

Trial HSC Extension 1, 2010 - Solutions

Question 1 (12 marks)

(a) $m_1 = m$ $m_2 = 2$

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$1 = \left| \frac{2 - m}{1 + 2m} \right| \quad (2)$$

$$\frac{2 - m}{1 + 2m} = 1 \quad \text{OR} \quad \frac{2 - m}{1 + 2m} = -1$$

$$2 - m = 1 + 2m$$

$$2 - m = -1 - 2m$$

$$3m = 1$$

$$-m = 3$$

$$\underline{\underline{m = \frac{1}{3}}}$$

$$\underline{\underline{m = -3}}$$

(b)

A(-4, -6)

B(6, -1)

-3

2

$$x = \frac{(-3 \times 6) + (2 \times -4)}{-3 + 2} = 26$$

$$y = \frac{(-3 \times -1) + (2 \times -6)}{-3 + 2} = 9$$

P(26, 9)

(2)

(c) $\frac{2x+1}{x-1} \geq 3$

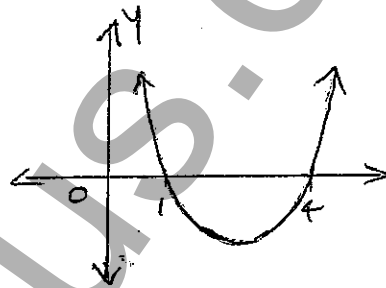
$$(x-1)(2x+1) \geq 3(x-1)^2, x \neq 1$$

$$3(x-1)^2 - (x-1)(2x+1) \leq 0$$

$$(x-1)[3(x-1) - (2x+1)] \leq 0$$

$$(x-1)(3x-3-2x-1) \leq 0$$

$$(x-1)(x-4) \leq 0$$



(2)

$$\underline{\underline{1 < x \leq 4}}$$

(d) $y = x \tan^{-1} \frac{x}{2}$

$$y' = x \times \frac{1}{1 + \frac{x^2}{4}} \times \frac{1}{2} + \tan^{-1} \frac{x}{2} \times 1$$

$$= x \times \frac{1}{\frac{4+x^2}{4}} \times \frac{1}{2} + \tan^{-1} \frac{x}{2}$$

$$= x \times \frac{4}{4+x^2} \times \frac{1}{2} + \tan^{-1} \frac{x}{2}$$

$$= \underline{\underline{\frac{2x}{4+x^2} + \tan^{-1} \frac{x}{2}}}$$

(3)

$$(c) \int_1^4 \frac{dx}{x + \sqrt{x}}, u = \sqrt{x}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}; \frac{dx}{du} = 2\sqrt{x}$$

$$dx = 2\sqrt{x} du = 2u du$$

$$\text{When } x=1, u=\sqrt{1}=1$$

$$\text{When } x=4, u=\sqrt{4}=2$$

$$\int_1^2 \frac{2u du}{u^2+u} = \int_1^2 \frac{2u du}{u(u+1)}$$

$$= \int_1^2 \frac{2 du}{u+1} = 2 \int_1^2 \frac{du}{u+1}$$

$$= 2 [\log(u+1)]_1^2 \quad (3)$$

$$= 2 [\log 3 - \log 2]$$

$$= 2 \log \frac{3}{2}$$

Question 2 (12 marks)

$$(a) T_{r+1} = 12 C_r (x^2)^{12-r} \left(\frac{2}{x}\right)^r$$

$$= 12 C_r x^{24-2r} \frac{2^r}{x^r}$$

$$= 12 C_r x^{24-3r} 2^r$$

$$24-3r = 9$$

$$3r = 15$$

$$r = 5$$

$$T_6 = 12 C_5 x^{24-15} 2^5$$

$$= 12 C_5 x^9 2^5$$

$$\text{Coefficient of } x^9 = \underline{\underline{12 C_5 \times 2^5}} \quad (2)$$

$$(b) \lim_{x \rightarrow 0} \frac{\sin 6x}{7x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin 6x}{6x} \times \frac{6x}{7x} \quad (2)$$

$$= \frac{6}{7} \left(\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right)$$

$$(c) f(x) = 4 \cos^{-1} \frac{x}{3}$$

$$(i) D: -1 \leq \frac{x}{3} \leq 1$$

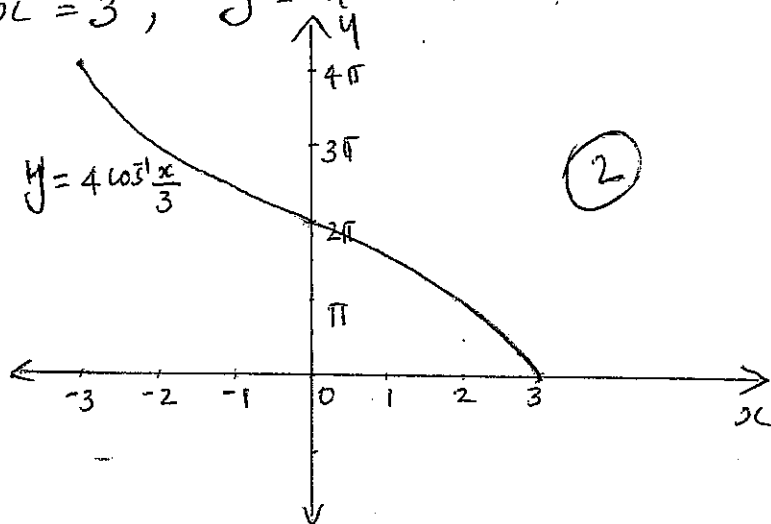
$$-3 \leq x \leq 3 \quad (2)$$

$$R: 0 \leq y \leq 4\pi$$

$$x = -3, y = 4 \cos^{-1}(-1) = 4\pi$$

$$x = 0, y = 4 \cos^{-1} 0 = 4 \times \frac{\pi}{2} = 2\pi$$

$$x = 3, y = 4 \cos^{-1} 1 = 0$$



(b) Total number of ways in which 16 people can be seated = $8P_4 \times 8P_2 \times 10!$ (2)

$$(i) P(x=3) = {}^5C_3 (0.4)^3 (0.6)^2 = 0.2304$$

$$(ii) P(x=3) + P(x=4) + P(x=5) = 0.2304 + {}^5C_4 (0.4)^4 (0.6)^1 + {}^5C_5 (0.4)^5 (0.6)^0 = 0.31744$$
 (3)

$$(iii) P(x=0) + P(x=1) + P(x=2) = 1 - 0.31744 = 0.68256$$

Question 4 (12 marks)

(a) when $n=2$,

$$LHS = \log \frac{2}{2-1} = \log 2$$

$$RHS = \log 2$$

$$LHS = RHS$$

$\therefore$ the result is true for $n=2$

Assume the result is true for $n=k$

page 4

$$i.e. \log 2 + \log \left(\frac{3}{2}\right) + \dots + \log \left(\frac{k}{k-1}\right) = \log k \text{ --- (1)}$$

To prove that the result is true for $n=k+1$

$$i.e. \log 2 + \log \left(\frac{3}{2}\right) + \log \left(\frac{4}{3}\right) + \dots + \log \left(\frac{k}{k-1}\right) + \log \left(\frac{k+1}{k}\right) = \log k+1$$

Now

$$\log 2 + \log \frac{3}{2} + \dots + \log \left(\frac{k}{k-1}\right) + \log \left(\frac{k+1}{k}\right) = \log k + \log \left(\frac{k+1}{k}\right) \text{ by assumption (1)}$$

$$= \log \left(k \times \frac{k+1}{k}\right)$$

$$= \log (k+1) \text{ (3)}$$

$\therefore$ the result is true for $n=k+1$

Hence by the principle of mathematical induction, the result is true for $n \geq 2$

(b) (i) Normal at P

$$x + py = 4p^3 + 8p \text{ --- (1)}$$

Normal at Q

$$x + qy = 4q^3 + 8q \text{ --- (2)}$$

① - ② gives

$$y(p-q) = 4(p^3-q^3) + 8(p-q)$$

$$y(p-q) = 4(p-q)(p^2+pq+q^2) + 8(p-q)$$

$$y = 4(p^2+pq+q^2) + 8$$

$$= 4(p^2-1+q^2) + 8$$

$$= 4p^2 - 4 + 4q^2 + 8$$

$$= 4p^2 + 4q^2 + 4$$

Substitute in ①

$$2x + p(4p^2 + 4q^2 + 4) = 4p^3 + 8p$$

$$2x + 4p^3 + 4pq^2 + 4p = 4p^3 + 8p$$

$$2x + 4pq^2 = 4p$$

$$2x + 4pq \times q = 4p$$

$$2x - 4q = 4p$$

$$2x = 4p + 4q$$

$$= 4(p+q)$$

$$2x = 4(p+q) \text{ --- ③}$$

$$y = 4p^2 + 4q^2 + 4 \text{ --- ④}$$

From ③ $p+q = \frac{2x}{4}$

From ④ $\frac{y}{4} = p^2 + q^2 + 1$

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$$p^2 + q^2 = \frac{y}{4} - 1$$

$$p^2 + q^2 = (p+q)^2 - 2pq$$

$$= (p+q)^2 - 2x - 1$$

$$= (p+q)^2 + 2$$

$$\frac{y}{4} - 1 = \left(\frac{2x}{4}\right)^2 + 2$$

$$\frac{y}{4} - 1 = \frac{2x^2}{16} + 2$$

$$4y - 16 = 2x^2 + 32 \text{ --- ③}$$

$$2x^2 = 4y - 48$$

$$2x^2 = 4(y - 12)$$

$$\textcircled{c} \quad y = \frac{2x^2 - 2}{x^2 - 9} = \frac{2(x+1)(x-1)}{(x+3)(x-3)}$$

Vertical asymptotes

$$x = -3 \text{ and } x = 3$$

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 2}{x^2 - 9}$$

$$= \lim_{x \rightarrow \infty} \frac{2 - \frac{2}{x^2}}{1 - \frac{9}{x^2}} \text{ --- ③}$$

$$= 2 \text{ (since } \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0)$$

Horizontal asymptote is

$$y = 2$$

x intercepts

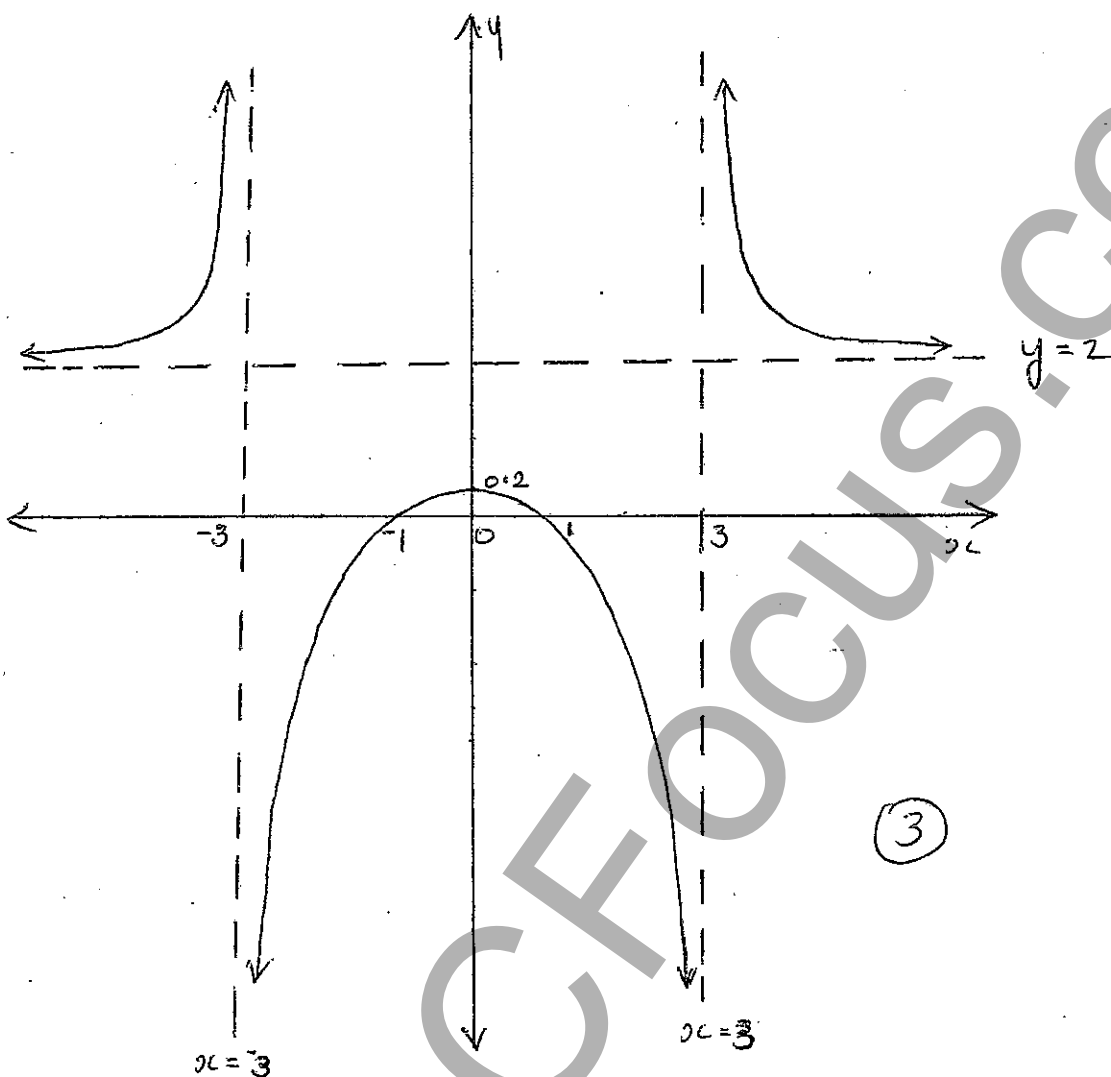
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$$y=0 \Rightarrow 2(x+1)(x-1)=0$$

$$x=-1 \text{ or } x=1$$

y intercept

$$x=0 \Rightarrow y = \frac{-2}{-9} = \frac{2}{9} = 0.2$$



(3)

Question 5 (12 marks)

$$(a) (1+x)^5 (1+x)^5 = (1+x)^{10}$$

$$\begin{aligned} & [5C_0 + 5C_1x + 5C_2x^2 + 5C_3x^3 + 5C_4x^4 + 5C_5x^5] [5C_0 + 5C_1x + 5C_2x^2 + \\ & 5C_3x^3 + 5C_4x^4 + 5C_5x^5] = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + {}^{10}C_3x^3 + {}^{10}C_4x^4 \\ & + {}^{10}C_5x^5 + \dots + {}^{10}C_{10}x^{10} \end{aligned}$$

Equating coefficients of x^5 on both sides, page 7

$${}^5C_0 \times {}^5C_5 + {}^5C_1 \times {}^5C_4 + {}^5C_2 \times {}^5C_3 + {}^5C_3 \times {}^5C_2 + {}^5C_4 \times {}^5C_1 \\ + {}^5C_5 \times {}^5C_0 = {}^{10}C_5$$

$${}^5C_0 \times {}^5C_0 + {}^5C_1 \times {}^5C_1 + {}^5C_2 \times {}^5C_2 + {}^5C_3 \times {}^5C_3 + {}^5C_4 \times {}^5C_4 \\ + {}^5C_5 \times {}^5C_5 = {}^{10}C_5 \quad (\text{since } nC_r = nC_{n-r})$$

$$({}^5C_0)^2 + ({}^5C_1)^2 + ({}^5C_2)^2 + ({}^5C_3)^2 + ({}^5C_4)^2 + ({}^5C_5)^2 = {}^{10}C_5$$

$$\text{i.e. } \sum_{k=0}^5 ({}^5C_k)^2 = {}^{10}C_5 \quad (3)$$

$$(b) (i) (1+x)^n = nC_0 + nC_1 x + nC_2 x^2 + \dots + nC_r x^r + \dots + nC_n x^n \quad (1)$$

$$(ii) \int_0^1 (1+x)^n dx = \int_0^1 (nC_0 + nC_1 x + nC_2 x^2 + \dots + nC_n x^n) dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1 = nC_0 x + nC_1 \frac{x^2}{2} + nC_2 \frac{x^3}{3} + \dots + nC_n \frac{x^{n+1}}{n+1} \Big|_0^1$$

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = nC_0 + \frac{nC_1}{2} + \frac{nC_2}{3} + \dots + \frac{nC_n}{n+1}$$

(3)

$$\underline{\underline{nC_0 + \frac{nC_1}{2} + \frac{nC_2}{3} + \dots + \frac{nC_n}{n+1} = \frac{2^{n+1} - 1}{n+1}}}$$

(1) (i) $T = A + Ce^{kt}$

$$\frac{dT}{dt} = Ce^{kt} \times k$$

$$= k \times Ce^{kt} \quad (2)$$

$$= k(T-A) \text{ (since } e^{kt} = T-A \text{)}$$

$\therefore T = A + Ce^{kt}$ is a solution of the equation.

$$\frac{dT}{dt} = k(T-A)$$

(ii) $T = 25 + Ce^{kt}$
when $t=0$, $T = 10^\circ\text{C}$

$$10 = 25 + C$$

$$C = 10 - 25 = -15$$

$$\therefore T = 25 - 15e^{kt}$$

when $t=20$, $T = 15^\circ$

$$15 = 25 - 15e^{20k}$$

$$15e^{20k} = 10$$

$$e^{20k} = \frac{10}{15}$$

$$20k = \log\left(\frac{10}{15}\right)$$

$$k = \frac{1}{20} \log\left(\frac{10}{15}\right)$$

when $t=50$, $T = ?$

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$$T = 25 - 15e^{50k}$$

$$= 25 - 15e^{50 \times \frac{1}{20} \log\left(\frac{10}{15}\right)} \quad (2)$$

$$= 20^\circ\text{C (to the nearest degree)}$$

(iii) $k = \frac{1}{20} \log\left(\frac{10}{15}\right) = -0.02$

$$T = 25 - 15e^{-0.02t}$$

Since $k < 0$, as $t \rightarrow \infty$, $e^{kt} \rightarrow 0$

$\therefore$ as t increases indefinitely the object's temperature approaches air temperature. ①

Question 6 (12 marks)

(a) (i) $f(x) = x \sin^{-1}x + \sqrt{1-x^2}$

$$f'(x) = x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \times 1 + \frac{1}{2\sqrt{1-x^2}} x^{-2x}$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

$$\frac{d}{dx} (x \sin^{-1}x + \sqrt{1-x^2}) = \sin^{-1}x \quad (3)$$

$$\frac{1}{2} \int \sin^{-1}x dx = \left[x \sin^{-1}x + \sqrt{1-x^2} \right]_0^{\frac{1}{2}}$$

$$= \left(\frac{1}{2} \sin^{-1}\left(\frac{1}{2}\right) + \sqrt{1-\frac{1}{4}} \right) - (0 + \sqrt{1})$$

$$= \frac{1}{2} \times \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}$$

$$= 0.128$$

②

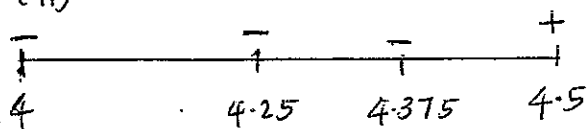
(b)(i) $\tan x - x = 0$

$\tan 4 - 4 = -2.8$

$\tan 4.5 - 4.5 = 0.14$

Since $f(4) < 0$ and $f(4.5) > 0$ there is a root of $f(x) = 0$ ① between $x = 4$ and

(ii) $x = 4.5$



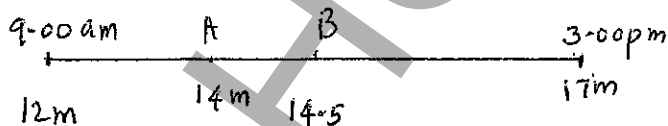
$f(4.25) = \tan 4.25 - 4.25 = -2.24$

$f(4.375) = \tan 4.375 - 4.375 = -1.52$

Approximate value of the root ②

$= \frac{4.375 + 4.5}{2} = 4.4$

(c)(i)



$T = 2 \times (3:00 \text{ pm} - 9:00 \text{ am})$

$= 2 \times 6 \text{ h}$

$= 12 \text{ h}$

$T = \frac{2\pi}{n} \quad n = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}$

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Let 9:00 am denote $t = 0$

$x = a \cos(nt + \pi)$

$= 2.5 \cos\left(\frac{\pi}{6}t + \pi\right)$

when the harbour is 14m deep

$x = -0.5$

$-0.5 = 2.5 \cos\left(\frac{\pi}{6}t + \pi\right)$

$\cos\left(\frac{\pi}{6}t + \pi\right) = -\frac{1}{5}$

$\frac{\pi}{6}t + \pi = \pi - \cos^{-1}\left(\frac{1}{5}\right), \pi + \cos^{-1}\left(\frac{1}{5}\right)$

$\frac{\pi t}{6} = -1.3694, 1.3694$

$\frac{\pi t}{6} = 1.3694$ (t can't be negative)

$t = 1.3694 \times \frac{6}{\pi}$

$= 2.6154$

$= 2 \text{ hr } 37 \text{ minutes.}$ ③

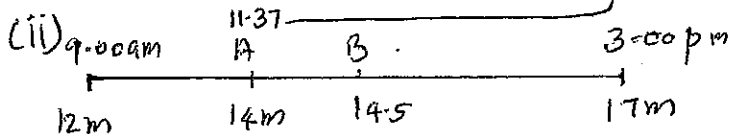
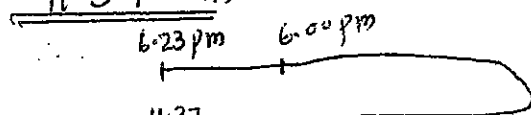
Time taken from A to B

$= 3 \text{ hr} - 2 \text{ hr } 37 \text{ min}$

$= 23 \text{ min}$

The ship can go into the harbour

at 11:37 am

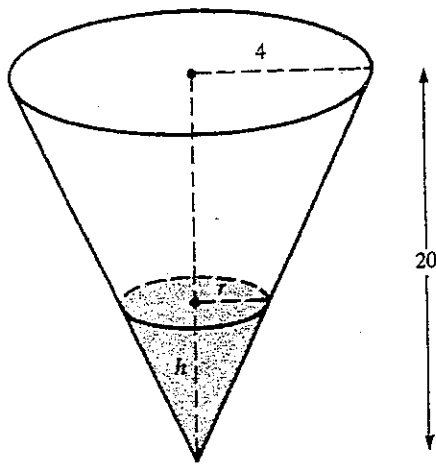


The ship must depart before

5:53 pm. ①

Question 7 (12 marks)

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$$V = \frac{1}{3} \pi r^2 h$$

$$\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$$

By similar triangles

$$\frac{r}{4} = \frac{h}{20}$$

$$r = \frac{4h}{20} = \frac{h}{5}$$

$$V = \frac{1}{3} \pi \times \frac{h^2}{25} \times h$$

$$= \frac{\pi h^3}{75}$$

(3)

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\frac{dV}{dh} = \frac{\pi \times 3h^2}{75} = \frac{\pi h^2}{25}$$

$$12 = \frac{\pi h^2}{25} \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{12 \times 25}{\pi h^2}$$

when $h = 5$,

$$\frac{dh}{dt} = \frac{12 \times 25}{\pi \times 25} = 3.82$$

The fluid level is dropping at the rate of 3.82 cm/s .

$$(b) f(x) = x^3 + x^2 - 10$$

$$f'(x) = 3x^2 + 2x$$

$$a = 2$$

$$a_1 = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \left[\frac{8 + 4 - 10}{3 \times 4 + 4} \right] = 2 - \frac{1}{8} = 1.875$$

$$= 1.9$$

$$a_2 = 1.875 - \frac{f(1.875)}{f'(1.875)}$$

$$= 1.875 - \left[\frac{(1.875)^3 + (1.875)^2 - 10}{3 \times (1.875)^2 + 2 \times 1.875} \right]$$

$$= 1.867 = 1.9$$

(2)

$$a_1 = a_2 = 1.9$$

$\therefore$ the root of $f(x) = 0$ correct to one decimal place is 1.9

(c) (i) When the particle strikes the ground $y = 0$

$$Vt \sin \alpha - \frac{1}{2} g t^2 = 0$$

$$t(V \sin \alpha - \frac{1}{2} g t) = 0$$

$$t = 0 \quad \text{or} \quad V \sin \alpha = \frac{1}{2} g t$$

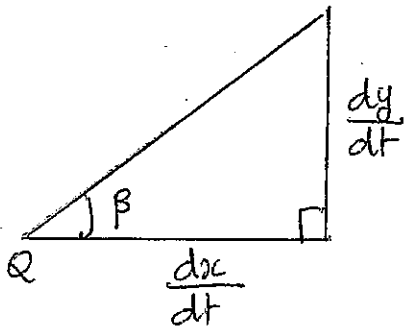
$$t=0 \text{ or } 2V\sin\alpha = gt$$

$$t=0 \text{ or } t = \frac{2V\sin\alpha}{g}$$

Now $t=0$ refers to the instant of projection (2) and hence $t = \frac{2V\sin\alpha}{g}$

is the required time, the time of flight.

(ii)



$$\tan\beta = \frac{dy/dt}{dx/dt}$$

$$= \frac{V\sin\alpha - gt}{V\cos\alpha}$$

$$\text{iii) } \frac{\sin\beta}{\cos\beta} = \frac{V\sin\alpha - gt}{V\cos\alpha}$$

$$V\sin\beta\cos\alpha = V\sin\alpha\cos\beta - gt\cos\beta$$

$$gt\cos\beta = V\sin\alpha\cos\beta - V\sin\beta\cos\alpha$$

$$gt\cos\beta = V(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$gt\cos\beta = V\sin(\alpha - \beta)$$

$$t = \frac{V\sin(\alpha - \beta)}{g\cos\beta}$$

(iv) when $\beta = \frac{\alpha}{2}$ we have

$$t = \frac{V\sin(\alpha - \frac{\alpha}{2})}{g\cos\frac{\alpha}{2}} = \frac{V\sin\frac{\alpha}{2}}{g\cos\frac{\alpha}{2}}$$

$$= \frac{V}{g} \tan\frac{\alpha}{2}$$

$$\text{Given that } \frac{V}{g} \tan\frac{\alpha}{2} = \frac{1}{3} \frac{2V\sin\alpha}{g}$$

$$\tan\frac{\alpha}{2} = \frac{1}{3} \times 2\sin\alpha$$

$$3\tan\frac{\alpha}{2} = 2\sin\alpha$$

$$= \frac{2 \times 2\tan\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}}$$

$$3 = \frac{4}{1 + \tan^2\frac{\alpha}{2}}$$

$$4 = 3 + 3\tan^2\frac{\alpha}{2}$$

$$3\tan^2\frac{\alpha}{2} = 1$$

$$\tan^2\frac{\alpha}{2} = \frac{1}{3}$$

$$\tan\frac{\alpha}{2} = \frac{1}{\sqrt{3}} \left(\frac{\alpha}{2} \text{ is acute} \right)$$

$$\frac{\alpha}{2} = \frac{\pi}{6}$$

$$\underline{\underline{\alpha = \frac{\pi}{3}}}$$

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