Catholic Schools Trial Examinations

2005 Mathematics Extension 1

05 CT 1a Find the value of $\lim_{x \to 0} \frac{\sin 2x}{5x}$. **2**

05 CT 1b The polynomial P(x) is given by $P(x) = x^3 + ax + b$ for some real numbers a and b. 2 is a zero of P(x). When P(x) is divided by (x + 1) the remainder is -15.

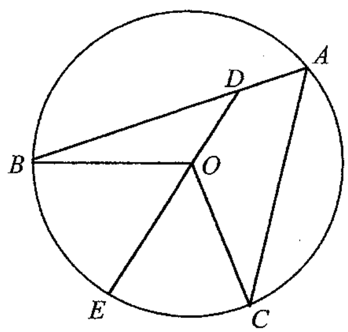
(i) Write down two equations in a and b. **2**

(ii) Hence find the values of a and b. **1**

05 CT 1c (i) Find the exact values of the gradients of the tangent to the curve $y = e^x$ at the points where $x = 0$ and $x = 1$. **1**

(ii) Find the acute angle between these tangent correct to the nearest degree. **2**

05 CT 1d



In the diagram, A, B and C are points on a circle with centre O. D is a point on AB such that ADOC is a cyclic quadrilateral DO produced meets the circle again at E.

(i) Copy the diagram.

(ii) Give a reason why $\angle$CAD = $\angle$COE. **1**

(iii) Show that DOE bisects $\angle$COB. **3**

05 CT 2a Evaluate $\log_2 7$ correct to two decimal places. **2**

05 CT 2b (i) Show that $\frac{1}{1-\tan x} + \frac{1}{1+\tan x} = \tan 2x$ **2**

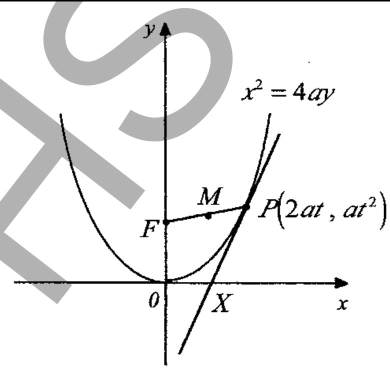
(ii) Evaluate $\frac{1}{1-\tan\frac{\pi}{6}} + \frac{1}{1+\tan\frac{\pi}{6}}$ **1**

05 CT 2c A(x, 10 and B(x^2, 6) are two fixed points for some real number x. The point P(5, 4) divides the interval AB externally in the ratio 3:1.

(i) Show that $3x^2 - x = 10$ **1**

(ii) Find any values of x. **2**

05 CT 2d



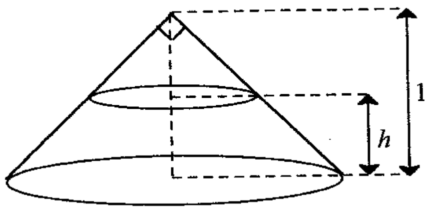
P($2at$, at^2) is a point on the parabola $x^2 = 4ay$ with focus F. The tangent to the parabola at P cuts the x axis at X. M is the midpoint of PF.

(i) Show that the tangent to the parabola at P has equation $tx - y - at^2 = 0$. **2**

(ii) Show that MX is parallel to the y axis. **2**

05 CT	**3a**	Consider the function $f(x) = 1 + \ln x$.		
		(i)	Show that the function $f(x)$ is increasing and the curve $y = f(x)$ is concave down for all values of x in the domain of the function.	**2**
		(ii)	Find the equation of the tangent to the curve $y = f(x)$ at the point on the curve where $x = 1$.	**2**
		(iii)	Find the equation of the inverse function $f^{-1}(x)$.	**1**
		(iv)	On the same diagram sketch the graph of the curves $y = f(x)$ and $y = f^{-1}(x)$. Show clearly the coordinates of any points of intersection of the two curves and any intercepts made on the coordinate axes.	**3**
05 CT	**3b**	(i)	Show that $\frac{d}{dx}(x\sqrt{1-x^2} + \sin^{-1}x) = 2\sqrt{1-x^2}$.	**2**
		(ii)	Evaluate $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\, dx$, giving the answer in simplest exact form.	**2**
05 CT	**4a**	The equation $x^3 - 3x - 3 = 0$ has exactly one real root a.		
		(i)	Show that $2 < \alpha < 3$.	**2**
		(ii)	Starting with an initial approximation $\alpha = 2$, use one application of Newton's method and find a further approximation of α correct to one decimal place.	**2**
05 CT	**4b**	Use the substitution $u = \sin^2 x$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin 2x}{1-\sin^2 x}\, dx$, giving the answer in simplest form.		**4**
05 CT	**4c**	A particle is moving in a horizontal straight line. At time t seconds, the displacement of the particle from a fixed point O on the line is x metres, its velocity is v ms^{-1}, and its acceleration a ms^{-2} is given by $a = 8x - 2x^3$. When the particle is 2m to the right of O, it is observed to be traveling right with a speed of $6ms^{-1}$.		
		(i)	Show that $v^2 = 20 + 8x^2 - x^4$.	**2**
		(ii)	Find the set of possible values of x.	**2**
05 CT	**5a**	A bag contains nine balls labelled 1, 2, 3, .., 9 but otherwise identical. Three balls are chosen at random from the bag. Find the probability that exactly two even numbered balls are chosen		
		(i)	if the balls are selected without replacement.	**2**
		(ii)	if each ball is replaced before the next is selected.	**2**

05 CT **5b**



A closed, right, hollow cone has a height of 1 metre and the semi-vertical angle 45°. The cone stands with its base on a horizontal surface. Water is poured into the cone through a hole in its apex at a constant rate of 0.1 m^3 per minute.

(i) Show that when the depth of water in the cone is h metres ($0 < h < 1$) the volume of water V m^3 in the cone is given by $V = \frac{\pi}{3}(h^3 - 3h^2 + 3h)$. **2**

(ii) Find the rate at which the depth of water in the cone is increasing when $h = 0.5$. **2**

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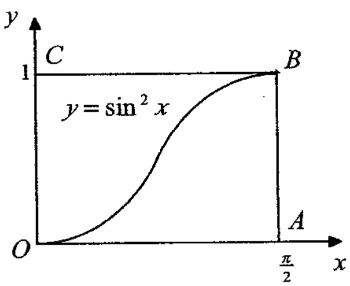
05 CT **5c**



The rectangle OABC has vertices O(0, 0), A($\frac{\pi}{2}$, 0), B($\frac{\pi}{2}$, 1) and C(0, 1). **4**

The curve $y = \sin^2 x$ is shown passing through the points O and B. Show that this curve divides the rectangle OABC into regions of equal area.

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05 CT **6a** A particle is performing Simple Harmonic Motion about a fixed point O on a straight line. At time t seconds it has displacement x metres from O given by $x = \cos 2t - \sin 2t$.

(i) Express x in the form $R\cos(2t + \alpha)$ for some $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Find the amplitude and the period of the motion. **2**

(iii) Determine whether the particle is initially moving toward O or away from O and whether it is initially speeding up or slowing down. **2**

(iv) Find the time at which the particle first returns to its starting point. **2**

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05 CT **6b** Use Mathematical Induction to show that for all positive integers $n \geq 1$. **4**

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}.$$

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05 CT **7a** A particle is projected from a point O with velocity V ms^{-1} at an angle θ above the horizontal. At time t seconds it has horizontal and vertical displacements x metres and y metres respective from O. The acceleration due to gravity is g ms^{-2}.

(i) Write down expressions for x and y in terms of V, θ and t. **2**

(ii) Show that $y = x\tan\theta - \frac{gx^2}{2V^2}(1 + \tan 2\theta)$ **2**

05 CT **7b** A particle is projected from O with velocity 60ms^{-1} at an angle α above the horizontal T seconds later, another particle is projected from O with velocity 60ms^{-1} at an angle β above the horizontal, where $\beta < \alpha$. The two particles collide 240 metres horizontally from O and at a height of 80 metres above O.
Taking $g = 10\ \text{ms}^{-2}$ and using results from question **7a**,

(i) Show that $\tan\alpha = 2$ and $\tan\beta = 1$ **2**

(ii) Find the value of T in simplest exact form. **2**

05 CT **7c** The real number x is a solution of the equation $x^2 - x - 1 = 0$. **4**
Use the Binomial Theorem to show that the sum S of the series $1 + x + x^2 + \ldots + x^{2n-1}$ ($n = 1, 2, 3, \ldots$) is given by $S = \sum_{r=1}^{n} {}^nC_r\, x^{r+1}$.

A **1a.** $\frac{2}{5}$ **1b.(i)** $8 + 2a + b = 0$ and $-1 - a + b = -15$ **(ii)** $a = 2$, $b = -12$ **1c.(i)** $x = 0$, grad is 1; $x = 1$ grad is e **(ii)** 25° **1d.(iii)** ext $\angle$ equals opp int $\angle$ in cyc. quad. **2a.** 2.81 **2b.(ii)** $\sqrt{3}$ **2c.(ii)** $x = -\frac{5}{3}, 2$ **3a.(ii)** $y = x$ **(iii)** $f^{-1}(x) = e^{x-1}$ **3b.(ii)** $\frac{\sqrt{3}}{8} + \frac{\pi}{12}$ **4a.(ii)** 2.1 **4b.** ln2 **4c.(ii)** $-\sqrt{10} \le x \le \sqrt{10}$ **5a.(i)** $\frac{3\times5\times4\times3}{9\times8\times7} = \frac{5}{14}$ **(ii)** $\frac{3\times5\times4^2}{9^3} = \frac{80}{243}$ **6a.(i)** $\sqrt{2}\cos(2t + \frac{\pi}{4})$ **(ii)** $a = \sqrt{2}$ m and per = π sec **(iii)** initially 1m to right of O moving towards O and speeding up. **7a.(i)** $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ **7b.(ii)** $T = 4(\sqrt{5} - \sqrt{2})$