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IN OMNIBUS GLORIFICETUR DEUS

**2008**
**TRIAL HIGHER SCHOOL CERTIFICATE**
EXAMINATION

# Mathematics Extension 1



**General Instructions**

- Reading Time- 5 minutes
- Working Time – 2 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Use a separate answer booklet for each question

Total marks (**84**)

- Attempt Questions 1-7
- All questions are of equal value

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**Question 1 (12 marks)**

**Marks**

(a) Evaluate $\int_0^{2\sqrt{3}} \frac{3}{4+x^2}\,dx$ **2**

(b) Evaluate $\lim_{x \to 0} \frac{\sin(2x)}{3x}$ **2**

(c) The point $P(x, y)$ divides the interval joining $A(5,3)$ to $B(-1,0)$ externally in the ratio $2:5$. **2**

Find the coordinates of the point $P$

(d) Solve $\frac{5}{2x-1} \leq 1$ **3**

(e) Use the substitition $u = 2 - x$ to evaluate $\int_0^1 \frac{x}{2-x}\,dx$ **3**

**<u>Question 2</u> (12 marks) – Use a SEPARATE writing booklet**

**Marks**

(a) State the domain and range of $y = 2\cos^{-1}(\frac{x}{3})$ **2**

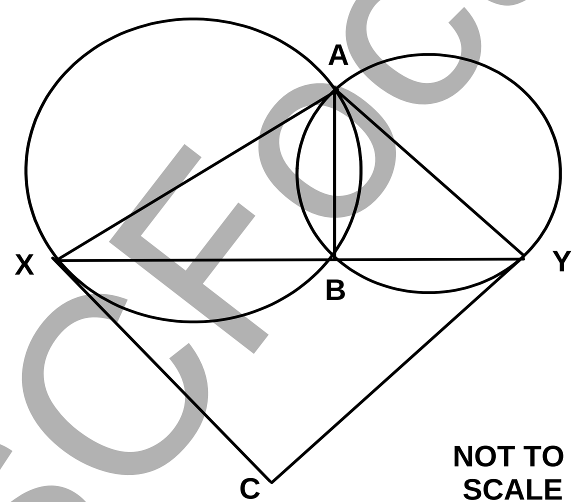
(b) Find the size of the acute angle between the lines whose equations are **2**
$y = 2x - 3$ and $y = 4 - 3x$

(c) Find the coefficient of the term in $x^3$ in the expansion of $(\frac{1}{x^2} - x)^9$ **2**

(d) (i) Find the $\frac{d}{dx}(x^2 \ln x)$ **1**

(ii) Hence (or otherwise) find $\int x \ln x \, dx$ **2**

(e) In the diagram below, $AB$ is a common chord of the two circles.
A straight line through $B$ intersects the circles at $X$ and $Y$ as shown.
The tangents to the circles at $X$ and $Y$ intersect at $C$.
Copy or trace the diagram onto your own paper.
Let $\angle BYC = \alpha$ and $\angle CXY = \beta$




(i) Explain why $\angle BYC = \angle BAY$ **1**
(ii) Hence, prove that $AXCY$ is a cyclic quadrilateral **2**

**<u>Question 3</u> (12 marks) Use a SEPARATE writing booklet**

**Marks**

(a) Find $\int \cos^2(\frac{x}{2})dx$ **2**

(b) Find the derivative of $\sin^4 3x$ **2**

(c) The function $f(x) = x - \cos x$ has a zero near $x = 0.7$

Taking $x = 0.7$ as a first approximation, use one application of Newton's method to find a second approximation to the zero. **2**

Give your answer correct to three decimal places.

(d) Let $P(x) = x^3 + 3x^2 + Ax + B$.

$(x+2)$ is a factor of $P(x)$. **3**

When $P(x)$ is divided by $(x-1)$, the remainder is 9.

Find the values of A and B.

(e) Use mathematical induction to prove that $9^n - 4^n$ is divisible by 5 **3**

for all integers $n \geq 1$

**<u>Question 4</u> (12 marks) Use a SEPARATE writing booklet**

**Marks**

(a) (i) Express $\sin\theta + \sqrt{3}\cos\theta$ in the form $\boldsymbol{R}\sin(\theta + \alpha)$ where R > 0 and $0 \le \alpha \le \frac{\pi}{2}$ **2**

(ii) Hence, or otherwise, solve $\sin\theta + \sqrt{3}\cos\theta = 1$ for $0 \le \theta \le 2\pi$ **2**

(b) Consider the function $\boldsymbol{y} = \dfrac{1}{\boldsymbol{x}^2 - 4}$.

You are given that this is an even function.

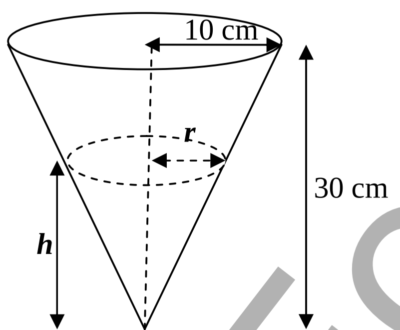
(i) Make a neat sketch of the graph of y = $\dfrac{1}{\boldsymbol{x}^2 - 4}$, clearly showing any asymptotes and points of intersection with the coordinate axes. **3**

(ii) Hence , or otherwise, determine the values of $\boldsymbol{k}$ so that the equation

$$\frac{1}{\boldsymbol{x}^2 - 4} = \boldsymbol{k}$$ **1**

has solutions that are real and different.

(c)




NOT TO SCALE

The sketch shows a conical container whose height is 30 cm and radius 10 cm.
The container is initially full of water.
The water now drains out of the container at a constant rate of $2\pi$ cm$^3$ per second.
After $\boldsymbol{t}$ seconds, the height of the water in the container is $\boldsymbol{h}$ cm and the radius of the surface of the water is $\boldsymbol{r}$ cm, as shown.

(i) Using similar triangles, show that r = $\dfrac{\boldsymbol{h}}{3}$

(Note: you do NOT need to prove the triangles are similar) **1**

(ii) Find the rate at which the height of water in the container is decreasing when the height of the water is 3 cm. **3**

(Volume of a cone = $\frac{1}{3}\pi \boldsymbol{r}^2 \boldsymbol{h}$)

**Question 5** (12 marks) Use a SEPARATE writing booklet

**Marks**

(a) Let $\alpha$, $\beta$ and $\gamma$ be the roots of $x^3 + 4x^2 + kx - 36 = 0$

(i) Find the value of $\alpha+\beta+\gamma$ and $\alpha\beta\gamma$ **2**

(ii) Given that two of the roots are equal in magnitude but opposite in sign, find the third root and hence find the value of $k$ **2**

(b) A cup full of hot water is cooling in a room where the temperature is a constant 20°C.

At time $t$ minutes, its temperature is decreasing according to the equation

$$\frac{dT}{dt} = -k(T-20),$$ where $k$ is a positive constant.

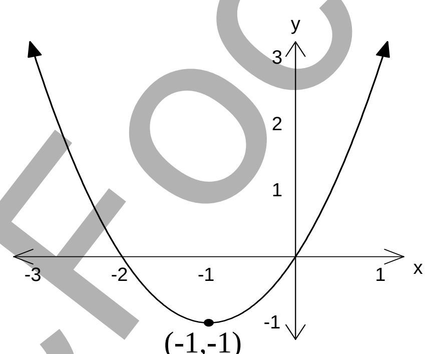
(i) Show that $T = 20 + Ae^{-kt}$ satisfies the above equation. **1**

(ii) The initial temperature of the cup of water is 80°C. **3**

After 10 minutes its temperature has decreased to 50°C.

Find the temperature of the cup of water after 20 minutes. Give your answer correct to the nearest degree.

(c)




NOT TO SCALE

The sketch shows the graph of $f(x) = x^2 + 2x$

(i) Explain why $f(x)$ does not have an inverse function. **1**

(ii) Let $g(x) = x^2 + 2x$ where $x \geq -1$ **1**

State the domain of $g^{-1}(x)$, the inverse function of $g(x)$

(iii) Find an expression for $y = g^{-1}(x)$ in terms of $x$ **2**

**Question 6 (12 marks) Use a SEPARATE writing booklet**

**Marks**

(a) A body is moving with simple harmonic motion along the $x$-axis.

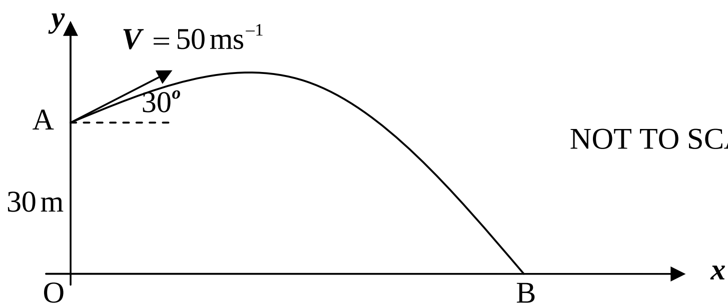
Its velocity, $v$ ms$^{-1}$, is given by $v^2 = 8 - 2x - x^2$, where $x$ is in metres.

(i) Find the endpoints of the motion **2**

(ii) Find the maximum speed of the body. **1**

(iii) Find an expression for the acceleration of the body in terms of $x$. **2**

(b)




NOT TO SCALE

A body is projected from a point 30 metres above level ground with velocity 50 ms$^{-1}$ and inclined at an angle of 30$^{o}$ to the horizontal as shown in the sketch.
The body lands on the ground at the point B.
The equations of motion of the body are $\ddot{y} = -10$ and $\ddot{x} = 0$

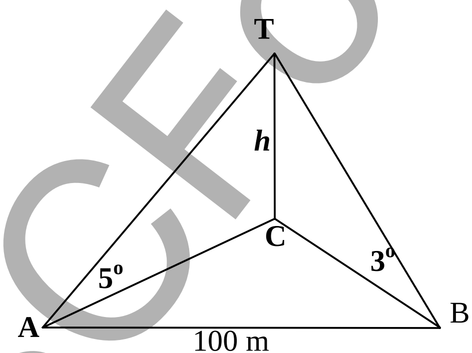
Hence the vertical and horizontal displacements of the body are given by

$$y = 30 + 25t - 5t^2 \quad \text{and} \quad x = 25\sqrt{3}\,t$$

(i) Find the maximum height above ground level that is reached by the body. **2**

(ii) Find the range of the projectile (ie the distance from O to B) **2**

(c)




NOT TO SCALE

In the diagram, TC represents a vertical tower that is due North of the point A.
The point B is 100 m due East of A.
From A the angle of elevation of the top of the tower is 5$^{o}$, and from B the angle of elevation of the top of the tower is 3$^{o}$.
Let the height of the tower be $h$ metres.

(i) Show that $AC = \dfrac{h}{\tan 5^o}$ **1**

(ii) Using a similar expression for $BC$, find the height of the tower, correct to the nearest metre. **2**

**<u>Question 7</u> (12 marks) – Use a SEPARATE writing booklet**

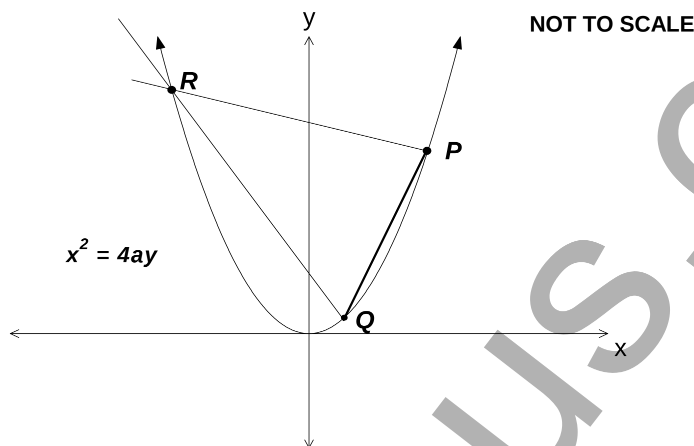
**Marks**

(a) (i) Show that in the binomial expansion of $\left(x-\frac{1}{x}\right)^{2n}$, the term that is independent of $x$ is $(-1)^n\ {}^{2n}C_n$ **1**

(ii) Show that $(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n}=\left(x-\frac{1}{x}\right)^{2n}$ **1**

(iii) Deduce that $\left({}^{2n}C_0\right)^2-\left({}^{2n}C_1\right)^2+\left({}^{2n}C_2\right)^2 \ldots\ldots\ldots\ldots + \left({}^{2n}C_{2n}\right)^2=(-1)^n\ {}^{2n}C_n$ **2**

(b)




$P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $R(2ar, ar^2)$ are points on the parabola $x^2=4ay$.
The normals at the points $P$ and $Q$ intersect at $R$.
$PQ$ is a chord of the parabola.

(i) Show that the equation of the normal at $P$ is $x+py=2ap+ap^3$ **2**

(ii) Hence show that $r=-p-\frac{2}{p}$ **2**

(iii) Show that $pq=2$ **2**

(iv) Hence find the equation of the locus of the midpoint of the chord $PQ$ **2**

**END OF EXAMINATION**

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# STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$