \binom{3n}{n}$

HE3
- Correct demonstration 2
- Makes some progress, e.g. identifies the significance of x^n to the solution 1

Question 7 (Continued) Sample answer	Syllabus outcomes and marking guide
(c) (i) Compounded monthly $A = P\left(1 + \frac{0.06}{12}\right)^{12}$ $= P \times 1.0616778\ldots$ Effective interest rate is 6.168% to 3 decimal places.	H5 • Obtains the correct solution 1
(ii) (α) Gradient of $PQ = \dfrac{\log_e\left(1 + \frac{r}{n}\right)}{\frac{r}{n}}$ $= \frac{n}{r}\log_e\left(1 + \frac{r}{n}\right)$ $= \frac{1}{r}\log_e\left(1 + \frac{r}{n}\right)^n$	P3, H3 • Obtains the correct solution 1
(β) Using $\frac{d}{dx}(\log_e x) = \frac{1}{x}$, the gradient of the tangent P, where $x = 1$, will be 1. Using the definition of a derivative, the gradient of the tangent at P: $\lim\limits_{n \to \infty} \frac{1}{r}\log_e\left(1 + \frac{r}{n}\right)^n$ $\lim\limits_{n \to \infty} \frac{1}{r}\log_e\left(1 + \frac{r}{n}\right)^n = 1$ $\lim\limits_{n \to \infty} \log_e\left(1 + \frac{r}{n}\right)^n = r$ $\lim\limits_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r$	H3, P6, P8 • Obtains the correct solution . . . . 3 • Makes significant progress towards the solution. . . . . 2 • Attempts to use the gradient at $x = 1$ for the derivative and the limiting position of the secant 1
(γ) $A = P\left(1 + \frac{0.06}{n}\right)^n$ when interest is compounded continuously, i.e. $n \to \infty$. $\lim\limits_{n \to \infty} \left(1 + \frac{0.06}{n}\right)^n = e^{0.06}$ $A = P \times e^{0.06}$ $1 + r = e^{0.06}$ The effective rate when the investment is compounded continuously is $e^{0.06} - 1$.	H5, H9 • Obtains the correct solution 1

2009
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

Student name / number ....

Marks

Question 1 **Begin a new booklet**

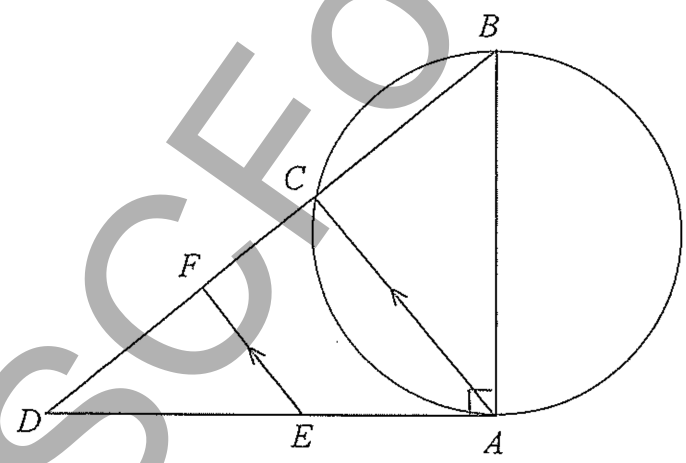
(a) Find $\lim_{x \to 0} \frac{\sin 3x}{2x}$. **2**

(b) Find the limiting sum of the geometric series $\left(\frac{e}{e+1}\right)+\left(\frac{e}{e+1}\right)^2+\left(\frac{e}{e+1}\right)^3+\ldots$. **2**

(c) The equation $x^3+2x^2+3x+6=0$ has roots α, β and γ. Find the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. **2**

(d) Find the acute angle between the lines $y=2x$ and $x+y-3=0$, giving the answer correct to the nearest degree. **2**

(e)



AB is a diameter of the circle and C is a point on the circle. The tangent to the circle at A meets BC produced at D. E is a point on AD and F is a point on CD such that EF is parallel to AC.

(i) Give a reason why $\angle EAC = \angle ABC$. **1**

(ii) Hence show that $EABF$ is a cyclic quadrilateral. **2**

(iii) Show that BE is a diameter of the circle through E, A, B and F. **1**

Student name / number ....

Marks

Question 2 **Begin a new booklet**

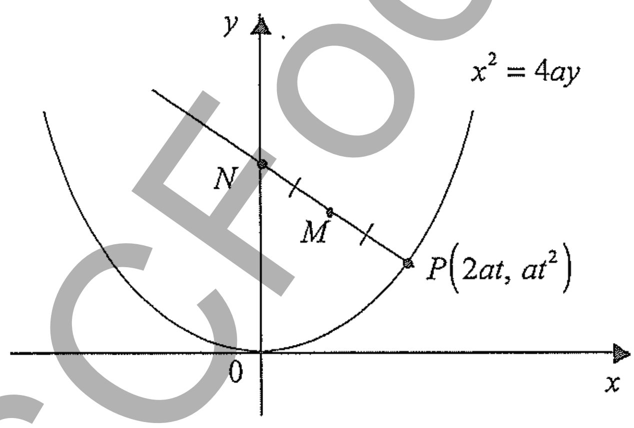
(a) Evaluate $\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx$, giving the answer in simplest exact form. **2**

(b) Find the number of ways in which 3 boys and 3 girls can be arranged in a line so that the two end positions are occupied by boys and no two boys are next to each other. **2**

(c) $A(-2, 3)$ and $B(6, -1)$ are two points. Find the coordinates of the point P that divides the interval AB internally in the ratio 3 : 2. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to show that $\dfrac{\sin x}{1-\cos x} = \cot\frac{x}{2}$. **2**

(e)



$P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$. The normal to the parabola at P cuts the y-axis at N. M is the midpoint of PN.

(i) Use differentiation to show that the normal to the parabola at P has equation $x + ty = 2at + at^3$. **2**

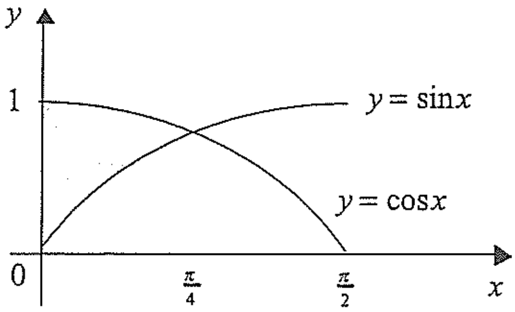
(ii) Find the equation of the locus of M as P moves on the parabola. **2**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a)



The region bounded by the curves $y = \cos x$ and $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution around the x-axis. Find the volume of the solid of revolution. **2**

(b) Use Mathematical Induction to show that for all positive integers $n \geq 2$, **4**

$$2\times1 + 3\times2 + 4\times3 + \ldots + n(n-1) = \frac{n(n^2-1)}{3}.$$

(c) Consider the function $f(x) = (x+2)^2 - 9$, $-2 \leq x \leq 2$.

(i) Find the equation of the inverse function $f^{-1}(x)$. **1**

(ii) On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the coordinates of the endpoints and the intercepts on the coordinate axes. **3**

(iii) Find the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = f^{-1}(x)$, giving the answer in simplest exact form. **2**

Student name / number ....

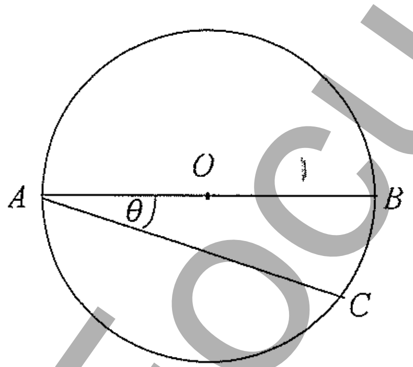
Marks

Question 4 **Begin a new booklet**

(a) Bob chooses six numbers from the numbers 1 to 40 inclusive. A machine then chooses six numbers at random from the numbers 1 to 40 inclusive. Find the probability that none of Bob's numbers match the numbers chosen by the machine, giving the answer correct to 2 decimal places. **2**

(b) Use the substitution $u = \sin^2 x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x}\,dx$, giving the answer in simplest exact form. **4**

(c)



AOB is a diameter of a circle with centre O and radius 1 metre. AC is a chord of the circle such that $\angle BAC = \theta$, where $0 < \theta < \frac{\pi}{2}$. The area of that part of the circle contained between the diameter AB and the chord AC is equal to one quarter of the area of the circle.

(i) Show that $\theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4} = 0$. **2**

(ii) Show that $0\cdot 4 < \theta < 0\cdot 5$. **2**

(iii) Use one application of Newton's method with an initial approximation $\theta_0 = 0\cdot 4$ to find the next approximation to the value of θ, giving your answer correct to 2 decimal places. **2**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a) Consider the function $f(x) = \tan^{-1}(x-1)$.

(i) Sketch the curve $y = f(x)$ showing clearly the equations of any asymptotes and the intercepts on the coordinate axes. **2**

(ii) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 1$. **2**

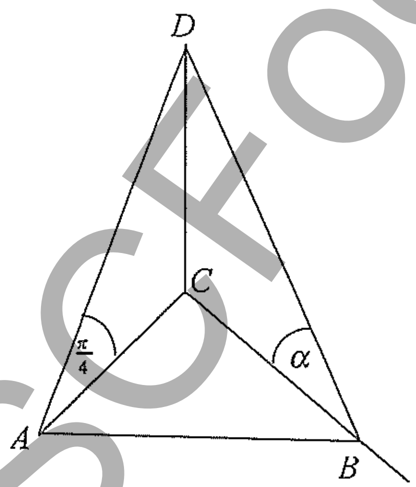
(b) A particle is moving in a straight line. After time t seconds, it has displacement x metres from a fixed point O in the line, velocity $v\ \text{ms}^{-1}$ given by $v = \sqrt{x}$ and acceleration $a\ \text{ms}^{-2}$. Initially the particle is 1 metre to the right of O.

(i) Show that a is independent of x. **1**

(ii) Express x in terms of t. **2**

(iii) Find the distance travelled by the particle during the third second of its motion. **1**

(c)



A vertical tower CD of height 15 metres stands with its base C on horizontal ground. A is a point on the ground due South of C such that the angle of elevation of the top D of the tower from A is $\frac{\pi}{4}$ radians. B is a variable point on the ground due East of C such that the angle of elevation of the top D of the tower from B is α radians, where $0 < \alpha < \frac{\pi}{2}$. The value of α is increasing at a constant rate of $0 \cdot 01$ radians per second.

(i) show that $AB = 15\operatorname{cosec}\alpha$. **2**

(ii) Find the rate at which the length AB is changing when $\alpha = \frac{\pi}{3}$. **2**

Student name / number ....

Marks

Question 6 **Begin a new booklet**

(a) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = -12\sin(2t + \frac{\pi}{3})$ and acceleration $\ddot{x}$ ms^{-2}. Initially the particle is 5 metres to the right of O.

(i) Show that $\ddot{x} = -4(x-2)$. **3**

(ii) Find the period and the extremities of the motion. **2**

(iii) Find the time taken by the particle to return to its starting point for the first time. **1**

(b) After t hours, the number of individuals in a population is given by $N = 500 - 400e^{-0\cdot 1t}$.

(i) Sketch the graph of N as a function of t, showing clearly the initial population size and the limiting population size. **2**

(ii) Show that $\dfrac{dN}{dt} = 0\cdot 1\,(500 - N)$. **1**

(iii) Find the population size for which the rate of growth of the population is half the initial rate of growth. **1**

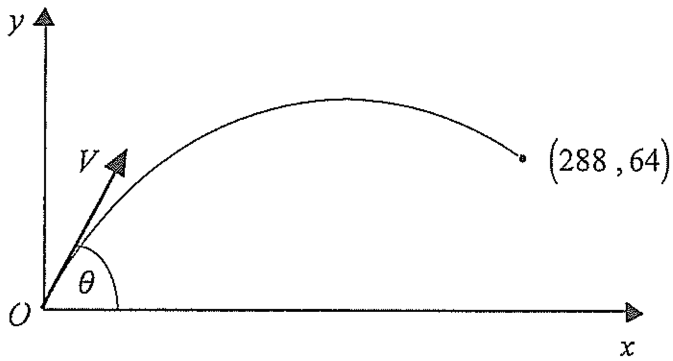
(c) If $\cos^{-1}x - \sin^{-1}x = k$, where $-\frac{\pi}{2} \le k \le \frac{3\pi}{2}$, show that $x = \frac{1}{\sqrt{2}}\left(\cos\frac{k}{2} - \sin\frac{k}{2}\right)$. **2**

Student name / number ....

Marks

Question 7 **Begin a new booklet**

(a)



A toy rocket is projected from a point O with speed V ms^{-1} at an angle θ above the horizontal, where $0 < \theta < \frac{\pi}{2}$. The rocket moves in a vertical plane under gravity where the acceleration due to gravity is 10 ms^{-2}. After 8 seconds the rocket hits a target at a horizontal distance 288 metres from O and at a height 64 metres above O.

(i) Use integration to show that after time t seconds, the horizontal and vertical displacements of the rocket from O, x metres and y metres respectively, are given by $x = Vt\cos\theta$ and $y = Vt\sin\theta - 5t^2$. **2**

(ii) Find the exact values of V and θ. **3**

(iii) Find the velocity of the rocket just before impact with the target, giving the speed correct to the nearest integer and the angle to the horizontal correct to the nearest degree. **3**

(b)(i) By considering the term in x^r on both sides of the identity $(1+x)^{m+n} = (1+x)^m(1+x)^n$, show that ${}^{m+n}C_r = \sum_{k=0}^{r} {}^mC_k\,{}^nC_{r-k}$, for $0 \le r \le m$ and $0 \le r \le n$. **2**

(ii) Hence show that ${}^{m+1}C_0\,{}^nC_2 + {}^{m+1}C_1\,{}^nC_1 + {}^{m+1}C_2\,{}^nC_0 = {}^mC_0\,{}^{n+1}C_2 + {}^mC_1\,{}^{n+1}C_1 + {}^mC_2\,{}^{n+1}C_0$ for $m \ge 2$ and $n \ge 2$. **2**

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• rearranges in terms of known trigonometric limit	1
• evaluates limit	1

Answer

$$\lim_{x\to 0}\frac{\sin 3x}{2x}=\tfrac{3}{2}\lim_{x\to 0}\frac{\sin 3x}{3x}=\tfrac{3}{2}\times 1=\tfrac{3}{2}$$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• identifies a and r for the G.P	1
• applies formula for limiting sum	1

Answer

$\left(\frac{e}{e+1}\right)+\left(\frac{e}{e+1}\right)^2+\left(\frac{e}{e+1}\right)^3+\dots$ is G.P. with $a=\frac{e}{e+1}$, and $r=\frac{e}{e+1}\Rightarrow 0<r<1$

$\therefore$ Limiting sum is $\frac{a}{1-r}=\frac{e}{e+1}\div\frac{1}{e+1}=e$

c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• expresses sum of reciprocals of roots in terms of sums of products	1
• evaluates using relationships between roots and coefficients	1

Answer

α, β and γ roots of $x^3+2x^2+3x+6=0$. $\quad\therefore\ \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=\frac{3}{-6}=-\frac{1}{2}$

d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• substitutes values of gradients into formula for tangent of acute angle between the lines	1
• evaluates required angle	1

Answer

Acute angle θ between lines $y=2x$ and $x+y-3=0$ is given by $\tan\theta=\left|\frac{2-(-1)}{1+2\,.\,(-1)}\right|=3$

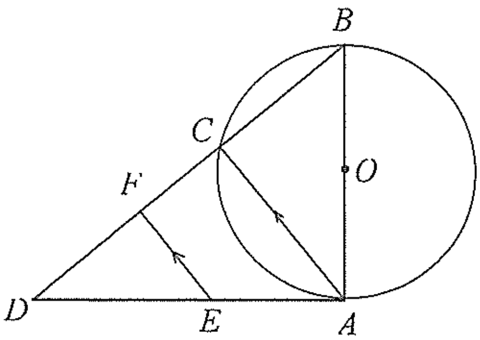
$\therefore \theta\approx 72°$ (to the nearest degree)

e. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • quotes alternate segment theorem	1
ii • gives a sequence of deductions resulting in a test for a cyclic quadrilateral	1
• justifies these deductions by quoting geometric properties and tests	1
iii • explains why BE subtends a right angle at A or at F	1

Answer



Let O be the centre of the circle.

i. The angle between the tangent at A and the chord AC is equal to the angle subtended by that chord in the alternate segment, hence $\angle EAC = \angle ABC$.

ii. $\angle EAC = \angle DEF$ (*Corresp. $\angle$'s with parallel lines AC, EF are equal*)

$\therefore \angle DEF = \angle ABC$ (*Both equal to $\angle EAC$*)

$\therefore EABF$ is cyclic (*Exterior $\angle$ equal to interior opp. $\angle$*)

iii. $\angle BAE = 90°$ (*Tangent to circle ABC at A is perpendicular to radius OA drawn to point of contact*)

$\therefore BE$ is a diameter of circle $EABF$. (*subtends right $\angle$ at circumference*)

Question 2

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• finds primitive	1
• evaluates in surd form	1

Answer

$$\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx = \tfrac{1}{2}\left[\sec 2x\right]_0^{\frac{\pi}{8}} = \tfrac{1}{2}\left(\sqrt{2}-1\right)$$

b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• counts arrangements for one possible pattern of B's and G's	1
• adds number of arrangements for the second possible pattern of B's and G's	1

Answer

B G B G G B or B G G B G B $\quad \therefore 2\times 3!\times 3! = 72$ ways

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• finds x coordinate of P	1
• finds y coordinate of P	1

Answer

$A(-2\,,3) \qquad B(6\,,-1)$

$3 \quad : \quad 2$

$$P\left(\frac{3\times 6+2\times(-2)}{3+2}\,,\ \frac{3\times(-1)+2\times 3}{3+2}\right)$$

$\therefore$ P has coordinates $P\left(\frac{14}{5}\,,\frac{3}{5}\right)$

d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• simplifies $1-\cos x$ in terms of t	1
• completes simplification of given expression in terms of t to establish required result	1

Answer

$t=\tan\frac{x}{2}$

$$1-\cos x = 1-\frac{1-t^2}{1+t^2} = \frac{2t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\therefore \frac{\sin x}{1-\cos x} = \frac{2t}{1+t^2}\times\frac{1+t^2}{2t^2} = \frac{1}{t} = \cot\frac{x}{2}$$

e. Outcomes assessed : PE3, PE4

Marking Guidelines

Criteria	Marks
i • finds $\frac{dy}{dx}$ as a function of t	1
• finds equation of normal in required form	1
ii • finds coordinates of M	1
• finds equation of locus of M	1

Answer

i.

$$y=at^2 \Rightarrow \frac{dy}{dt}=2at$$

$$x=2at \Rightarrow \frac{dx}{dt}=2a$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=t$$

$\therefore$ Normal at P has gradient $-\frac{1}{t}$ and equation

$$y-at^2=-\tfrac{1}{t}(x-2at)$$

$$ty-at^3=-x+2at$$

$$x\ +\ ty=2at+at^3$$

ii. $N\left(0\,,2a+at^2\right) \qquad \therefore M\left(at\,,a+at^2\right)$

$P\left(2at\,,at^2\right)$

Locus of M has equation $y=a+a\left(\frac{x}{a}\right)^2$

$$x^2=a(y-a)$$

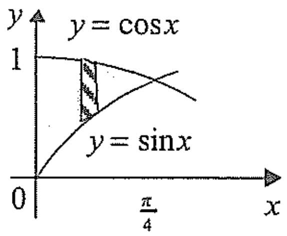
Question 3

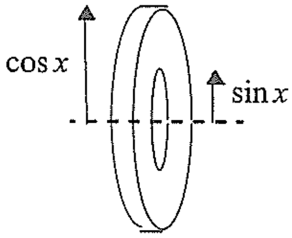
a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• writes definite integral for the volume in terms of $\cos x$ and $\sin x$	1
• evaluates the integral.	1

Answer





$$V = \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \cos 2x \, dx$$

$$= \tfrac{1}{2}\pi \left[\sin 2x \right]_0^{\frac{\pi}{4}}$$

$$= \tfrac{1}{2}\pi\, (1 - 0)$$

Volume is $\frac{\pi}{2}$ cubic units.

b. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• defines an appropriate sequence of statements $S(n)$ and shows the first member is true	1
• writes the LHS of $S(k+1)$ in terms of RHS of $S(k)$, conditional on truth of $S(k)$	1
• rearranges conditional expression for LHS of $S(k+1)$ to obtain RHS	1
• completes proof by Mathematical Induction	1

Answer

Let $S(n)$, $n = 2, 3, 4, \ldots$, be the sequence of statements defined by

$$S(n): \quad 2\times 1 + 3\times 2 + 4\times 3 + \ldots + n(n-1) = \frac{n(n^2-1)}{3}$$

Consider $S(2)$: $\quad LHS = 2\times 1 = 2$; $\quad RHS = \dfrac{2(2^2-1)}{3} = 2$. $\quad$ Hence $S(2)$ is true.

If $S(k)$ is true : $\quad 2\times 1 + 3\times 2 + 4\times 3 + \ldots + k(k-1) = \dfrac{k(k^2-1)}{3} \quad *$

Consider $S(k+1)$: $\quad LHS = \left\{2\times 1 + 3\times 2 + 4\times 3 + \ldots + k(k-1)\right\} + (k+1)k$

$$= \frac{k(k^2-1)}{3} + (k+1)k \qquad \text{if } S(k) \text{ is true, using } *.$$

$$= \frac{k(k+1)\left\{(k-1)+3\right\}}{3}$$

$$= \frac{(k+1)\left\{k^2+2k\right\}}{3}$$

$$= \frac{(k+1)\left\{(k+1)^2-1\right\}}{3}$$

$$= RHS$$

Hence if $S(k)$ is true then $S(k+1)$ is true. But $S(2)$ is true, and hence $S(3)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n \geq 2$.

c. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i • rearranges and interchanges x and y to obtain equation of inverse function	1
ii • sketches graph of $y = f(x)$ showing endpoints and intercepts	1
• sketches inverse function by reflection in $y = x$	1
• shows endpoints and intercepts for inverse function	1
iii • writes equation for x	1
• solves for x in simplest exact form	1

Answer

i. $f(x) = (x+2)^2 - 9\,,\quad -2 \le x \le 2.$

$(x+2)^2 = y+9 \qquad \text{and} \quad 0 \le x+2 \le 4$

$x+2 = +\sqrt{y+9}$

$\therefore \quad x = -2 + \sqrt{y+9}\;,\quad -9 \le y \le 7$

$\therefore x \leftrightarrow y \;\Rightarrow\; f^{-1}(x) = -2 + \sqrt{x+9}\,,\ -9 \le x \le 7$

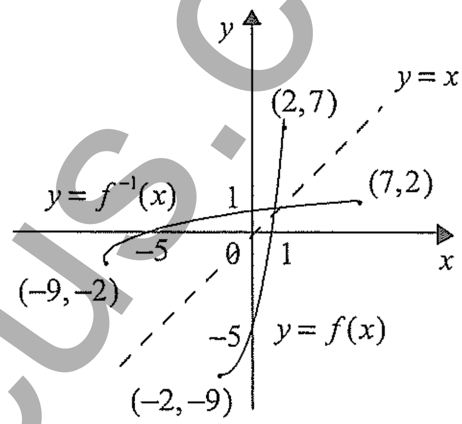
iii. Graphs intersect on the line $y = x$.

Hence $(x+2)^2 - 9 = x$

$x^2 + 3x - 5 = 0$

$\therefore x > 0 \Rightarrow x = \dfrac{-3+\sqrt{29}}{2}$

ii. Graphs of inverse functions are reflections of each other in $y = x$



Question 4

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
• writes expression for probability in terms of binomial coefficients	1
• evaluates required probability	1

Answer

$P(\textit{none in common}) = \dfrac{^{34}C_6}{^{40}C_6} \approx 0\cdot 35$ (to 2 decimal places)

b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes du in terms of dx and converts limits for x into limits for u	1
• finds equivalent definite integral in terms of u	1
• finds primitive and substitutes limits	1
• simplifies exact answer	1

Answer

$u = \sin^2 x$

$du = 2\sin x \cos x \; dx$

$du = \sin 2x \; dx$

$x = \frac{\pi}{4} \Rightarrow u = \frac{1}{2}$

$x = \frac{\pi}{3} \Rightarrow u = \frac{3}{4}$

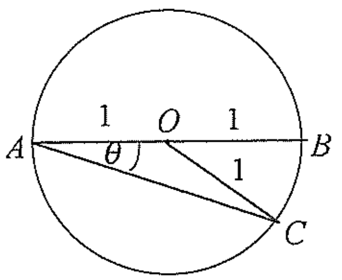
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x}\, dx = \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{1+u}\, du$$
$$= \left[\ln(1+u)\right]_{\frac{1}{2}}^{\frac{3}{4}}$$
$$= \ln\tfrac{7}{4} - \ln\tfrac{3}{2}$$
$$= \ln\tfrac{7}{6}$$

c. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • finds area of ΔAOC in terms of $\sin 2\theta$	1
• uses area information to complete equation for θ	1
ii • shows that $f(0\cdot 4)$, $f(0\cdot 5)$ have opposite signs	1
• notes that f is continuous, and deduces equation has one root θ, $0\cdot 4 < \theta < 0\cdot 5$	1
iii • applies Newton's rule to write numerical expression for next approximation	1
• evaluates this approximation	1

Answer



i. $\angle OCA = \theta$ *($\angle$'s opp. equal sides are equal in ΔAOC)*

$\angle AOC = \pi - 2\theta$ *($\angle$ sum of Δ is π)*

$\angle BOC = 2\theta$ *(adj. supp. $\angle$'s add to π)*

Area sector BOC + Area ΔAOC = $\frac{1}{4}$ Area circle

$$\therefore \tfrac{1}{2}\times 1^2 \times 2\theta + \tfrac{1}{2}\times 1^2 \times \sin(\pi - 2\theta) = \tfrac{1}{4}\times \pi \times 1^2$$
$$\theta + \tfrac{1}{2}\sin 2\theta - \tfrac{\pi}{4} = 0$$

ii. Let $f(\theta) = \theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4}$

$f(0\cdot 4) \approx -0\cdot 03 < 0$

$f(0\cdot 5) \approx \; 0\cdot 14 > 0$ and f is continuous

Also $f'(\theta) = 1 + \cos 2\theta > 0 \Rightarrow f$ monotonic increasing

$\therefore f(\theta) = 0$ for exactly one value of θ, $0\cdot 4 < \theta < 0\cdot 5$

iii. Since $f'(\theta) = 1 + \cos 2\theta$,

$$\theta \approx 0\cdot 4 - \frac{f(0\cdot 4)}{f'(0\cdot 4)}$$
$$\approx 0\cdot 4 - \frac{-0\cdot 0267}{1\cdot 6967}$$
$$\approx 0\cdot 42 \quad \text{(to 2 dec. pl.)}$$

Question 5

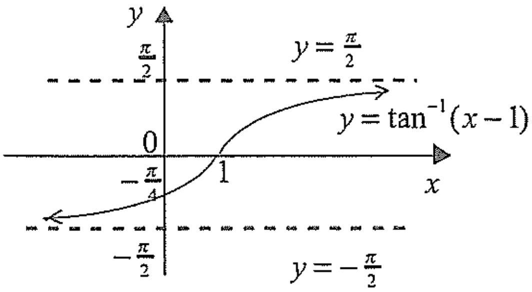
a. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i • shows correct shape and asymptotes	1
• shows intercepts on coordinate axes	1
ii • finds $\frac{dy}{dx}$ and evaluates for $x = 1$	1
• finds equation of tangent	1

Answer

i.



ii. $y = \tan^{-1}(x-1)$

$$\frac{dy}{dx} = \frac{1}{1+(x-1)^2}$$

$$\therefore \frac{dy}{dx} = 1 \quad \text{when} \quad x = 1$$

$\therefore$ Tangent at $(1, 0)$ has gradient 1 and equation $y = x - 1$

b. Outcomes assessed : HE5

Marking Guidelines

Criteria	Marks
i • shows by differentiation that a is constant	1
ii • integrates to find a primitive function for t in terms of x	1
• evaluates constant of integration using initial conditions then writes x as a function of t	1
iii • evaluates x at $t=2$ and $t=3$ to find distance travelled in third second.	1

Answer

i. $v = \sqrt{x} \quad \Rightarrow \tfrac{1}{2}v^2 = \tfrac{1}{2}x$

$\therefore a = \dfrac{d}{dx}(\tfrac{1}{2}v^2) = \tfrac{1}{2}$ for all x

Hence a is independent of x.

ii. $$\frac{dx}{dt} = x^{\frac{1}{2}}$$

$$\frac{dt}{dx} = x^{-\frac{1}{2}}$$

$$t = 2x^{\frac{1}{2}} + c$$

$$\left.\begin{matrix} t=0 \\ x=1 \end{matrix}\right\} \Rightarrow c = -2$$

$$\therefore t = 2\sqrt{x} - 2$$

$$x = \tfrac{1}{4}(t+2)^2$$

iii. Between $t=2$ and $t=3$, particle moves right from $x=4$ to $x = \frac{25}{4}$

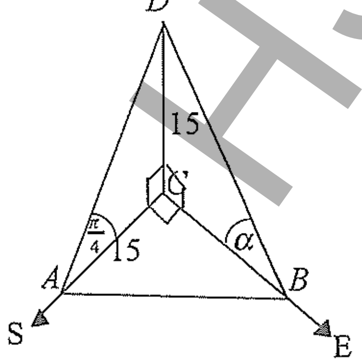
Distance travelled in third second is $2 \cdot 25$ m.

c. Outcomes assessed : H5, HE5, HE7

Marking Guidelines

Criteria	Marks
i • finds AC and finds BC in terms of $\cot\alpha$	1
• uses Pythagoras' theorem and an appropriate trig. identity to find AB in terms of $\operatorname{cosec}\alpha$	1
ii • differentiates AB with respect to t using chain rule or implicit differentiation	1
• substitutes given values and interprets result	1

Answer



i. In ΔACD,

$\angle DAC = \angle ADC = \frac{\pi}{4}$

$\therefore AC = 15$.

In ΔBCD, $BC = 15\cot\alpha$.

$\therefore$ In ΔABC,

$$AB^2 = 15^2 + 15^2\cot^2\alpha$$
$$= 15^2(1+\cot^2\alpha)$$
$$= 15^2\operatorname{cosec}^2\alpha$$

$\therefore AB = 15\operatorname{cosec}\alpha$

ii. When $\alpha = \frac{\pi}{3}$,

$$\frac{dAB}{dt} = -15\operatorname{cosec}\alpha\cot\alpha\,\frac{d\alpha}{dt}$$

$$= -15 \times \tfrac{2}{\sqrt{3}} \times \tfrac{1}{\sqrt{3}} \times 0 \cdot 01$$

$$= -0 \cdot 1$$

$\therefore AB$ is decreasing at a rate of $0 \cdot 1 \text{ ms}^{-1}$

Question 6

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • integrates v with respect to t to find expression for x	1
• uses initial conditions to evaluate the constant of integration, giving x as a function of t	1
• differentiates v with respect to t to get $\ddot{x}$ then expresses $\ddot{x}$ in terms of x	1
ii • states period	1
• states extremities	1
iii • solves trig. equation to find time to first return	1

Answer

i.
$$v = -12\sin(2t + \tfrac{\pi}{3})$$

$x = 6\cos(2t + \tfrac{\pi}{3}) + c$

$t = 0,\ x = 5 \Rightarrow c = 2$

$\therefore x = 2 + 6\cos(2t + \tfrac{\pi}{3})$

$\ddot{x} = -24\cos(2t + \tfrac{\pi}{3})$

$\therefore \ddot{x} = -4(x - 2)$

ii. Period is π seconds. $-4 \le x \le 8$

iii. $x = 5 \Rightarrow \cos(2t + \tfrac{\pi}{3}) = \tfrac{1}{2}$

$2t + \tfrac{\pi}{3} = \tfrac{\pi}{3},\ 2\pi - \tfrac{\pi}{3}, \ldots$

$t = 0,\ \tfrac{2\pi}{3},\ \ldots$

First return after $\tfrac{2\pi}{3}$ seconds.

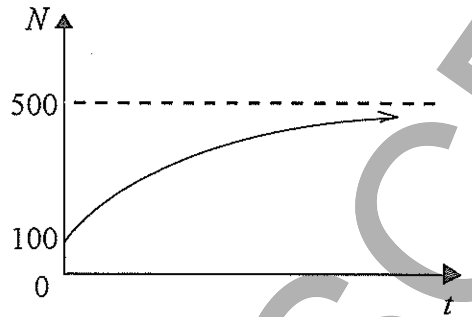
b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • sketches graph of correct shape with correct vertical intercept	1
• shows asymptote for limiting population size	1
ii • differentiates with respect to t	1
iii • writes and solves equation for N	1

Answer

i.



ii.

$N = 500 - 400e^{-0\cdot 1t}$

$\dfrac{dN}{dt} = 0\cdot 1 \times 400e^{-0\cdot 1t}$

$= 0\cdot 1\,(500 - N)$

iii.

Initial rate of growth is

$0\cdot 1\,(500 - 100) = 0\cdot 1 \times 400$

$\therefore$ want N such that

$0\cdot 1\,(500 - N) = 0\cdot 1 \times 200$

$500 - N = 200$

$N = 300$

c. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
• uses inverse trig. identity to simplify equation	1
• uses trig. expansion to evaluate x in terms of k	1

Answer

$\cos^{-1} x - \sin^{-1} x = k, \quad -\tfrac{\pi}{2} \le k \le \tfrac{3\pi}{2}$

$\cos^{-1} x + \sin^{-1} x = \tfrac{\pi}{2}$

$\therefore 2\cos^{-1} x = k + \tfrac{\pi}{2}$

$\cos^{-1} x = \tfrac{k}{2} + \tfrac{\pi}{4}$

$x = \cos(\tfrac{k}{2} + \tfrac{\pi}{4})$

$\therefore x = \cos\tfrac{k}{2}\cos\tfrac{\pi}{4} - \sin\tfrac{k}{2}\sin\tfrac{\pi}{4}$

$= \tfrac{1}{\sqrt{2}}(\cos\tfrac{k}{2} - \sin\tfrac{k}{2})$

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • uses integration to find expressions for $\dot{x}$ and x	1
• uses integration to find expressions for $\dot{y}$ and y	1
ii • writes simultaneous equations for V and θ	1
• finds the value of V	1
• finds the value of θ	1
iii • finds the values of $\dot{x}$ and $\dot{y}$ just before impact	1
• uses Pythagoras' theorem to find the magnitude of v	1
• uses trigonometry to find the direction of v as an angle relative to the horizontal	1

Answer

i.

$$\ddot{x}=0 \qquad \dot{x}=c_1 \qquad \therefore \dot{x}=V\cos\theta$$

$$x=Vt\cos\theta+c_2 \qquad \left.\begin{aligned}t&=0\\x&=0\end{aligned}\right\}\Rightarrow c_2=0 \qquad \therefore x=Vt\cos\theta$$

$$\ddot{y}=-10 \qquad \dot{y}=-10t+c_3 \qquad \left.\begin{aligned}t&=0\\\dot{y}&=V\sin\theta\end{aligned}\right\}\Rightarrow c_3=V\sin\theta \qquad \therefore \dot{y}=-10t+V\sin\theta$$

$$y=-5t^2+Vt\sin\theta+c_4 \qquad \left.\begin{aligned}t&=0\\y&=0\end{aligned}\right\}\Rightarrow c_4=0 \qquad \therefore y=Vt\sin\theta-5t^2$$

ii. When $t=8$

$$\left.\begin{aligned}x&=288\\y&=64\end{aligned}\right\}\Rightarrow \begin{aligned}8V\cos\theta&=288\\8V\sin\theta&=384\end{aligned}$$

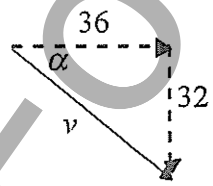
$$\therefore V^2(\cos^2\theta+\sin^2\theta)=36^2+48^2 \qquad V=60$$

$$\tan\theta=\tfrac{48}{36}=\tfrac{4}{3} \qquad \theta=\tan^{-1}\tfrac{4}{3}$$

iii. When $t=8$

$$\dot{x}=60\times\tfrac{3}{5}=36$$

$$\dot{y}=-80+60\times\tfrac{4}{5}=-32$$



$$v^2=36^2+32^2 \qquad v=4\sqrt{145}$$

$$\tan\alpha=\tfrac{8}{9} \qquad \alpha\approx 41\cdot 6^\circ$$

Velocity of rocket just before impact is approximately $48\ \text{ms}^{-1}$ inclined at 42° below the horizontal.

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes a typical term in x^r in the expansion of the RHS of the identity	1
• collects like terms to find coefficient of x^r, then equates to coefficient of x^r on LHS	1
ii • writes single binomial coefficient for sum on LHS	1
• writes single binomial coefficient for sum on RHS then deduces result	1

Answer

i. $(1+x)^{m+n}=(1+x)^m(1+x)^n$

For $0\le r\le m$ and $0\le r\le n$,

terms in x^r in expansion of the RHS have the form ${}^mC_k x^k\times{}^nC_{r-k}x^{r-k}$, $k=0,1,2,\ldots,r$.

Collecting such like terms gives the coefficient of x^r as $\sum_{k=0}^{r}{}^mC_k\,{}^nC_{r-k}$.

The coefficient of x^r in the expansion of the LHS is ${}^{m+n}C_r$.

Hence equating coefficients of x^r on both sides of the identity gives ${}^{m+n}C_r=\sum_{k=0}^{r}{}^mC_k\,{}^nC_{r-k}$.

ii. Using i., for $m\ge 2$ and $n\ge 2$,

${}^{m+1}C_0\,{}^nC_2+{}^{m+1}C_1\,{}^nC_1+{}^{m+1}C_2\,{}^nC_0={}^{(m+1)+n}C_2$ and ${}^mC_0\,{}^{n+1}C_2+{}^mC_1\,{}^{n+1}C_1+{}^mC_2\,{}^{n+1}C_0={}^{m+(n+1)}C_2$

$\therefore\ {}^{m+1}C_0\,{}^nC_2+{}^{m+1}C_1\,{}^nC_1+{}^{m+1}C_2\,{}^nC_0={}^mC_0\,{}^{n+1}C_2+{}^mC_1\,{}^{n+1}C_1+{}^mC_2\,{}^{n+1}C_0={}^{m+n+1}C_2$

Independent Trial Examination 2009 Mathematics Extension 1 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1 a	2	Trigonometric functions	H5	E2-E3
b	2	Series and applications	H5	E2-E3
c	2	Polynomials	PE3	E2-E3
d	2	Angle between two lines	H5	E2-E3
e i	1	Circle geometry	PE3	E2-E3
ii	2	Circle geometry	PE2, PE3	E2-E3
iii	1	Circle geometry	PE3	E2-E3
2 a	2	Trigonometric functions	H5	E2-E3
b	2	Permutations and combinations	PE3	E2-E3
c	2	Division of an interval	H5	E2-E3
d	2	Trigonometric functions	H5	E2-E3
e i	2	Parametric representation	PE3, PE4	E2-E3
ii	2	Parametric representation	PE3, PE4	E2-E3
3 a	2	Trigonometric functions	H5	E2-E3
b	4	Induction	HE2	E3-E4
c i	1	Inverse functions	HE4	E2-E3
ii	3	Inverse functions	HE4	E2-E3
iii	2	Inverse functions	HE4	E2-E3
4 a	2	Further probability	HE3	E2-E3
b	4	Methods of integration	HE6	E2-E3
c i	2	Trigonometric functions	H5	E2-E3
ii	2	Polynomials	PE3	E2-E3
iii	2	Polynomials	PE3	E2-E3
5 a i	2	Inverse functions	HE4	E2-E3
ii	2	Inverse functions	HE4	E2-E3
b i	1	Velocity and acceleration as functions of displacement	HE5	E2-E3
ii	2	Velocity and acceleration as functions of displacement	HE5	E2-E3
iii	1	Velocity and acceleration as functions of displacement	HE5	E2-E3
c i	2	Trigonometric functions	H5	E3-E4
ii	2	Rates of change	HE5, HE7	E3-E4
6 a i	3	Simple harmonic motion	HE3	E3-E4
ii	2	Simple harmonic motion	HE3	E3-E4
iii	1	Simple harmonic motion	HE3	E3-E4
b i	2	Exponential growth and decay	HE3	E2-E3
ii	1	Exponential growth and decay	HE3	E2-E3
iii	1	Exponential growth and decay	HE3	E2-E3
c	2	Trigonometric functions, inverse functions	H5, HE4	E2-E3
7 a i	2	Projectile motion	HE3	E3-E4
ii	3	Projectile motion	HE3	E3-E4
iii	3	Projectile motion	HE3	E3-E4
b i	2	Binomial theorem	HE3	E3-E4
ii	2	Binomial theorem	HE3	E3-E4