Trial HSC Papers 2012

Mathematics Extension 1

Centre Number

Student Number

CSSA

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2012
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Morning Session
Friday, 10 August 2012

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a separate sheet
- Show all necessary working for Questions 11-14
- Write your Centre Number and Student Number at the top of this page and page 6

Total marks – 70

Section I Pages 2–5

10 marks
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–12

60 marks
- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I
10 marks

Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Let $P(x) = 2x^3 - x^2 + 2$. Find the remainder when $P(x)$ is divided by $x + 1$.

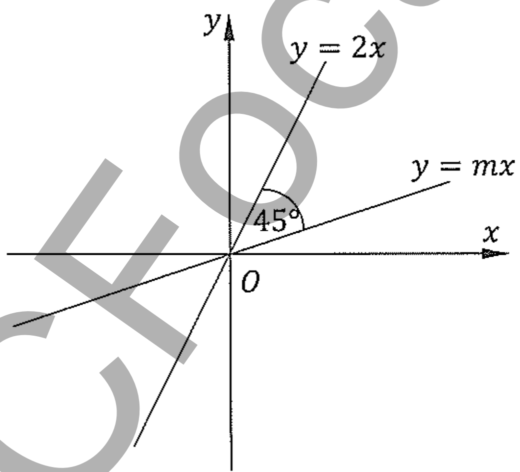
(A) -1

(B) 1

(C) 3

(D) 5

2 The angle between the lines $y = mx$ and $y = 2x$ is 45°, where $m > 0$, as shown in the diagram below.



NOT TO SCALE

Find the value of m.

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) 1

(D) 3

3 Let $t = \tan\frac{\theta}{2}$ where $0 < \theta < \pi$. Which of the following gives the correct expression for $\sin\theta + \cos\theta$?

(A) $\dfrac{3t - 1}{1 + t}$

(B) $\dfrac{2t - 1 + t^2}{1 + t^2}$

(C) $\dfrac{1 + 2t - t^2}{1 + t^2}$

(D) $\dfrac{t^2 - 1 - 2t}{1 + t^2}$

4 Let A be the point $(-2, 3)$ and B be the point $(3, -4)$. Find the coordinates of the point which divides AB externally in the ratio $3:2$.

(A) $\left(0, \frac{1}{5}\right)$

(B) $\left(1, -\frac{6}{5}\right)$

(C) $(-12, 17)$

(D) $(13, -18)$

5 From six girls and four boys, a committee of 3 girls and 2 boys is to be chosen. How many different committees can be formed?

(A) 26

(B) 120

(C) 252

(D) 1440

6 Consider the function $f(x) = \frac{2x}{x+1}$ and its inverse function $f^{-1}(x)$.
Evaluate $f^{-1}(3)$.

(A) -3

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) 3

7 The equation of the normal to the parabola $x^2 = 4ay$ at the variable point $P(2ap, ap^2)$ is given by $x + py = 2ap + ap^3$.

How many different values of p are there such that the normal passes through the focus of the parabola?

(A) 0

(B) 1

(C) 2

(D) 3

8 An advertisement claims that '8 out of 10 people prefer Winky Chocolate Bars'. If the advertisement's claim is accurate and a sample of six people is interviewed, what is the probability that at least five people prefer Winky Chocolate Bars?

(A) $1 - (0.8)^6$

(B) $2(0.8)^5$

(C) $5(0.2)^5$

(D) $(0.8)^5(0.2) + (0.8)^6$

9 What is the coefficient of x^2 in the expansion of $\left(x^2 + \frac{2}{x}\right)^7$?

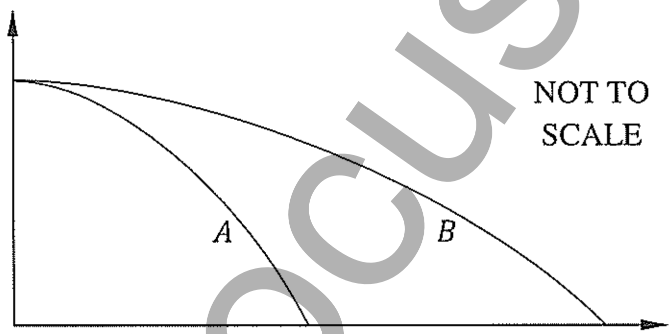
(A) 1

(B) 16

(C) 35

(D) 560

10 Two balls, A and B, are rolled horizontally off a 10 metre cliff at 10 ms^{-1} and 20 ms^{-1} respectively.



Which of the following statements is FALSE?

(A) A and B are in the air for the same length of time.

(B) A and B are travelling with the same vertical speed on impact.

(C) B is travelling at twice the speed of A on impact with the ground.

(D) B lands twice as far from the base of the cliff as A.

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Centre Number

Mathematics Extension 1

Student Number

Section II
60 marks

Attempt Questions 11–14
All questions are of equal value.
Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\lim_{x \to 0} \dfrac{\sin 2x}{x}$. **1**

(b) Use the table of standard integrals to evaluate $\displaystyle\int_0^4 \frac{dx}{\sqrt{x^2+9}}$. **2**

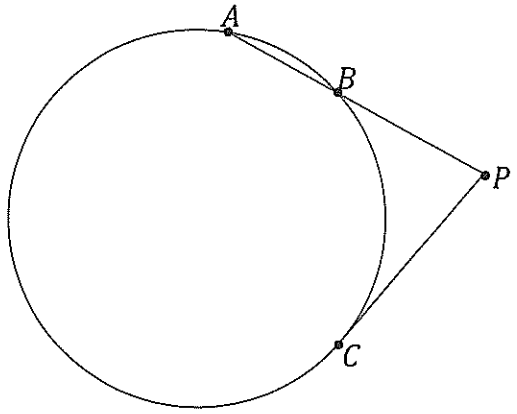
(c) Use the substitution $u = x - 8$ to find $\displaystyle\int_8^{8.5} \frac{dx}{\sqrt{(7-x)(x-9)}}$. **3**

(d) Solve $\dfrac{2t}{1-t} \geq t$. **3**

Question 11 continues on page 7

Question 11 (continued)

(e)



NOT TO SCALE

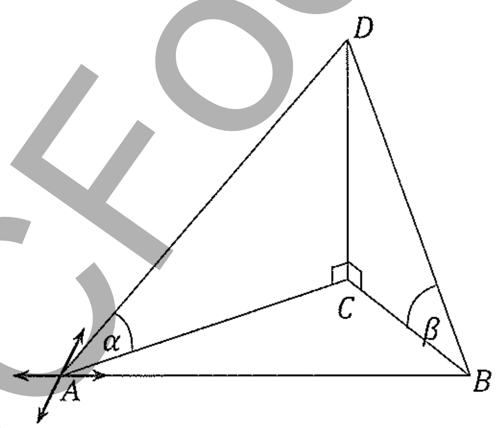
In the diagram the points A, B and C lie on the circle and AB produced meets the tangent from C at the point P.

(i) Given that $PC = 12$, $AB = 7$ and $PB = x$, find x. **2**

(ii) BC is the diameter of the circle passing through P, B and C. **1**

Find the length of BC.

(f) **3**



A vertical pole, CD, is positioned so that the angles of elevation of the top of the pole from the points A and B on the ground are α and β respectively.

The ground is a level horizontal surface and the triangle ABC is right-angled at C. Point B is due east of point A and point C is on a bearing of 060°T from point A.

Show that $\dfrac{\tan\alpha}{\tan\beta} = \dfrac{1}{\sqrt{3}}$.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are real numbers and $a \neq 0$. Let α, β and γ be zeros of $f(x)$.

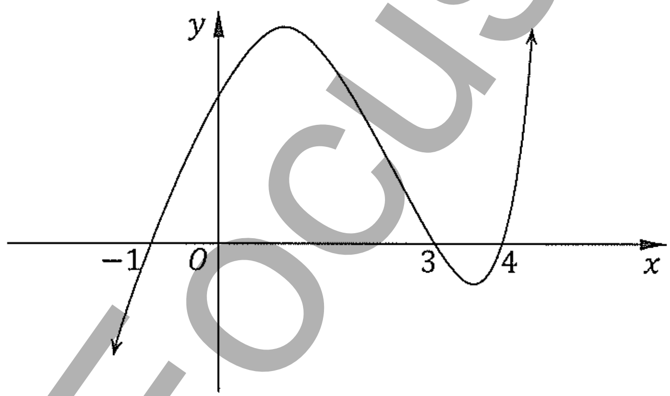
(i) Write down an expression for $\alpha + \beta + \gamma$. **1**

All cubic polynomial functions have a single point of inflexion when the second derivative is equal to zero.

(ii) Using part (i), or otherwise, show that the x-coordinate of the point of inflexion on the curve $y = f(x)$ is given by **2**

$$x = \frac{\alpha + \beta + \gamma}{3}.$$

(iii) The cubic polynomial below has x-intercepts at -1, 3 and 4. Find the x-coordinate of the point of inflexion of the cubic polynomial. **1**



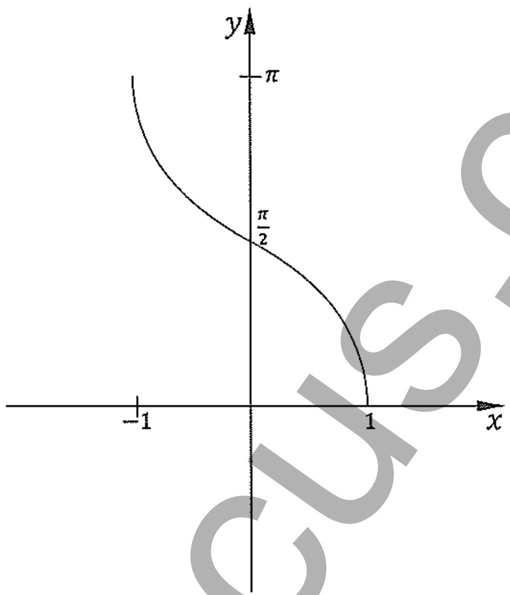
Question 12 continues on page 9

Question 12 (continued)

(b) (i) Given $\sin^{-1}x = \alpha$ and $\cos^{-1}x = \frac{\pi}{2} - \alpha$, solve $\sin^{-1}x = \cos^{-1}x$. **2**

(ii) The curve $y = \cos^{-1}x$ is sketched below. **1**

Copy the diagram into your writing booklet and draw a sketch of the curve $y = \sin^{-1}x$ on the same set of axes. Clearly show the point of intersection of the two curves.



The region bounded by $y = \sin^{-1}x$, $y = \cos^{-1}x$ and the y-axis is rotated about the y-axis to form a solid of revolution.

(iii) Explain why the volume V of the solid formed is given by: **2**

$$V = 2\pi \int_0^{\frac{\pi}{4}} \sin^2 y \, dy.$$

(iv) Hence, find the volume V of the solid formed. **2**

(c) (i) Use mathematical induction to prove that for $n \geq 2$ **3**

$$\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \ldots \times \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

(ii) Hence evaluate $\frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \ldots \times \frac{9999}{10000}$. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) When an egg is placed in a pot of boiling water the rate at which the temperature, T (in degrees Celsius), of the egg increases after t minutes is given by $\frac{dT}{dt} = 0.2(100 - T)$.

(i) Show that $T = 100 - Ae^{-0.2t}$ satisfies this equation, for some constant A. **1**

(ii) Find the value of the constant A, for an egg taken from the refrigerator with an initial temperature of 4°C. **1**

(iii) It is known from experience that it will take $4\frac{1}{2}$ minutes for an egg taken from the refrigerator to cook. **1**

Determine the temperature of the egg after $4\frac{1}{2}$ minutes. Give your answer correct to 3 significant figures.

(iv) If an egg is initially at room temperature, the temperature of the egg can be modelled by the equation $T = 100 - 79e^{-0.2t}$. **2**

How much less time will it take for an egg initially at room temperature to reach the temperature of part (iii)?

(b) A particle moves in Simple Harmonic Motion. Initially, the particle is 4 metres to the right of the origin, moving with a velocity of $-8\sqrt{3}$ ms^{-1}. The displacement x is given by $x = A\cos(2t + \alpha)$ for some constants A and α.

(i) Find the values of A and α. **3**

(ii) When does the particle first reach the centre of motion? **2**

(c) The acceleration of a particle P is given by the equation $\frac{d^2x}{dt^2} = 2x^3 + 18x$, where x is its displacement in metres from the origin after t seconds. Initially the particle is at the origin and has velocity 9 ms^{-1}.

(i) Show that the velocity of the particle is given by $v = x^2 + 9$. **3**

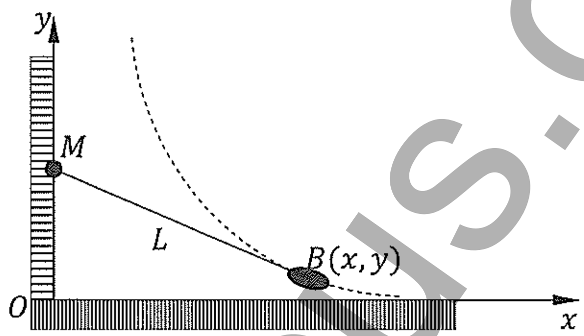
(ii) Hence, find an expression for x as a function of t. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $3\sin x = \ln x$ has a number of positive solutions, with the smallest solution being close to $x = 3$. **3**

Use ONE application of Newton's Method to find another approximation to the smallest positive solution. Give this approximation correct to TWO decimal places.

(b) A man M walks along a pier, represented by the positive y-axis, pulling on a boat B by a rope of length L. The man is initially at the origin O and the boat is initially on the x-axis, L metres from O. The man keeps the rope taut and the path followed by the boat is such that the rope is always tangent to the curve tracing its path.



(i) Let the path followed by the boat be the graph of the function $y = f(x)$. By considering the gradient of the line MB, show that **1**

$$\frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}.$$

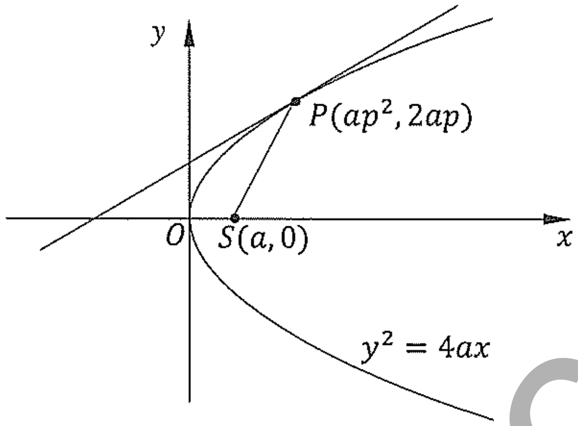
(ii) The man walks along the pier such that the boat moves in the y direction at a constant rate of 3 ms^{-1}. **2**

Find the rate $\frac{dx}{dt}$ at which the boat approaches the pier, when it is a distance $\frac{L}{2}$ metres horizontally from the pier.

Question 14 continues on page 12

Question 14 (continued)

(c) The tangent at the point $P(ap^2, 2ap)$ on the parabola $y^2 = 4ax$, with the focus $S(a, 0)$ is shown in the diagram below. **3**



The gradient function is given by $\frac{dy}{dx} = \frac{2a}{y}$. (Do NOT prove this.)
Prove that the tangent to the parabola at P is equally inclined to the axis of the parabola and the focal chord through P.

(d) (i) Show that $\binom{n}{r} = \binom{n}{n-r}$ for $r = 0, 1, 2, \ldots, n$. **1**

(ii) Let $f(r) = \binom{n}{0}\binom{n}{r} + \binom{n}{1}\binom{n}{r+1} + \cdots + \binom{n}{n-r}\binom{n}{n}$ for $r = 0, 1, 2, \ldots, n$. **3**

By considering the coefficient of x^{n-r} in the expansions of $(1+x)^{2n}$ and $(1+x)^n(1+x)^n$ show that

$$f(r) = \binom{2n}{n-r}$$

(iii) Show that $\binom{n}{0}f(0) + \binom{n}{1}f(1) + \cdots + \binom{n}{n}f(n) = \binom{3n}{n}$. **2**

End of Paper

Examiners

Gerry Sozio (Convenor)	St Mary Star of the Sea College, Wollongong
Jenny Bell	St Joseph's Catholic High School, Albion Park
Frank Reid	University of New South Wales, Australian Catholic University
Thanom Shaw	SCEGGS, Darlinghurst
Greg Wagner	Kesser Torah College, Dover Heights

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CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW

2012 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 1

Section I
10 marks

Questions 1–10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: PE2
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$P(-1) = -1$	A	1

Question 2 (1 mark)
Outcomes Assessed: H5, HE7
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$\left\lvert\frac{2-m}{1+2m}\right\rvert = \tan 45°, \mathrm{m} > o$ $m = \frac{1}{3}$	A	1

Question 3 (1 mark)
Outcomes Assessed: H5
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ $\sin\theta + \cos\theta = \frac{1+2t-t^2}{1+t^2}$	C	1

Question 4 (1 mark)
Outcomes Assessed: P4, PE2
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$(x,y)=\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right)$ $=\left(\frac{3\times3+(-2)\times(-2)}{3+(-2)},\ \frac{3\times(-4)+(-2)\times3}{3+(-2)}\right)$ $=(13,-18)$	D	1

Question 5 (1 mark)
Outcomes Assessed: PE3
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$\binom{6}{3}\times\binom{4}{2}=120$	B	1

Question 6 (1 mark)
Outcomes Assessed: HE4
***Targeted Performance Bands*: E3**

Solution	Answer	Mark
$f^{-1}(3)=x$ $f(x)=3$ $\frac{2x}{x+1}=3$ $x=-3$	A	1

Question 7 (1 mark)
Outcomes assessed: PE3, PE4
Targeted Performance Bands: E2

Solution	Answer	Mark
$x + py = 2ap + ap^3$ When $x = 0, y = a$: $ap = 2ap + ap^3$ $ap(1 + p^2) = 0$ $p = 0$ is the only solution. Therefore there exists only one solution.	B	1

Question 8 (1 mark)
Outcomes Assessed: HE3, HE7
Targeted Performance Bands: E3

Solution	Answer	Mark
Let $p = 0.8$ and $q = 0.2$ $P(X \geq 5) = P(X = 5) + P(X = 6)$ $= \binom{6}{5}(0.2)(0.8)^5 + (0.8)^6$ $= (0.8)^5(6 \times 0.2 + 0.8)$ $= 2(0.8)^5$	B	1

Question 9 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E3

Solution	Answer	Mark
$T_{k+1} = \binom{7}{k}(x^2)^{7-k}\left(\frac{2}{x}\right)^k$ $= \binom{7}{k}2^k x^{14-3k}$ When $14 - 3k = 2$, $k = 4$. The coefficient of $x^2 = \binom{7}{4}2^4$ $= 560.$	D	1

Question 10 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E3, E4

Solution	Answer	Mark
The FALSE statement is C	C	1

Section II
60 marks

Question 11 (15 marks)
(a) (1 mark)
Outcomes assessed: H5
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:

$$\lim_{x\to 0}\frac{\sin 2x}{x} = 2\lim_{x\to 0}\frac{\sin 2x}{2x}$$
$$= 2$$

(b) (2 marks)
Outcomes assessed: HE6
Targeted Performance Bands: E2

Criteria	Marks
• Correct answer	2
• Recognising the correct primitive	1

Sample answer:

$$\int_0^4 \frac{dx}{\sqrt{x^2+9}} = \left[\ln\left(x+\sqrt{x^2+9}\right)\right]_0^4$$
$$= \ln 9 - \ln 3$$
$$= \ln 3$$

(c) (3 marks)

Outcomes assessed: HE4, HE6

Targeted Performance Bands: E2–E3

Criteria	Marks
• Correct answer	3
• Significant progress towards solution	2
• Correct integral with correct limits, after applying given substitution	1

Sample answer:

Let $u = x - 8$

$du = dx$

$x = 8.5 \Rightarrow u = 0.5$ and $x = 8 \Rightarrow u = 0$.

Upon substituting: $\int_0^{0.5} \frac{du}{\sqrt{(7-(u+8))(u+8-9)}}$

$= \int_0^{0.5} \frac{du}{\sqrt{1-u^2}}$

$= [\sin^{-1} u]_0^{0.5}$

$= \frac{\pi}{6}$

(d) (3 marks)

Outcomes assessed: PE3

Targeted Performance Bands: E2–E3

Criteria	Marks
• Correct range of solutions given	3
• Significant progress towards solution (e.g. inequality if using solution presented below or points of intersection if using graphical techniques)	2
• Some progress towards answer (considering technique used e.g. graphical or algebraic)	1

Sample answer:

$\frac{2t}{1-t} \geq t \qquad t \neq 1$

$2t(1-t) \geq t(1-t)^2$

$t(1-t)(1+t) \geq 0$

$\therefore t \leq -1, 0 \leq t < 1$

(e) (i) (2 marks)
Outcomes assessed: PE2, PE3
Targeted Performance Bands: E2

Criteria	Marks
• Correct answer	2
• Recognises the relationship between the tangent and the intercepts from an external point	1

Sample answer:

$$PC^2 = AP.BP$$
$$12^2 = (x+7).x$$
$$x = 9 \quad \text{(since } x > 0\text{)}$$

(e) (ii) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:
$\angle BPC = 90°$ since BC is the diameter of the circle passing through P, B and C.
$\therefore$ ΔBPC is a right-angled triangle.

$$BC^2 = BP^2 + PC^2$$
$$BC^2 = 9^2 + 12^2$$
$$BC = 15$$

(f) (3 marks)
Outcomes assessed: PE2, PE6, HE7
Targeted Performance Bands: E3

Criteria	Marks
• Correct proof logically presented	3
• Progress towards solution	2
• Correctly identifying a relevant trigonometric relationship in one of the triangles	1

Sample answer:
Let $CD = h$

$$\text{In } \Delta ADC, \tan\alpha = \frac{h}{AC}$$
$$\text{In } \Delta BDC, \tan\beta = \frac{h}{BC}$$
$$\therefore \frac{\tan\alpha}{\tan\beta} = \frac{BC}{AC}$$

ΔABC is right-angled at C and $\angle CAB = 90° - 60° = 30°$ (from the bearings given), hence

$$\frac{BC}{AC} = \tan 30° = \frac{1}{\sqrt{3}}$$
$$\therefore \frac{\tan\alpha}{\tan\beta} = \frac{1}{\sqrt{3}}.$$

Question 12 (15 marks)

(a) (i) (1 mark)

Outcomes assessed: PE3

Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

(a) (ii) (2 marks)

Outcomes assessed: H5, HE7

Targeted Performance Bands: E3–E4

Criteria	Marks
• Correctly using the result in part (i) to obtain the result	2
• Solves $f''(x) = 0$	1

Sample answer:

$$f'(x) = 3ax^2 + 2bx + c$$
$$f''(x) = 6ax + 2b$$

Solving $f''(x) = 0$

$$6ax + 2b = 0$$
$$x = \frac{-b}{3a}$$
$$= \frac{\alpha+\beta+\gamma}{3}$$

(a) (iii) (1 mark)

Outcomes assessed: HE7

Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	1

Sample answer:

$$x = \frac{\alpha+\beta+\gamma}{3}$$
$$= \frac{-1+3+4}{3}$$
$$= 2$$

(b) (i) (2 marks)

Outcomes assessed: HE4, HE7

Targeted Performance Bands: E3

Criteria	Marks
• Correct value of x	2
• Correct value of α	1

Sample answer:

$$\sin^{-1}x = \alpha,\ \cos^{-1}x = \frac{\pi}{2} - \alpha$$

$$\text{Now,}\quad \sin^{-1}x = \cos^{-1}x$$

$$\text{i.e.}\quad \alpha = \frac{\pi}{2} - \alpha$$

$$\alpha = \frac{\pi}{4}$$

$$\therefore \sin^{-1}x = \frac{\pi}{4}$$

$$x = \sin\frac{\pi}{4}$$

$$\therefore \text{The solution is } x = \frac{\sqrt{2}}{2}.$$

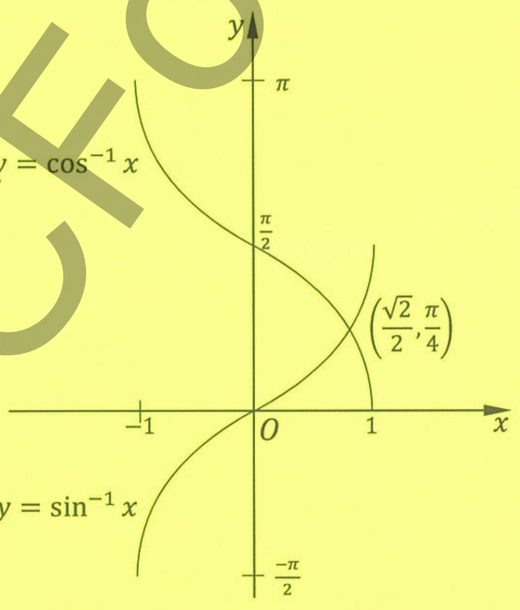
(b) (ii) (1 marks)

Outcomes assessed: HE4, HE7

Targeted Performance Bands: E2

Criteria	Marks
• Sketches curves correctly, showing the point of intersection	1

Sample answer:



(b) (iii) (2 marks)

Outcomes assessed: HE2

Targeted Performance Bands: E3

Criteria	Mark
• Correct explanation	2
• Reference to symmetry of region	1

Sample answer:

The region is symmetrical about the line $y = \frac{\pi}{4}$.

Therefore the volume of the rotated region bounded by $y = \sin^{-1} x$ between $y = 0$ and $y = \frac{\pi}{4}$ (given by $\pi \int_0^{\frac{\pi}{4}} \sin^2 y \, dy$) is equal to the volume of the rotated region bounded by $y = \cos^{-1} x$ between $y = \frac{\pi}{4}$ and $y = \frac{\pi}{2}$ (given by $\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 y \, dy$).

$$\therefore V = \pi \int_0^{\frac{\pi}{4}} \sin^2 y \, dy + \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 y \, dy$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sin^2 y \, dy$$

(b) (iv) (2 marks)

Outcomes assessed: H8, HE6

Targeted Performance Bands: E3

Criteria	Marks
• Correct answer	2
• Substantial progress towards integrating $\sin^2 y$	1

Sample answer:

$$V = 2\pi \int_0^{\frac{\pi}{4}} \sin^2 y \, dy$$

$$= \pi \int_0^{\frac{\pi}{4}} (1 - \cos 2y) dy$$

$$= \pi \left[y - \frac{\sin 2y}{2} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4}(\pi - 2)$$

$\therefore$ The volume is $\frac{\pi}{4}(\pi - 2)$ units^3.

(c) (i) (3 marks)
Outcomes assessed: HE2, HE7
Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	3
• Establishes the induction step	2
• Verifies the result for n=2	1

Sample answer:

(1) When $n = 2$:

$$\text{LHS} = \left(1 - \frac{1}{2^2}\right) = \frac{3}{4}$$

$$\text{RHS} = \frac{2+1}{2\times 2} = \frac{3}{4} = \text{LHS}$$

Assume true for $n = k$: $\left(1-\frac{1}{2^2}\right)\times\left(1-\frac{1}{3^2}\right)\times\left(1-\frac{1}{4^2}\right)\times\ldots\times\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$

Prove true for $n = k+1$:

i.e. $\left(1-\frac{1}{2^2}\right)\times\left(1-\frac{1}{3^2}\right)\times\left(1-\frac{1}{4^2}\right)\times\ldots\times\left(1-\frac{1}{k^2}\right)\times\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$

Now $\quad \text{LHS} = \frac{k+1}{2k}\times\left(1-\frac{1}{(k+1)^2}\right)$ (using assumption)

$$= \frac{k+1}{2k}\times\frac{(k+1)^2-1}{(k+1)^2}$$

$$= \frac{k^2+2k}{2k(k+1)}$$

$$= \frac{k+2}{2(k+1)}$$

$$= \text{RHS}$$

Hence, by the Principle of Mathematical Induction, the result holds true for all integers $n \geq 2$.

(c) (ii) (1 mark)
Outcomes assessed: HE2, HE7
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:

$n = 100$, hence the product equals $\frac{101}{200}$.

Question 13 (15 marks)

(a) (i) (1 mark)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E2–E3

Criteria	Mark
• Correct proof	1

Sample answer:

$$T = 100 - Ae^{-0.2t}$$
$$\frac{dT}{dt} = 0.2Ae^{-0.2t}$$
$$= 0.2(100 - T) \qquad \text{as } Ae^{-0.2t} = 100 - T.$$

(a) (ii) (1 mark)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E2–E3

Criteria	Mark
• Correct value for A	1

Sample answer:

When $t = 0, T = 4$.

$4 = 100 - Ae^{-0.2\times0}$

$A = 96$

(a) (iii) (1 mark)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E2–E3

Criteria	Mark
• Correct value for temperature	1

Sample answer:

$T = 100 - 96e^{-0.2t}$

Substitute $t = 4.5$

$T = 100 - 96e^{-0.2\times4.5}$

$T = 60.9693\ldots$

Therefore, the temperature correct to three significant figures is 61.0°C.

(a) (iv) (2 marks)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E2–E3

Criteria	Marks
• Correct answer	2
• Correct value of *t* for an egg boiled from room temperature	1

Sample answer:

$$T = 100 - 79e^{-0.2t}$$

Substituting $T = 60.9693\ldots$

$$60.9693\ldots = 100 - 79e^{-0.2t}$$

$$t = \ln\left(\frac{100 - 60.9693\ldots}{79}\right) \div -0.2$$

$$= 3.525498\ldots \approx 3.5 \text{ minutes}$$

Therefore, the egg at room temperature will cook approximately one minute faster than an egg taken from the refrigerator.

(b) (i) (3 marks)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct answers for A and α	3
• Correct value for either A or α	2
• Significant progress towards correct equations involving A and α	1

Sample answer:

$$x = A\cos(2t + \alpha)$$

When $t = 0, x = 4$: $\quad A\cos\alpha = 4 \quad$ (i)

Now $\quad \frac{dx}{dt} = -2A\sin(2t + \alpha)$

When $t = 0, \frac{dx}{dt} = -8\sqrt{3}$ $\quad A\sin\alpha = 4\sqrt{3} \quad$ (ii)

From (i) and (ii) above: $\quad \frac{A\sin\alpha}{A\cos\alpha} = \frac{4\sqrt{3}}{4}$

$$\tan\alpha = \sqrt{3}$$

$$\alpha = \frac{\pi}{3}$$

Substitute $\alpha = \frac{\pi}{3}$ into (i): $\quad A\cos\frac{\pi}{3} = 4$

$$A = 8$$

Therefore, $A = 8$ and $\alpha = \frac{\pi}{3}$.

(b) (ii) (2 marks)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct answer	2
• Correct equation with ONE unknown, t	1

Sample answer:

Particle is at the centre of motion when $x = 0$:

$$8\cos\left(2t + \frac{\pi}{3}\right) = 0$$

$$2t + \frac{\pi}{3} = \frac{\pi}{2}, \quad \text{for the first time.}$$

$$t = \frac{\pi}{12}$$

Therefore, the particle arrives at the origin after $\frac{\pi}{12}$ seconds.

(c) (i) (3 marks)

Outcomes assessed: HE5, HE7

Targeted Performance Bands: E3–E4

Criteria	Marks
• Correctly shows the result	3
• Correct expression for v^2, including the value of the constant	2
• Progress towards expressing the result in terms of v and x, e.g. using the correction version of acceleration $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$	1

Sample answer:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 18x$$

$$\frac{1}{2}v^2 = \frac{1}{2}x^4 + 9x^2 + c$$

When $x = 0, v = 9$: $\quad c = \frac{81}{2}$

Therefore, $\quad \frac{1}{2}v^2 = \frac{1}{2}x^4 + 9x^2 + \frac{81}{2}$

$$v^2 = x^4 + 18x^2 + 81$$

$$v^2 = (x^2 + 9)^2$$

$$v = \pm(x^2 + 9)$$

However, since initially the particle has a positive velocity and there are no solutions to $v = 0$, the particle must always travel in a positive direction.

Hence, $v = x^2 + 9$.

(c) (ii) (2 marks)

Outcomes assessed: HE5, HE7

Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct equation for x in terms of t	2
• Progress towards an expression for t in terms of x	1

Sample answer:

$$\frac{dx}{dt} = x^2 + 9$$

$$\frac{dt}{dx} = \frac{1}{x^2+9}$$

$$t = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$$

When $x = 0$, $t = 0$, therefore $c = 0$.

$$t = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

$$x = 3\tan(3t)$$

Question 14 (15 marks)

(a) (3 marks)

Outcomes assessed: H5, HE7

Targeted Performance Bands: E2–E3

Criteria	Marks
• Correct answer	3
• Correct substitution into the formula of Newton's Method	2
• Establishing an appropriate function and determining its derivative	1

Sample answer:

$$3\sin x = \ln x$$

$$\text{Let } f(x) = 3\sin x - \ln x$$

$$f'(x) = 3\cos x - \frac{1}{x}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3 - \frac{3\sin 3 - \ln 3}{3\cos 3 - \frac{1}{3}}$$

$$= 2.79558\ldots$$

$$= 2.80 \text{ (2 d.p.)}$$

(b) (i) (1 mark)

Outcomes assessed: H5

Targeted Performance Bands: E3, E4

Criteria	Mark
• Correct justification for result	1

Sample answer:

For the gradient of MB: applying Pythagoras' Theorem, rise $= \sqrt{L^2 - x^2}$; run $= x$.

The gradient of $MB = \frac{-\text{rise}}{\text{run}} = \frac{-\sqrt{L^2 - x^2}}{x}$.

Since the rope is always tangent to the curve, the line MB is a tangent to $y = f(x)$

$$\therefore \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}.$$

(b) (ii) (2 marks)

Outcomes assessed: HE5, HE7

Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct answer	2
• Substantial progress towards expressing $\frac{dx}{dt}$ as a function of x	1

Sample answer:

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$
$$= \frac{-x}{\sqrt{L^2 - x^2}} \times 3$$
$$= \frac{\frac{-L}{2}}{\sqrt{L^2 - \frac{L^2}{4}}} \times 3 \text{ when } x = \frac{L}{2}$$
$$= -\sqrt{3}$$

Therefore, the boat approaches the pier at a rate of $\sqrt{3}\ \mathrm{ms}^{-1}$.

(c) (3 marks)

Outcomes assessed: PE3, PE4, HE2

Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct proof	3
• Substantial progress towards determining both required distances or both required angles	2
• Recognising the gradient at P is given by $\frac{1}{p}$ and progress towards determining a relevant distance or a relevant angle	1

Sample answer:

The gradient of the tangent at $P = \frac{2a}{2ap} = \frac{1}{p}$.

The equation of the tangent at P is given by $y - 2ap = \frac{1}{p}(x - ap^2)$

Let the tangent at P intersect the x-axis at G.

$\therefore$ G has coordinates $(-ap^2, 0)$. P has coordinates $(ap^2, 2ap)$. S has coordinates $(a, 0)$.

$SG = ap^2 + a$

$$SP = \sqrt{(ap^2 - a)^2 + (2ap)^2}$$
$$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$$
$$= \sqrt{(ap^2 + a)^2}$$
$$= ap^2 + a$$

Since $SG = SP$, ΔSPG is isosceles.

$\therefore \angle SGP = \angle SPG$ (angles opposite equal sides of an isosceles triangle are equal), thus the angle of inclination to the axis of the parabola equals the angle between the tangent and the focal chord.

Hence the tangent to the parabola at P is equally inclined to the axis of the parabola and the focal chord through P.

(d) (i) (1 mark)
Outcomes assessed: HE2, HE7,
Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	1

Sample answer:
The coefficient of x^r in the expansion $(1+x)^n$ is $\binom{n}{r}$.
The coefficient of x^r in the expansion $(x+1)^n$ is $\binom{n}{n-r}$.
Since $(1+x)^n = (x+1)^n$, the expansions are the same. Therefore, $\binom{n}{n-r} = \binom{n}{r}$.

(d) (ii) (3 marks)
Outcomes assessed: HE2, HE7
Targeted Performance Bands: E4

Criteria	Marks
• Correct proof	3
• Correctly identifies and matches coefficients of x^{n-r} in the two binomial expansions	2
• Expansion of ONE relevant binomial expression	1

Sample answer:
$f(r) = \binom{n}{0}\binom{n}{r} + \binom{n}{1}\binom{n}{r+1} + \cdots + \binom{n}{n-r}\binom{n}{n}$.
Therefore, $f(r) = \binom{n}{0}\binom{n}{n-r} + \binom{n}{1}\binom{n}{n-r-1} + \cdots + \binom{n}{n-r}\binom{n}{0}$, using the result in part (i).

Now, consider $(1+x)^n(1+x)^n = (1+x)^{2n}$

$$(1+x)^n(1+x)^n = \left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right)\left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right)$$

The coefficient of x^{n-r} in this expansion is:

$$\binom{n}{0}\binom{n}{n-r} + \binom{n}{1}\binom{n}{n-r-1} + \binom{n}{2}\binom{n}{n-r-2} + \cdots + \binom{n}{n-r}\binom{n}{0} = f(r) \text{ from above}$$

But the coefficient of x^{n-r} in the expansion of $(1+x)^{2n}$ is $\binom{2n}{n-r}$.
Therefore, $f(r) = \binom{2n}{n-r}$, for $r = 0, 1, 2, \ldots, n$

(d) (iii) (2 marks)
Outcomes assessed: HE2, HE7
Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct proof	2
• Correctly identifies and matches coefficients in relevant binomial expansions	1

Sample answer:

Consider $(1+x)^{3n} = (1+x)^{2n}(1+x)^n$

$$(1+x)^{2n}(1+x)^n = \left(\binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \cdots + \binom{2n}{2n}x^{2n}\right)\left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right)$$

The coefficient of x^n in this expansion is:

$$\binom{2n}{0}\binom{n}{n} + \binom{2n}{1}\binom{n}{n-1} + \binom{2n}{2}\binom{n}{n-2} + \cdots + \binom{2n}{n}\binom{n}{0}$$
$$= f(n)\binom{n}{n} + f(n-1)\binom{n}{n-1} + f(n-2)\binom{n}{n-2} + \cdots + f(0)\binom{n}{0}, as\ f(r) = \binom{2n}{n-r} \text{ for } r = 0, 1, 2 \ldots, n$$

But the coefficient of x^n in the expansion of $(1+x)^{3n}$ is $\binom{3n}{n}$.

Therefore, $\binom{n}{0}f(0) + \binom{n}{1}f(1) + \cdots + \binom{n}{n}f(n) = \binom{3n}{n}$.

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2012
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in Question 11 – 14
- Write your student number and/or name at the top of every page

Total marks – 70

Section I - Pages 3 – 4

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 5 – 9

60 marks

Attempt Questions 11 – 14

Allow about 1 hour 45 minutes for this section

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ……………………..

STUDENT NUMBER/NAME: ……………………..

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

Student name / number ....

Marks

Section I

1. What is the value of $\lim_{x \to 0} \dfrac{\sin 3x}{4x}$? **1**

(A) 0
(B) $\frac{3}{4}$
(C) 1
(D) $\frac{4}{3}$

2. The equation $x^3 + 2x^2 - 3x - 6 = 0$ has roots $\alpha, -\alpha, \beta$. What is the value of β? **1**

(A) -6
(B) -2
(C) 2
(D) 6

3. If $y = e^{x^2}$, which of the following is an expression for $\dfrac{d^2y}{dx^2}$? **1**

(A) $2xe^{x^2}$
(B) $2e^{x^2}$
(C) $4x^2e^{x^2}$
(D) $(4x^2 + 2)e^{x^2}$

4. Which of the following is an expression for $\int \sin^2 2x \ dx$? **1**

(A) $\frac{1}{2}x - \frac{1}{8}\sin 4x + c$
(B) $\frac{1}{2}x - \frac{1}{4}\sin 4x + c$
(C) $\frac{1}{2}x + \frac{1}{8}\sin 4x + c$
(D) $\frac{1}{2}x + \frac{1}{4}\sin 4x + c$

5. What is the value of $\cos n\pi$ for $n = 1, 2, 3, \ldots$? **1**

(A) -1
(B) $(-1)^n$
(C) $(-1)^{n+1}$
(D) $\dfrac{(-1) + (-1)^n}{2}$

Student name / number ………………………………

Marks

6. If $\frac{dN}{dt} = 0\cdot1\left(N-100\right)$ and $N = 300$ when $t = 0$, which of the following is an expression for N? **1**

(A) $200 + 100e^{0\cdot1t}$
(B) $300 + 100e^{0\cdot1t}$
(C) $100 + 200e^{0\cdot1t}$
(D) $100 + 300e^{0\cdot1t}$

7. Which of the following represents the inverse of $f(x) = e^x - 2$? **1**

(A) $f^{-1}(x) = \frac{1}{e^x - 2}$
(B) $f^{-1}(x) = e^{-x} - \frac{1}{2}$
(C) $f^{-1}(x) = \log_e x + 2$
(D) $f^{-1}(x) = \log_e(x+2)$

8. Which of the following is an expression for $\int \frac{2x}{\sqrt{1+x^2}}\,dx$? **1**

(A) $\log_e(1+x^2) + c$
(B) $\log_e\sqrt{1+x^2} + c$
(C) $\sqrt{1+x^2} + c$
(D) $2\sqrt{1+x^2} + c$

9. If $t = \tan\frac{x}{2}$ which of the following is an expression for $\frac{dx}{dt}$? **1**

(A) $\frac{1}{2}(1+t^2)$
(B) $1+t^2$
(C) $\frac{2}{1+t^2}$
(D) $\frac{1}{1+t^2}$

10. What is the value of the term independent of x in the expansion of $\left(x^2 + \frac{2}{x}\right)^6$? **1**

(A) 60
(B) 160
(C) 192
(D) 240

Student name / number ....

Marks

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 11 **Begin a new booklet**

(a) Find the number of ways in which 3 boys and 3 girls can be arranged in a straight line so that the girls are all next to each other, but the boys are not all next to each other. **2**

(b) Find the coordinates of the point $P(x, y)$ which divides the interval joining the points $A(-2, 5)$ and $B(4, 1)$ externally in the ratio 3 : 1. **2**

(c) Solve the inequality $\dfrac{1}{x-1} < 1$. **2**

(d) Consider the function $f(x) = x + e^{-x}$.

(i) Find the coordinates and nature of the stationary point on the curve $y = f(x)$. **2**

(ii) Find the equation of the asymptote on the graph of the curve $y = f(x)$. **1**

(e) $P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$ with focus $F(0, a)$.

(i) Use differentiation to show that the normal to the parabola at P has gradient $-\frac{1}{t}$. **1**

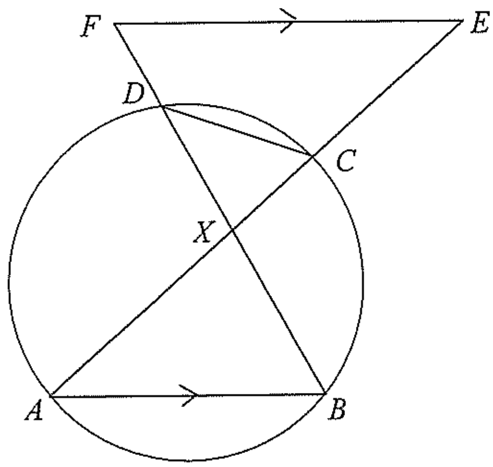
(ii) If θ is the acute angle between the normal to the parabola at P and the line PF show that $\tan\theta = |t|$. **2**

Student name / number ....

Marks

Question 11 (cont)

(f)



AC and BD are two chords of a circle which intersect at point X inside the circle. E is a point on AC produced and F is a point on BD produced such that $FE \parallel AB$. Show that $DCEF$ is a cyclic quadrilateral. **3**

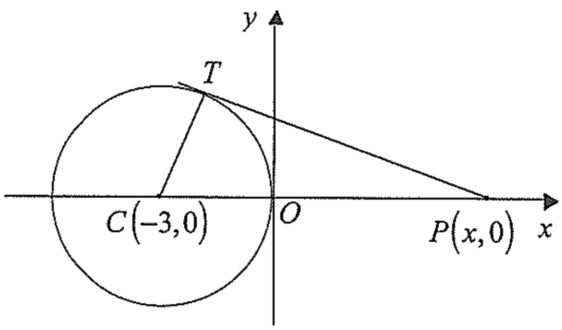
Student name / number ....

Marks

Question 12 **Begin a new booklet**

(a) Show that the equation $\log_e x + x = 0$ has a root between $x = 0\cdot5$ and $x = 1$. **2**

(b)



$P(x, 0)$ is a point on the positive x-axis. T is the point of contact of a tangent drawn from P to the circle with centre $C(-3, 0)$ and radius 3.

(i) Show that $PT = \sqrt{x^2 + 6x}$. **1**

(ii) If the units in the above diagram are cm, and P is moving along the x-axis away from O at a constant rate of $0\cdot1 \text{ cm s}^{-1}$, find the rate of change of PT when $x = 2$ cm. **2**

(c) Use the substitution $u = x + 1$ to evaluate $\int_1^3 \frac{x+2}{(x+1)^2}\,dx$, giving the answer in simplest exact form. **3**

(d) Use Mathematical Induction to show that for all positive integers $n \geq 2$, **3**

$$2\times1 + 3\times2 + 4\times3 + \ldots + n(n-1) = \tfrac{1}{3}n(n^2-1) \ .$$

(e)(i) If $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{3}$, show that $\sin^{-1}x = \frac{5\pi}{12}$. **2**

(ii) Hence solve $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{3}$, giving the answer in simplest surd form. **2**

Student name / number ....

Marks

Question 13 **Begin a new booklet**

(a) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} given by $v = \frac{2}{3\sqrt{x}}$ and acceleration a ms^{-2}. Initially the particle is 1 metre to the right of O.

(i) Show that $a = \frac{-2}{9x^2}$. **1**

(ii) Show that $x = (t+1)^{\frac{2}{3}}$. **2**

(iii) Show that the particle is always moving away from O and slowing down. **2**

(iv) Find the time taken for the speed of the particle to drop to 10% of its initial speed. **2**

(b) Consider the function $f(x) = 2\cos^{-1}(x-1)$ where $1 \le x \le 2$.

(i) Sketch the curve $y = f(x)$ showing clearly the coordinates of the endpoints. **2**

(ii) Find the equation of the inverse function $f^{-1}(x)$ and state its domain. **2**

(c) On any roll of a biased die there is a probability p of getting a 'six'. If the die is rolled 6 times the probability of getting at least 5 'sixes' is $\frac{1}{2}$.

(i) Show that $10p^6 - 12p^5 + 1 = 0$. **2**

(ii) Use one application of Newton's Method with an initial approximation of $p_0 = 0 \cdot 75$ to find the next approximation to the value of p, giving your answer correct to 2 decimal places. **2**

Student name / number ....

Marks

Question 14 **Begin a new booklet**

(a) A particle is moving in a straight line and performing Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point O on the line, given by $x = 2\cos\left(2t - \frac{\pi}{4}\right)$, velocity v ms^{-1} and acceleration $\ddot{x}$ ms^{-2}.

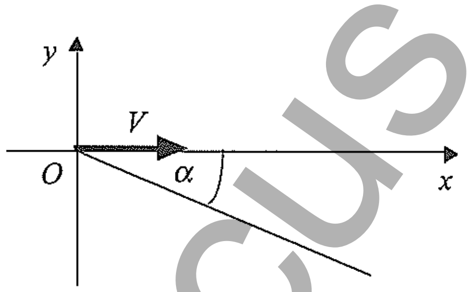
(i) Show that $v^2 - x\ddot{x} = 16$. **2**

(ii) Sketch the graph of x as a function of t for $0 \le t \le \pi$ showing clearly the coordinates of the endpoints. **2**

(iii) Show that the particle first returns to its starting point after one quarter of its period. **1**

(iv) Find the time taken by the particle to travel the first 100 metres of its motion. **2**

(b)



A particle is projected horizontally from a point O with speed V ms^{-1} down a slope which is inclined at an angle $\alpha = \tan^{-1}\frac{1}{2}$ below the horizontal. The particle moves in a vertical plane under gravity where the acceleration due to gravity is g ms^{-2}. At time t seconds the horizontal and vertical displacements from O, x metres and y metres respectively, are given by $x = Vt$ and $y = -\frac{1}{2}gt^2$. (DO NOT PROVE THESE RESULTS.)

(i) Show the particle hits the slope after time $\dfrac{V}{g}$ seconds. **2**

(ii) Show that the particle hits the slope with velocity $V\sqrt{2}$ ms^{-1} at an angle of $45°$ to the vertical. **2**

(c)(i) Show that $\displaystyle\sum_{r=1}^{n}(1+x)^{r-1} = \sum_{r=1}^{n} {}^{n}C_r\, x^{r-1}$. **2**

(ii) Hence show that for $n \ge 3$, $\displaystyle\sum_{r=2}^{n-1} {}^{r}C_2 = {}^{n}C_3$. **2**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Independent Trial HSC 2012 Mathematics Extension 1 Marking Guidelines

Section I

Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1.	**B**	$\lim_{x\to 0}\frac{\sin 3x}{4x}=\frac{3}{4}\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right)=\frac{3}{4}\times 1=\frac{3}{4}$	H5
2.	**B**	$\alpha+(-\alpha)+\beta=-2 \quad \therefore \beta=-2$	PE3
3.	**D**	$\frac{d}{dx}\left(e^{x^2}\right)=2xe^{x^2} \qquad \therefore \frac{d^2}{dx^2}\left(e^{x^2}\right)=2e^{x^2}+2x.2xe^{x^2}=\left(4x^2+2\right)e^{x^2}$	PE5
4.	**A**	$\int \sin^2 2x\, dx=\int \tfrac{1}{2}(1-\cos 4x)\, dx=\tfrac{1}{2}x-\tfrac{1}{8}\sin 4x+c$	H5
5.	**B**	$\cos n\pi=\begin{cases}1, & n \text{ even}\\ -1, & n \text{ odd}\end{cases} \qquad \therefore \cos n\pi=(-1)^n, \quad n=1,2,3,\ldots$	HE2
6.	**C**	$\frac{dN}{dt}=0\cdot 1\left(N-100\right) \Rightarrow N-100=Ae^{0\cdot 1t}$ for some constant A Then $N=300,\ t=0 \Rightarrow 200=A \qquad \therefore N=100+200e^{0\cdot 1t}$	HE3
7.	**D**	$y=e^x-2 \Leftrightarrow \log_e(y+2)=x \qquad \therefore f^{-1}(x)=\log_e(x+2)$	HE4
8.	**D**	$\int \frac{2x}{\sqrt{1+x^2}}\,dx=\int (1+x^2)^{-\frac{1}{2}}.2x\ dx=2(1+x^2)^{\frac{1}{2}}+c=2\sqrt{1+x^2}+c$	HE6
9.	**C**	$t=\tan\tfrac{x}{2} \Rightarrow \frac{dt}{dx}=\tfrac{1}{2}\sec^2\tfrac{x}{2}=\tfrac{1}{2}\left(1+\tan^2\tfrac{x}{2}\right) \qquad \therefore \frac{dx}{dt}=\frac{2}{1+t^2}$	HE4
10.	**D**	$\left(x^2+\tfrac{2}{x}\right)^6$ has general term ${}^6C_r\left(\tfrac{2}{x}\right)^r\left(x^2\right)^{6-r}={}^6C_r 2^r x^{12-3r}$, $r=0,1,\ldots,6$ $r=4$ gives constant term ${}^6C_4\, 2^4=240$	HE3

Section II

Question 11

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• counts arrangements for one suitable pattern of B's and G's	1
• adds arrangements for the second possible pattern	1

Answer

With restriction, pattern B G G G B B or B B G G G B

$3 \times 3! \times 2! + 2! \times 3! \times 3 = 72$ arrangements.

b. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• finds the x coordinate of P	1
• finds the y coordinate of P	1

Answer

$A(-2, 5) \qquad B(4, 1)$

$3 \quad : \quad -1$

$$P\left(\frac{3\times 4+(-1)\times(-2)}{3+(-1)}, \frac{3\times 1+(-1)\times 5}{3+(-1)}\right) \qquad \therefore P(7, -1)$$

c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• obtains one inequality satisfied by x	1
• obtains the second inequality and indicates correctly how the two inequalities are combined	1

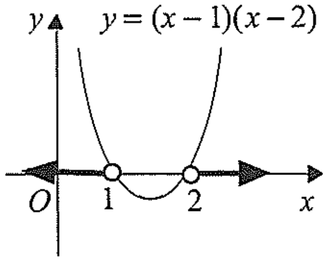
Answer

$$\frac{1}{x-1} < 1$$

$$x-1 < (x-1)^2 \quad , \quad x \neq 1$$

$$0 < (x-1)\{(x-1)-1\}$$

$$\therefore \quad 0 < (x-1)(x-2), \quad x \neq 1$$



$\therefore x < 1 \quad or \quad x > 2$

Question 11 (cont)

d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i. • considers the first derivative to find coordinates of the stationary point	1
• applies first or second derivative test to determine nature of the stationary point	1
ii. • writes the equation of the asymptote	1

Answer

i.

$f(x) = x + e^{-x}$

$f'(x) = 1 - e^{-x}$

$f''(x) = e^{-x} > 0$ for all x

$f'(x) = 0 \Rightarrow e^{-x} = 1$

$x = 0$

where $f''(0) > 0$

Hence there is one stationary point $(0, 1)$ which is a minimum turning point.

ii. $y = x + e^{-x}$ $\quad \therefore y - x = e^{-x} \to 0$ as $x \to +\infty$. Hence the line $y = x$ is an asymptote as $x \to +\infty$.

e. Outcomes assessed : H5, PE3, PE4

Marking Guidelines

Criteria	Marks
i. • finds gradient of normal by differentiation	1
ii. • finds gradient of PF	1
• substitutes gradients into formula for $\tan\theta$ and simplifies to obtain required result	1

Answer

i.

$y = at^2 \qquad \frac{dy}{dt} = 2at$

$x = 2at \qquad \frac{dx}{dt} = 2a$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = t$

Hence normal at $P(2at, at^2)$ has gradient $-\frac{1}{t}$.

ii.

$$m_{PF} = \frac{a(t^2-1)}{2at} = \frac{(t^2-1)}{2t}$$

$$\therefore \tan\theta = \left| \frac{\frac{1}{2t}\left\{(t^2-1)-(-2)\right\}}{1+\frac{1}{2t}(t^2-1).(-\frac{1}{t})} \right| = \left| \frac{t(t^2+1)}{2t^2-(t^2-1)} \right|$$

$$\therefore \tan\theta = \left| \frac{t(t^2+1)}{t^2+1} \right| = |t|$$

f. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
• quotes an appropriate test for a cyclic quadrilateral	1
• writes a sequence of deductions resulting in the application of this test	1
• quotes appropriate geometric properties to support these deductions	1

Answer

$\angle EFD = \angle DBA$ (*Alternate $\angle$'s between parallel lines are equal*)

$\angle DBA = DCA$ (*$\angle$'s subtended at the circumference of the circle by the same arc DA are equal*)

$\therefore \angle DCA = \angle EFD$

$\therefore$ EFDC is a cyclic quadrilateral (*an exterior $\angle$ is equal to the opposite interior $\angle$*)

Question 12

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• shows $f(0\cdot 5)$ and $f(1)$ have opposite signs	1
• notes the continuity of the function f	1

Answer

$f(x)=\log_e x+x$ is a continuous function for $x>0$ and $f(0\cdot 5)\approx -0\cdot 19<0$, $f(1)=1>0$

$\therefore f(x)=0$ for some x such that $0\cdot 5<x<1$.

b. Outcomes assessed : P4, HE5

Marking Guidelines

Criteria	Marks
i. • shows required result	1
ii. • finds required rate of change in terms of x and $\frac{dx}{dt}$	1
• substitutes given values to calculate rate of change in cm s^{-1}	1

Answer

i. $PT^2=PC^2-TC^2$ (by Pythagoras' theorem, since tangent $\perp$ radius drawn to point of contact)

$\therefore PT=\sqrt{(x+3)^2-3^2}=\sqrt{x^2+6x}$

ii. Let $s=PT$: $s^2=x^2+6x$

$$2s\frac{ds}{dt}=\left(2x+6\right)\frac{dx}{dt}$$

$$\frac{ds}{dt}=\frac{x+3}{\sqrt{x^2+6x}}\,\frac{dx}{dt}$$

$x=2$, $\frac{dx}{dt}=0\cdot 1$

$$\therefore \frac{ds}{dt}=\frac{5}{4}\times 0\cdot 1$$

$\therefore PT$ is increasing at $0\cdot 125$ cm s^{-1}.

c. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• converts to definite integral in terms of u	1
• finds primitive	1
• substitutes limits and simplifies	1

Answer

$u=x+1$

$du=dx$

$x=1\Rightarrow u=2$

$x=3\Rightarrow u=4$

$$I=\int_1^3\frac{x+2}{(x+1)^2}\,dx=\int_2^4\frac{u+1}{u^2}\,dx$$

$$\therefore I=\left[\ln u-\tfrac{1}{u}\right]_2^4=(\ln 4-\ln 2)-(\tfrac{1}{4}-\tfrac{1}{2})=\ln 2+\tfrac{1}{4}$$

Question 12 (cont)

d. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• verifies that the equality holds for $n=2$	1
• shows that the truth of the statement for $n=k$ implies the truth of the statement for $n=k+1$	1
• presents the proof in a way that shows an understanding of Mathematical Induction	1

Answer

Let $S(n)$, $n=2,3,4,...$ be the sequence of statements defined by $S(n)$: $\sum_{r=2}^{n} r(r-1)=\frac{1}{3}n(n^2-1)$

Consider $S(2)$: $LHS=2\times 1=2$ $RHS=\frac{1}{3}2(2^2-1)=2$ $\therefore S(2)$ is true.

If $S(k)$ is true : $\sum_{r=2}^{k} r(r-1)=\frac{1}{3}k(k^2-1)$ **

Consider $S(k+1)$: $\sum_{r=2}^{k+1} r(r-1)=\sum_{r=2}^{k} r(r-1)+(k+1)k$

$=\frac{1}{3}k(k^2-1)+(k+1)k$ if $S(k)$ is true, using **

$=\frac{1}{3}(k+1)\left\{k(k-1)+3k\right\}$

$=\frac{1}{3}(k+1)\left\{(k+1)^2-1\right\}$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(2)$ is true. Hence $S(3)$ is true, and then $S(4)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n\geq 2$.

e. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
i. • uses the identity $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$	1
• obtains the required result	1
ii. • writes x as $\sin(\alpha\pm\beta)$ where the exact trig. ratios of α, β are known as surds	1
• writes x in simplest surd form	1

Answer

i. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for all $-1\leq x\leq 1$.

$\therefore \sin^{-1}x-\cos^{-1}x=\frac{\pi}{3} \Rightarrow 2\sin^{-1}x=\frac{\pi}{2}+\frac{\pi}{3}$

$\sin^{-1}x=\frac{5\pi}{12}$

ii. $\sin^{-1}x=\frac{5\pi}{12}$

$x=\sin\frac{5\pi}{12}=\sin(\frac{\pi}{4}+\frac{\pi}{6})$

$x=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$

Question 13

a. Outcomes assessed : HE5

Marking Guidelines

Criteria	Marks
i. • derives $\frac{1}{2}v^2$ with respect to x to give a in terms of x	1
ii. • integrates to find t as a function of x	1
• uses the initial conditions to evaluate the constant of integration; writes x as a function of t	1
iii. • explains why particle is moving away from O	1
• explains why particle is slowing down	1
iv. • finds x when v has 10% of initial value	1
• finds t for this x value	1

Answer

i. $v^2 = \dfrac{4}{9x} \qquad \therefore a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{-2}{9x^2}$

ii.
$$\frac{dx}{dt} = \frac{2}{3\sqrt{x}}$$
$$\frac{dt}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$
$$t = x^{\frac{3}{2}} + c$$
$$\left.\begin{array}{l} t=0 \\ x=1 \end{array}\right] \Rightarrow \quad c = -1$$
$$t = x^{\frac{3}{2}} - 1$$
$$x = (t+1)^{\frac{2}{3}}$$

iii. $v = \dfrac{2}{3\sqrt{x}} > 0$ and $a = \dfrac{-2}{9x^2} < 0$ for all x, and initially $x = 1$. Hence particle initially moves to the right, away from O, and continues to move right away from O while slowing down (since v decreases as x increases, or alternatively since v and a have opposite signs)

iv. $t = 0 \Rightarrow v = \frac{2}{3}$

When $v = \frac{1}{10} \times \frac{2}{3}$, $x = \left(\frac{2}{3v}\right)^2 = 100$

$$t = 100^{\frac{3}{2}} - 1 = 999$$

$\therefore$ 10% of initial velocity after $999\text{ s} = 16\text{ min }39\text{ s}$.

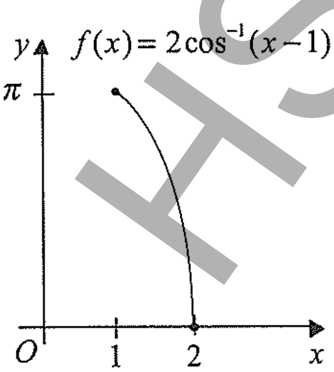
b. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i. • shows curve of correct shape with x-intercept 2	1
• shows coordinates of endpoint at $x = 1$	1
ii. • writes equation of inverse function	1
• states domain	1

Answer

i.



ii.

$$y = 2\cos^{-1}(x-1) \Leftrightarrow x = \cos(\tfrac{1}{2}y) + 1, \quad 0 \le y \le \pi$$
$$\therefore f^{-1}(x) = \cos(\tfrac{1}{2}x) + 1, \quad 0 \le x \le \pi$$

Question 13 (cont)

c. Outcomes assessed : PE3, HE3

Marking Guidelines

Criteria	Marks
i. • writes an expression for the probability of at least 5 sixes in terms of p	1
• evaluates binomial coefficients, writes and simplifies equation for p	1
ii. • substitutes $p_0 = 0\cdot 75$ into formula for finding next approximation by Newton's Method	1
• calculates next approximation	1

Answer

i.

$$^6C_5 p^5(1-p) + {}^6C_6 p^6 = \tfrac{1}{2}$$
$$2(6p^5 - 6p^6 + p^6) = 1$$
$$10p^6 - 12p^5 + 1 = 0$$

ii.

Let $f(p) = 10p^6 - 12p^5 + 1$ Then $f(0\cdot 75) \approx -0\cdot 0679$

$f'(p) = 60p^5 - 60p^4$ $f'(0\cdot 75) \approx -4.7461$

Next approximation is $p_1 = 0\cdot 75 - \dfrac{f(0\cdot 75)}{f'(0\cdot 75)} \approx 0\cdot 74$

Question 14

a. Outcomes assessed : HE3

Marking Guidelines

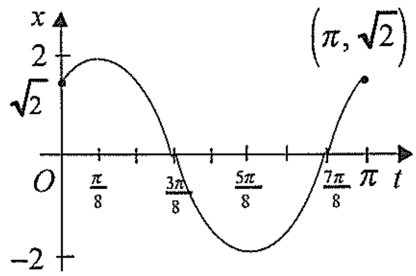
Criteria	Marks
i. • obtains $\dot{x}$, $\ddot{x}$ by differentiation	1
• uses trig. identity to establish result	1
ii. • sketches sinusoidal curve with correct amplitude and period	1
• shows appropriate lateral shift for the given phase angle with coordinates of endpoints	1
iii. • uses the symmetry of the graph to deduce the result	1
iv. • shows that 100 m of travel corresponds to $12\cdot 5$ complete oscillations	1
• deduces that time taken is $12\cdot 5$ periods	1

Answer

i.

$$x = 2\cos\left(2t - \tfrac{\pi}{4}\right)$$
$$\dot{x} = -4\sin\left(2t - \tfrac{\pi}{4}\right)$$
$$\ddot{x} = -8\cos\left(2t - \tfrac{\pi}{4}\right)$$
$$v^2 - x\,\ddot{x} = 16\left\{\sin^2\left(2t - \tfrac{\pi}{4}\right) + \cos^2\left(2t - \tfrac{\pi}{4}\right)\right\}$$
$$= 16$$

ii.



iii. Using the symmetry in the graph, first return to $x = \sqrt{2}$ is when $t = \frac{\pi}{4}$.

Period is $T = \frac{2\pi}{2} = \pi$ seconds. Hence first return to starting point is after $\frac{1}{4}T$.

iv. Particle travels 8m in one oscillation. $100 \text{ m} = 12 \times 8 \text{ m} + 4 \text{ m}$

Time taken is $12 \times \pi + \frac{\pi}{2} = \frac{25\pi}{2}$ seconds

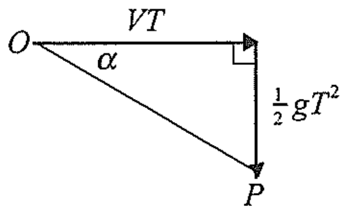
Question 14 (cont)
b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i. • shows the displacement of the particle down the slope at impact as vector sum of x and y	1
• uses $\tan\alpha = \frac{1}{2}$ to deduce result	1
ii. • expresses $\dot{x}$ and $\dot{y}$ at time of impact in terms of V	1
• deduces direction and magnitude of velocity at impact from vector sum of $\dot{x}$ and $\dot{y}$	1

Answer

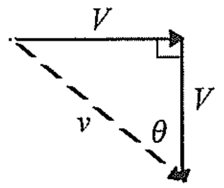
i. Let particle hit slope at time T seconds at point P.



$$\tan\alpha = \frac{\frac{1}{2}gT^2}{VT} = \frac{1}{2}\cdot\frac{gT}{V}$$

But $\tan\alpha = \frac{1}{2}$. $\quad \therefore T = \dfrac{V}{g}$

ii. Let particle hit slope at angle θ to vertical with velocity v ms^{-1}
Then v has horizontal and vertical components $\dot{x} = V$, $\dot{y} = -gT = -V$



Hence $\theta = 45°$ (isosceles right Δ has equal $\angle$'s of 45°)
and $v = V\sqrt{2}$ (applying Pythagoras' theorem)

c. Outcomes assessed : H5, HE3

Marking Guidelines

Criteria	Marks
i. • recognises sum of a GP to simplify LHS	1
• uses Binomial theorem to simplify RHS	1
ii. • chooses to equate coefficients of x^2 on both sides of identity in i., with correct RHS	1
• obtains coefficient of x^2 on LHS as a sum of binomial coefficients to deduce result	1

Answer

i. $$\sum_{r=1}^{n}(1+x)^{r-1} = 1 + (1+x) + (1+x)^2 + \dots + (1+x)^{n-1} = \frac{1.\left\{(1+x)^n - 1\right\}}{(1+x)-1} = \frac{1}{x}\left\{(1+x)^n - 1\right\} \quad \text{(using } S_n \text{ for GP)}$$

$$\sum_{r=1}^{n} {}^nC_r\, x^{r-1} = \frac{1}{x}\left\{\sum_{r=0}^{n} {}^nC_r\, x^r - {}^nC_0\, x^0\right\} = \frac{1}{x}\left\{(1+x)^n - 1\right\} \quad \text{(using Binomial theorem)}$$

Hence $\displaystyle\sum_{r=1}^{n}(1+x)^{r-1} = \sum_{r=1}^{n} {}^nC_r\, x^{r-1}$.

ii. For $n \geq 3$, equating coefficients of x^2 on both sides of the identity in (i) :

$${}^2C_2 + {}^3C_2 + \dots + {}^{n-1}C_2 = {}^nC_3$$

Hence $$\sum_{r=2}^{n-1} {}^rC_2 = {}^nC_3$$

Independent HSC Trial Examination 2012 Mathematics Extension 1 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1	1	Trigonometric functions	H5	E2
2	1	Polynomials	PE3	E2
3	1	Differentiation	PE5	E2
4	1	Integration; Trigonometric functions	H5	E2
5	1	Trigonometric functions	HE2	E3
6	1	Exponential growth and decay	HE3	E2
7	1	Inverse functions	HE4	E3
8	1	Methods of integration	HE6	E3
9	1	Inverse functions	HE4	E3
10	1	Binomial theorem	HE3	E3
11 a	2	Permutations and combinations	PE3	E2-E3
b	2	Division of an interval	P4	E2-E3
c	2	Inequalities	PE3	E2-E3
d i	2	Geometric applications of differentiation	H5	E2-E3
ii	1	Functions	H5	E2-E3
e i	1	Parametric representation	PE4	E2-E3
ii	2	Parametric representation; Angle between two lines	H5, PE3	E2-E3
f	3	Circle geometry	PE2, PE3	E2-E3
12 a	2	Polynomials	PE3	E2-E3
b i	1	Basic algebra	P4	E2-E3
ii	2	Rates of change	HE5	E2-E3
c	3	Methods of integration	HE6	E2-E3
d	3	Mathematical Induction	HE2	E3-E4
e i	2	Inverse functions	HE4	E2-E3
ii	2	Further trigonometry	H5	E2-E3
13 a i	1	Motion in a straight line	HE5	E2-E3
ii	2	Motion in a straight line	HE5	E2-E3
iii	2	Motion in a straight line	HE5	E2-E3
iv	2	Motion in a straight line	HE5	E2-E3
b i	2	Inverse functions	HE4	E2-E3
ii	2	Inverse functions	HE4	E2-E3
c i	2	Binomial probability	HE3	E2-E3
ii	2	Polynomials	PE3	E2-E3
14 a i	2	Simple harmonic motion	HE3	E3-E4
ii	2	Simple harmonic motion	HE3	E3-E4
iii	1	Simple harmonic motion	HE3	E3-E4
iv	2	Simple harmonic motion	HE3	E3-E4
b i	2	Projectile motion	HE3	E3-E4
ii	2	Projectile motion	HE3	E3-E4
c i	2	Series; Binomial theorem	H5, HE3	E3-E4
ii	2	Binomial theorem	HE3	E3-E4