

$$(a) \frac{n!}{(n-1)!} = n$$

$$(b) \frac{-2x}{\sqrt{1-x^4}}$$

$$(c) \int \frac{1}{40+x^2} dx = \frac{1}{2\sqrt{10}} \tan^{-1} \frac{x}{2\sqrt{10}} + c$$

$$(d) \ln e^{\frac{1}{2}} = \frac{1}{2} \ln e$$

$$= \frac{1}{2}$$

No penalty
for omission
of c.

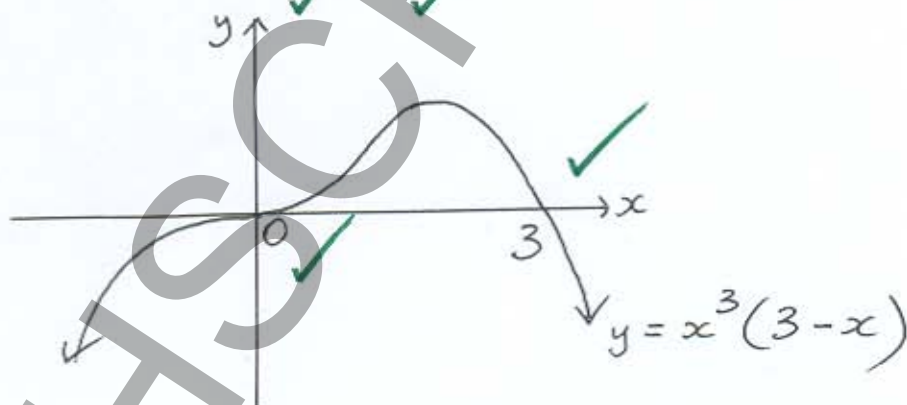
$$(e) \int 2x e^{x^2} dx = e^{x^2} + c$$

$$(f) \cos 2\theta = \frac{1-t^2}{1+t^2} \quad (\text{where } t = \tan \theta)$$

$$(g) P = \left(\frac{28-24}{11}, \frac{70+8}{11} \right)$$

$$= \left(\frac{4}{11}, 7\frac{1}{11} \right)$$

(h)



i) Substitute $x=1$ into the identity:

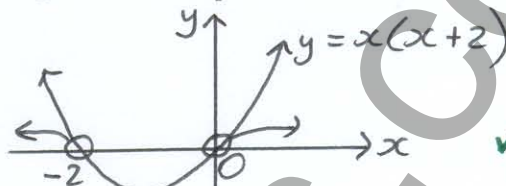
$$\sum_{r=0}^n {}^nC_r \cdot (1)^r = (1+1)^n$$

i.e. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$

(2) (a) $\int \frac{x}{(x+2)^2} dx = \int \frac{u-2}{u^2} du$ ✓
 $= \int \left(\frac{1}{u} - 2u^{-2} \right) du$ ✓
 $= \ln u + \frac{2}{u} + c$
 $= \ln(x+2) + \frac{2}{x+2} + c$ ✓

Let $x = u - 2$
 $\therefore \frac{dx}{du} = 1$
 $\therefore dx = du$

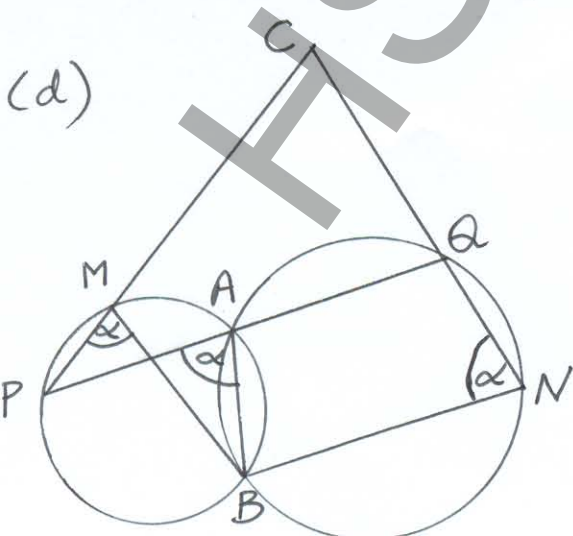
(b) $\frac{x}{x+2} > 0 \quad (x \neq -2)$
 Multiply both sides by $(x+2)^2$:
 $x(x+2) > 0$ ✓
 $x < -2$ or $x > 0$ ✓



(c) Let $\alpha = \tan^{-1} 2$ and $\beta = \tan^{-1} \sqrt{2}$.

$\therefore \tan \alpha = 2$, where $0 < \alpha < \frac{\pi}{2}$,
 and $\tan \beta = \sqrt{2}$, where $0 < \beta < \frac{\pi}{2}$.

$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$ ✓
 $= \frac{2 - \sqrt{2}}{1 + 2\sqrt{2}} \cdot \frac{1 - 2\sqrt{2}}{1 - 2\sqrt{2}}$ ✓
 $= \frac{2 - 4\sqrt{2} - \sqrt{2} + 4}{1 - 8}$
 $= \frac{6 - 5\sqrt{2}}{-7}$
 $= \frac{5\sqrt{2} - 6}{7}$ ✓



(d)

- (i) Exterior angle of cyclic quad $ABNQ$ is equal to the interior opposite angle. ✓
- (ii) $\angle PMB = \alpha$ (angles at circumference standing on same arc) ✓
 $\therefore \angle PMB = \angle BNQ = \alpha$
 $\therefore$ quad $CMBN$ is cyclic (converse of reason in (i)) ✓

(3)(a) Let $V \text{ mm}^3$ be the volume of the ice-cube, and $x \text{ mm}$ its edge length.

We are given $\frac{dx}{dt} = -2 \text{ mm/min}$.

We want $\frac{dV}{dt}$ when $x = 15$.

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} \checkmark, \text{ where } V = x^3.$$

$$\therefore \frac{dV}{dt} = 3x^2 \cdot (-2) \checkmark$$
$$= -6x^2$$

$$\text{So when } x = 15, \frac{dV}{dt} = -6(15)^2 \checkmark$$
$$= -1350.$$

So when the edge is 15 mm , the volume is decreasing at $1350 \text{ mm}^3/\text{min}$. $\checkmark$

(b) Let the roots be α , $-\frac{2}{\alpha}$ and β .

(i) The product of the roots is $-\frac{d}{a} = -1$.

$$\therefore \alpha \cdot -\frac{2}{\alpha} \cdot \beta = -1$$

$$\therefore \beta = \frac{1}{2}$$

So one of the roots is $\frac{1}{2}$.

(ii) The sum of the roots is $-\frac{b}{a} = \frac{17}{6}$.

$$\therefore \alpha - \frac{2}{\alpha} + \frac{1}{2} = \frac{17}{6} \checkmark$$

$$\alpha - \frac{2}{\alpha} = \frac{7}{3}$$

$$3\alpha^2 - 7\alpha - 6 = 0 \checkmark$$

$$(3\alpha + 2)(\alpha - 3) = 0$$

$$\alpha = -\frac{2}{3} \text{ or } 3 \checkmark$$

So the other two roots are $-\frac{2}{3}$ and 3 . $\checkmark$

$$(c) \int_0^{\frac{\pi}{2}} (\cos x - \cos^2 x) dx = \int_0^{\frac{\pi}{2}} \left(\cos x - \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) \right) dx \checkmark$$
$$= \left[\sin x - \frac{1}{2}x - \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}} \checkmark$$
$$= \sin \frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{4} \sin \pi - (0 - 0 - 0) \checkmark$$
$$= 1 - \frac{\pi}{4} \checkmark$$

(4)(a) When $n=1$, $LHS = 1 \times 2^2$
 $= 4$

$RHS = \frac{1}{12} \times 1 \times 2 \times 3 \times 8$
 $= 4$ ✓

So the result is true for $n=1$.

Assume that the result is true for $n=k$, where k is a positive integer.

i.e. assume that $1 \times 2^2 + 2 \times 3^2 + \dots + k(k+1)^2 = \frac{1}{12} k(k+1)(k+2)(3k+5)$.

Prove that the result is true for $n=k+1$.

i.e. prove that

$1 \times 2^2 + 2 \times 3^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2 = \frac{1}{12} (k+1)(k+2)(k+3)(3k+8)$

$LHS = 1 \times 2^2 + 2 \times 3^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2$

$= \frac{1}{12} k(k+1)(k+2)(3k+5) + (k+1)(k+2)^2$ ✓
 (using the assumption)

$= \frac{1}{12} (k+1)(k+2)(k(3k+5) + 12(k+2))$ }

$= \frac{1}{12} (k+1)(k+2)(3k^2 + 17k + 24)$ ✓

$= \frac{1}{12} (k+1)(k+2)(k+3)(3k+8)$

$= RHS$

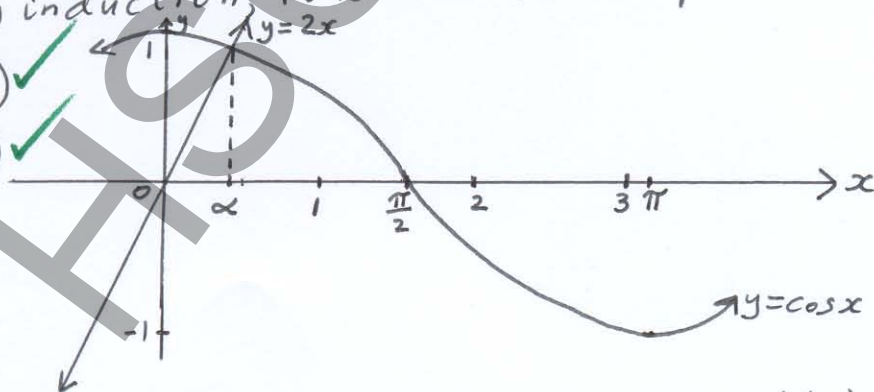
So the result is true for $n=k+1$ if it is true for $n=k$.

But the result is true for $n=1$.

So, by induction, it is true for all positive integer values of n .

(b)(i) ✓

(ii) ✓



(iii) Let $f(x) = 2x - \cos x$, so that $f'(x) = 2 + \sin x$. ✓

$x_2 = 0.5 - \frac{1 - \cos 0.5}{2 + \sin 0.5}$ ✓

$= 0.4506 \dots$ ✓

≈ 0.45

$$(4)(c) \text{ RHS of identity} = (1+x)^{100} \\ = \sum_{r=0}^{100} \binom{100}{r} x^r.$$

The coefficient of x^4 is $\binom{100}{4}$.

LHS of identity

$$= \left(\binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4 \right) \\ \cdot \left(\binom{96}{0} + \binom{96}{1}x + \binom{96}{2}x^2 + \binom{96}{3}x^3 + \binom{96}{4}x^4 + \dots + \binom{96}{96}x^{96} \right)$$

The coefficient of x^4 is

$$\binom{4}{0}\binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} + \binom{4}{4}\binom{96}{0} \\ = \binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} + 1, \\ \text{since } \binom{4}{0} = \binom{4}{4} = \binom{96}{0} = 1.$$

The coefficients of x^4 on both sides of the identity are equal, so

$$\binom{96}{4} + \binom{4}{1}\binom{96}{3} + \binom{4}{2}\binom{96}{2} + \binom{4}{3}\binom{96}{1} = \binom{100}{4} - 1.$$

$$\begin{aligned}
 (5)(a) \text{ General term} &= {}^{5n}C_r \cdot (ax^3)^{5n-r} \cdot (bx^{-2})^r \\
 &= {}^{5n}C_r \cdot a^{5n-r} \cdot b^r \cdot x^{15n-3r} \cdot x^{-2r} \\
 &= {}^{5n}C_r \cdot a^{5n-r} \cdot b^r \cdot x^{15n-5r}
 \end{aligned}$$

We require $15n - 5r = 0$,
i.e. $r = 3n$.

So the constant term is

$${}^{5n}C_{3n} \cdot a^{2n} \cdot b^{3n}$$

$$\begin{aligned}
 (b)(i) \quad \frac{dH}{dt} &= -kAe^{-kt} \\
 &= -k(H - S)
 \end{aligned}$$

(ii) When $t = 0$, $H = 80$.

$$\therefore 80 = A + 20$$

$$\therefore A = 60$$

When $t = 5$, $H = 70$.

$$\therefore 70 = 60e^{-5k} + 20$$

$$\frac{5}{6} = e^{-5k}$$

$$k = -\frac{1}{5} \ln \frac{5}{6}$$

$$\therefore H = 60e^{\frac{1}{5}t \ln \frac{5}{6}} + 20$$

$$= 20 + 60e^{\ln\left(\frac{5}{6}\right)^{\frac{t}{5}}}$$

$$= 20 + 60\left(\frac{5}{6}\right)^{\frac{t}{5}}, \text{ as required.}$$

(iii) When $t = 60$,

$$H = 20 + 60\left(\frac{5}{6}\right)^{12}$$

$$= 26.729 \dots$$

So after one hour, the temperature of the body is 26.7°C , correct to one decimal place

$$\begin{aligned}(c) (i) \quad P(-b) &= b^2(b+c) + b^2(c-b) + c^2(-b+b) - 2b^2c \\ &= b^3 + b^2c + b^2c - b^3 - 2b^2c \\ &= 0\end{aligned}$$

$\therefore a+b$ is a factor of $P(a)$

(ii) $P(a)$ is symmetric in a, b and c , so $b+c$ and $c+a$ are also factors of $P(a)$.

$$\text{So } P(a) = (a+b)(b+c)(c+a).$$

Other methods, such as long division, are acceptable.

$$(6)(a) \ddot{x} = -4\left(x + \frac{1}{x^3}\right)$$

$$\therefore \frac{1}{2}v^2 = -4 \int (x + x^{-3}) dx \quad \checkmark$$

$$= -4\left(\frac{x^2}{2} + \frac{x^{-2}}{-2}\right) + c$$

$$= -4\left(\frac{x^2}{2} - \frac{1}{2x^2}\right) + c \quad \checkmark$$

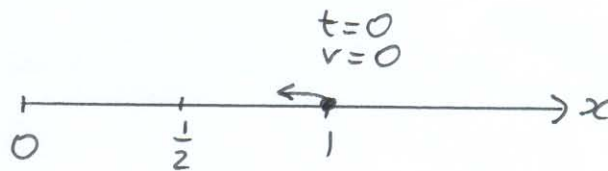
When $t=0$, $x=1$ and $v=0$.

$$\therefore 0 = -4\left(\frac{1}{2} - \frac{1}{2}\right) + c \quad \checkmark$$

$$\therefore c = 0$$

$$\therefore v^2 = -8\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)$$

$$= -4x^2 + \frac{4}{x^2}$$



When $t = \frac{1}{2}$,

$$v^2 = -4 \cdot \frac{1}{4} + \frac{4}{\frac{1}{4}}$$

$$= 15 \quad \checkmark$$

$\therefore v = -\sqrt{15}$, because the particle is travelling in the negative direction.

$$(6)(b)(i) \ y = \frac{x^2}{4a}$$

$$\therefore y' = \frac{x}{2a}$$

$$\text{When } x = 2ap,$$

$$y' = \frac{2ap}{2a}$$

$$= p.$$

So the normal at P has gradient $-\frac{1}{p}$.

So the normal at P has equation

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap$$

$$x + py = 2ap + ap^3$$

$$(ii) \text{ When } x = -ap \text{ and } y = 3a + ap^2,$$

$$\text{LHS} = x + py$$

$$= -ap + p(3a + ap^2)$$

$$= 2ap + ap^3$$

$$= \text{RHS}$$

So the normal at P passes through R.

(iii) The normal at Q has equation $x + qy = 2aq + aq^3$.
Substitute $x = -ap$ and $y = 3a + ap^2$:

$$-ap + 3aq + ap^2q = 2aq + aq^3$$

$$aq^3 - ap^2q - aq + ap = 0$$

$$aq(q^2 - p^2) - a(q - p) = 0$$

$$q(q - p)(q + p) - 1(q - p) = 0 \quad (a \neq 0)$$

$$(q - p)(q^2 + pq - 1) = 0$$

$q \neq p$ since P and Q are distinct points,
so $q^2 + pq - 1 = 0$.

(iv) Consider the equation $q^2 + pq - 1 = 0$ as a quadratic equation in q .

$\therefore \Delta = p^2 + 4 > 0$ for all real values of p .

So the equation ^{always} has two real roots.

(6)(b)(v) Consider again the quadratic equation $q^2 + pq - 1 = 0$. Let the roots be q_1 and q_2 ($q_1 \neq q_2$)

The product of the roots is -1 .

$$\therefore q_1 q_2 = -1$$

$$\therefore -\frac{1}{q_1} \cdot -\frac{1}{q_2} = -1$$

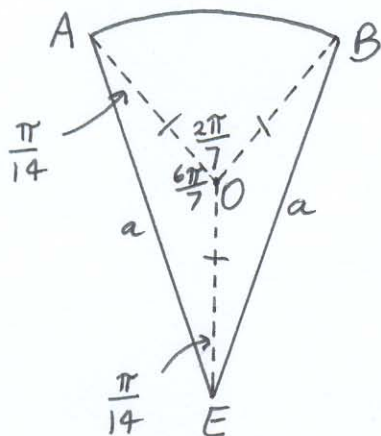
So the normals at the points $(2aq_1, aq_1^2)$ and $(2aq_2, aq_2^2)$ pass through R , and these normals (whose gradients are $-\frac{1}{q_1}$ and $-\frac{1}{q_2}$) are perpendicular.

From (ii), we also know that the normal at P passes through R .

(7)(a)

Let O be the centre of the coin.

$$\therefore OA = OB = OE$$



$$\begin{aligned} \angle AOB &= \frac{1}{7} \text{ of a revolution} \\ &= \frac{2\pi}{7} \end{aligned}$$

$$\therefore \angle AOE = \angle BOE = \frac{6\pi}{7} \text{ (angles at a point)}$$

$$\therefore \angle OAE = \angle OEA = \frac{\pi}{14} \text{ (angle sum of isosceles triangle)}$$

(i) $\angle AEB = \frac{\pi}{7}$ (with some justification) ✓

$$\begin{aligned} \text{So area of sector AEB} &= \frac{1}{2} r^2 \theta \\ &= \frac{1}{2} \cdot a^2 \cdot \frac{\pi}{7} \\ &= \frac{1}{14} \pi a^2 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \checkmark$$

(ii) In $\triangle OAE$,

$$\frac{h}{\frac{1}{2}a} = \tan \frac{\pi}{14}$$

$$\therefore h = \frac{1}{2}a \tan \frac{\pi}{14}$$

$$\text{So } \triangle OAE \text{ has area } \frac{1}{4}a^2 \tan \frac{\pi}{14} \cdot \checkmark$$

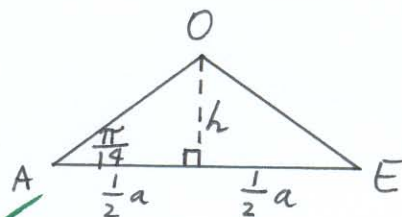
$$\begin{aligned} \text{So area of portion AOB} &= \text{area of sector AEB} \\ &\quad - 2 \times \text{area of } \triangle OAE \end{aligned}$$

$$= \frac{1}{14} \pi a^2 - \frac{1}{2} a^2 \tan \frac{\pi}{14} \cdot \checkmark$$

So area of coin is $7 \times$ area of AOB

$$= 7 \left(\frac{1}{14} \pi a^2 - \frac{1}{2} a^2 \tan \frac{\pi}{14} \right)$$

$$= \frac{1}{2} a^2 \left(\pi - 7 \tan \frac{\pi}{14} \right) \cdot \checkmark$$



(7)(b)(i) P has coordinates $(d \cos \beta, d \sin \beta)$. ✓

(ii) This point lies on the parabola, so

$$d \cos \beta = V t \cos \alpha \quad (1) \quad \text{and} \quad d \sin \beta = V t \sin \alpha - \frac{1}{2} g t^2 \quad (2)$$

From (1), $t = \frac{d \cos \beta}{V \cos \alpha}$.

Substitute into (2):

$$d \sin \beta = V \sin \alpha \cdot \frac{d \cos \beta}{V \cos \alpha} - \frac{g}{2} \cdot \frac{d^2 \cos^2 \beta}{V^2 \cos^2 \alpha}$$

Dividing by d ($d \neq 0$, since $d = 0$ corresponds to the particle being at the origin),

$$\sin \beta = \tan \alpha \cos \beta - d \cdot \frac{g \cos^2 \beta}{2 V^2 \cos^2 \alpha}$$

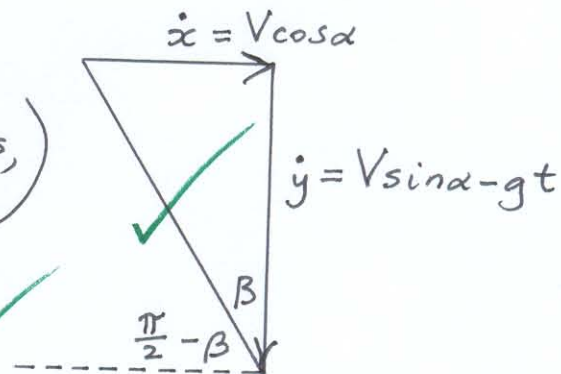
$$\therefore d = \frac{2 V^2 \cos^2 \alpha}{g \cos^2 \beta} (\tan \alpha \cos \beta - \sin \beta)$$

(iii) $\cot \beta = \frac{-\dot{y}}{\dot{x}}$ ($\dot{y}$ is negative because the particle is moving downwards, $\dot{x}$ is positive.)

$$= \frac{gt - V \sin \alpha}{V \cos \alpha}$$

$$= \frac{g}{V \cos \alpha} \cdot \frac{d \cos \beta}{V \cos \alpha} - \frac{V \sin \alpha}{V \cos \alpha}$$

$$= \frac{g d \cos \beta}{V^2 \cos^2 \alpha} - \tan \alpha$$



(iv) From (iii),

$$\tan \alpha = \frac{g d \cos \beta}{V^2 \cos^2 \alpha} - \cot \beta$$

Using (ii),

$$\tan \alpha = \frac{g \cos \beta}{V^2 \cos^2 \alpha} \cdot \frac{2 V^2 \cos^2 \alpha}{g \cos^2 \beta} (\tan \alpha \cos \beta - \sin \beta) - \cot \beta$$

$$= \frac{2}{\cos \beta} (\tan \alpha \cos \beta - \sin \beta) - \cot \beta$$

$$= 2 \tan \alpha - 2 \tan \beta - \cot \beta$$

$$\therefore \tan \alpha = \cot \beta + 2 \tan \beta$$