2009
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ……………………..

Marks

Question 1 **Begin a new booklet**

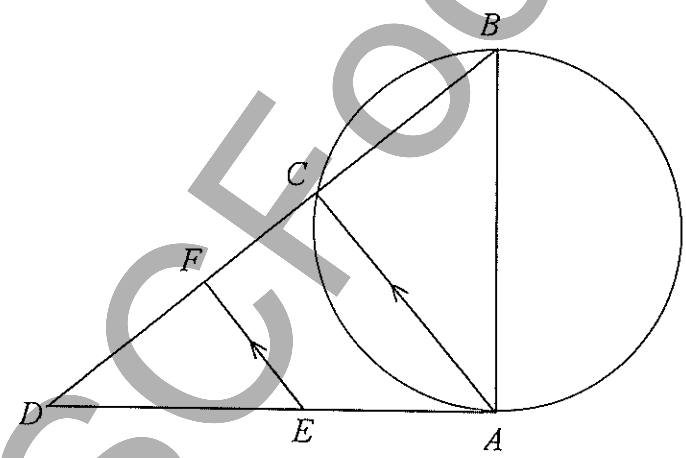
(a) Find $\lim_{x \to 0} \frac{\sin 3x}{2x}$. **2**

(b) Find the limiting sum of the geometric series $\left(\frac{e}{e+1}\right)+\left(\frac{e}{e+1}\right)^2+\left(\frac{e}{e+1}\right)^3+\ldots$. **2**

(c) The equation $x^3+2x^2+3x+6=0$ has roots α, β and γ. Find the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. **2**

(d) Find the acute angle between the lines $y=2x$ and $x+y-3=0$, giving the answer correct to the nearest degree. **2**

(e)



AB is a diameter of the circle and C is a point on the circle. The tangent to the circle at A meets BC produced at D. E is a point on AD and F is a point on CD such that EF is parallel to AC.

(i) Give a reason why $\angle EAC = \angle ABC$. **1**

(ii) Hence show that $EABF$ is a cyclic quadrilateral. **2**

(iii) Show that BE is a diameter of the circle through E, A, B and F. **1**

Student name / number ....

Marks

Question 2 **Begin a new booklet**

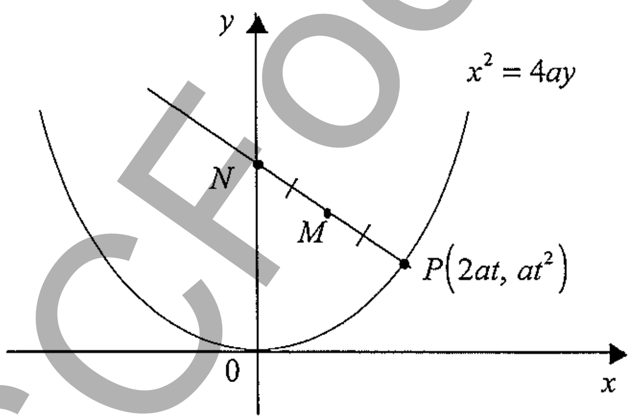
(a) Evaluate $\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \, dx$, giving the answer in simplest exact form. **2**

(b) Find the number of ways in which 3 boys and 3 girls can be arranged in a line so that the two end positions are occupied by boys and no two boys are next to each other. **2**

(c) $A(-2\,,3)$ and $B(6\,,-1)$ are two points. Find the coordinates of the point P that divides the interval AB internally in the ratio 3 : 2. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to show that $\dfrac{\sin x}{1-\cos x} = \cot\frac{x}{2}$. **2**

(e)



$P(2at\,,at^2)$ is a point on the parabola $x^2 = 4ay$. The normal to the parabola at P cuts the y-axis at N. M is the midpoint of PN.

(i) Use differentiation to show that the normal to the parabola at P has equation $x + ty = 2at + at^3$. **2**

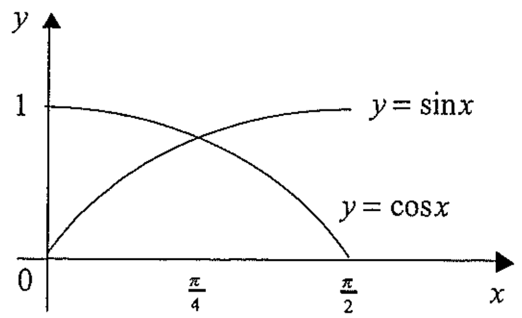
(ii) Find the equation of the locus of M as P moves on the parabola. **2**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a)



The region bounded by the curves $y = \cos x$ and $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution around the x-axis. Find the volume of the solid of revolution. **2**

(b) Use Mathematical Induction to show that for all positive integers $n \geq 2$, **4**

$$2\times1 + 3\times2 + 4\times3 + \ldots + n(n-1) = \frac{n(n^2-1)}{3}.$$

(c) Consider the function $f(x) = (x+2)^2 - 9$, $-2 \leq x \leq 2$.

(i) Find the equation of the inverse function $f^{-1}(x)$. **1**

(ii) On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the coordinates of the endpoints and the intercepts on the coordinate axes. **3**

(iii) Find the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = f^{-1}(x)$, giving the answer in simplest exact form. **2**

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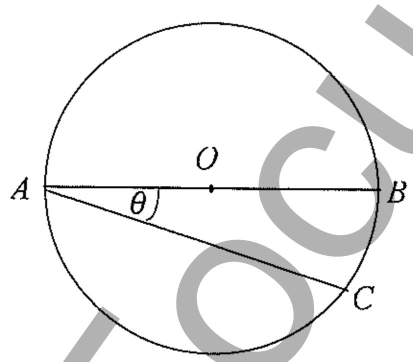
Marks

Question 4 **Begin a new booklet**

(a) Bob chooses six numbers from the numbers 1 to 40 inclusive. A machine then chooses six numbers at random from the numbers 1 to 40 inclusive. Find the probability that none of Bob's numbers match the numbers chosen by the machine, giving the answer correct to 2 decimal places. **2**

(b) Use the substitution $u = \sin^2 x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1+\sin^2 x}\, dx$, giving the answer in simplest exact form. **4**

(c)



AOB is a diameter of a circle with centre O and radius 1 metre. AC is a chord of the circle such that $\angle BAC = \theta$, where $0 < \theta < \frac{\pi}{2}$. The area of that part of the circle contained between the diameter AB and the chord AC is equal to one quarter of the area of the circle.

(i) Show that $\theta + \frac{1}{2}\sin 2\theta - \frac{\pi}{4} = 0$. **2**

(ii) Show that $0 \cdot 4 < \theta < 0 \cdot 5$. **2**

(iii) Use one application of Newton's method with an initial approximation $\theta_0 = 0 \cdot 4$ to find the next approximation to the value of θ, giving your answer correct to 2 decimal places. **2**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a) Consider the function $f(x) = \tan^{-1}(x-1)$.

(i) Sketch the curve $y = f(x)$ showing clearly the equations of any asymptotes and the intercepts on the coordinate axes. **2**

(ii) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 1$. **2**

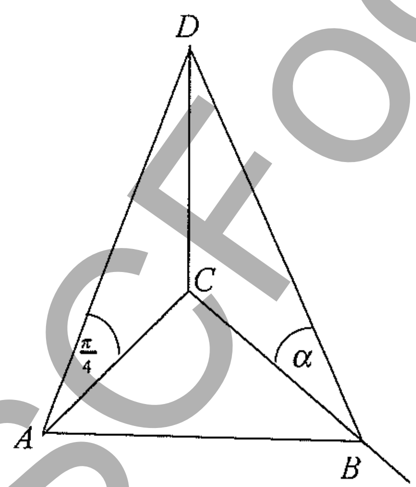
(b) A particle is moving in a straight line. After time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = \sqrt{x}$ and acceleration a ms^{-2}. Initially the particle is 1 metre to the right of O.

(i) Show that a is independent of x. **1**

(ii) Express x in terms of t. **2**

(iii) Find the distance travelled by the particle during the third second of its motion. **1**

(c)



A vertical tower CD of height 15 metres stands with its base C on horizontal ground. A is a point on the ground due South of C such that the angle of elevation of the top D of the tower from A is $\frac{\pi}{4}$ radians. B is a variable point on the ground due East of C such that the angle of elevation of the top D of the tower from B is α radians, where $0 < \alpha < \frac{\pi}{2}$. The value of α is increasing at a constant rate of $0 \cdot 01$ radians per second.

(i) show that $AB = 15\operatorname{cosec}\alpha$. **2**

(ii) Find the rate at which the length AB is changing when $\alpha = \frac{\pi}{3}$. **2**

Student name / number ....

Question 6 **Begin a new booklet** **Marks**

(a) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds, it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v = -12\sin(2t + \frac{\pi}{3})$ and acceleration $\ddot{x}$ ms^{-2}. Initially the particle is 5 metres to the right of O.

(i) Show that $\ddot{x} = -4(x-2)$. **3**

(ii) Find the period and the extremities of the motion. **2**

(iii) Find the time taken by the particle to return to its starting point for the first time. **1**

(b) After t hours, the number of individuals in a population is given by $N = 500 - 400e^{-0\cdot 1t}$.

(i) Sketch the graph of N as a function of t, showing clearly the initial population size and the limiting population size. **2**

(ii) Show that $\frac{dN}{dt} = 0\cdot 1(500 - N)$. **1**

(iii) Find the population size for which the rate of growth of the population is half the initial rate of growth. **1**

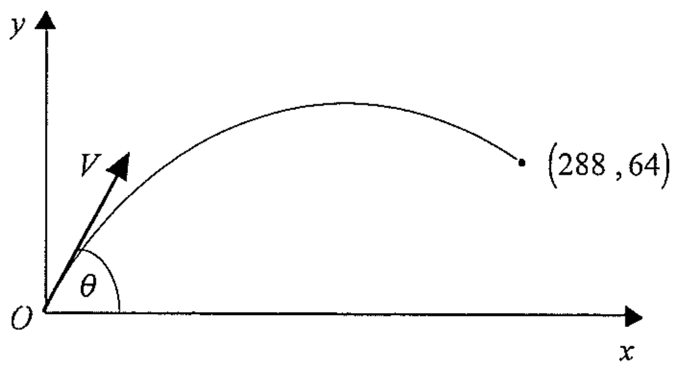
(c) If $\cos^{-1}x - \sin^{-1}x = k$, where $-\frac{\pi}{2} \le k \le \frac{3\pi}{2}$, show that $x = \frac{1}{\sqrt{2}}\left(\cos\frac{k}{2} - \sin\frac{k}{2}\right)$. **2**

Student name / number ....

Marks

Question 7 **Begin a new booklet**

(a)



A toy rocket is projected from a point O with speed V ms^{-1} at an angle θ above the horizontal, where $0 < \theta < \frac{\pi}{2}$. The rocket moves in a vertical plane under gravity where the acceleration due to gravity is 10 ms^{-2}. After 8 seconds the rocket hits a target at a horizontal distance 288 metres from O and at a height 64 metres above O.

(i) Use integration to show that after time t seconds, the horizontal and vertical displacements of the rocket from O, x metres and y metres respectively, are given by $x = Vt\cos\theta$ and $y = Vt\sin\theta - 5t^2$. **2**

(ii) Find the exact values of V and θ. **3**

(iii) Find the velocity of the rocket just before impact with the target, giving the speed correct to the nearest integer and the angle to the horizontal correct to the nearest degree. **3**

(b)(i) By considering the term in x^r on both sides of the identity $(1+x)^{m+n} = (1+x)^m(1+x)^n$, show that ${}^{m+n}C_r = \sum_{k=0}^{r} {}^mC_k\,{}^nC_{r-k}$, for $0 \le r \le m$ and $0 \le r \le n$. **2**

(ii) Hence show that ${}^{m+1}C_0\,{}^nC_2 + {}^{m+1}C_1\,{}^nC_1 + {}^{m+1}C_2\,{}^nC_0 = {}^mC_0\,{}^{n+1}C_2 + {}^mC_1\,{}^{n+1}C_1 + {}^mC_2\,{}^{n+1}C_0$ for $m \ge 2$ and $n \ge 2$. **2**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$