JRAHS 2007 Extension 1 Trial

EXTENSION 1 TRIAL 2007 JRAHS

1 a) $y = x\tan^{-1}x$

$\frac{dy}{dx} = \tan^{-1}x + \frac{x}{1+x^2}$ (Product Rule) 2

b) $f(x) = \sin^{-1}(1-2x)$

Let $u = 1-2x$, $\frac{du}{dx} = -2$,

$\therefore f'(x) = \frac{1}{\sqrt{1-(1-2x)^2}} \times -2$ 2

$= \frac{-2}{\sqrt{1-(1+4x^2-4x)}}$

$= \frac{-2}{\sqrt{4x-4x^2}} = \frac{-1}{\sqrt{x-x^2}}$ 1

c) $P(x) = Ax(x+4)(x-4)$ 2

1 1

$P(3) = 3A\times 7\times -1 = 21 \quad \therefore A = 1$

$\therefore P(x) = -x(x+4)(x-4)$ 1

$(= 16x - x^3)$

d) Let $I = \int_4^5 \frac{x(x-4)}{(x-2)}\,dx$

Let $u = x-2$ $\quad x=5 \Rightarrow u=3$, $x=4 \Rightarrow u=2$

$\frac{du}{dx} = 1 \quad \therefore$ "$dx = du$" 1

$\therefore I = \int_2^3 \frac{(u+2)(u-2)}{u}\,du$

$= \int_2^3 \frac{u^2-4}{u}\,du$

$= \int_2^3 u - \frac{4}{u}\,du$ 1

$= \left[\frac{u^2}{2} - 4\ln u\right]_2^3$ 1

$= \left(\frac{9}{2} - 4\ln 3\right) - \left(2 - 4\ln 2\right)$

$= \frac{5}{2} - 4\ln\left(\frac{3}{2}\right)$ 1

2 a) ABCD is a cyclic quadrilateral

$\angle CAD = 90°$ (Angle at circumference in semi circle) 1

$\angle BAC = x°$ (Alternate angles are equal $AB \parallel CD$) 1

$\therefore \angle BAD = x° + 90°$

$\therefore \angle BCD = 180° - (x°+90°)$ (Opposite angles in cyclic quad. supp^y)

$= 90° - x°$

$\therefore \angle BCA = \angle BCD - \angle ACD$

$= 90° - 2x°$ 1

b) When $n=2$,

$T_2 = 9^2 - 16 - 1$

$= 64$ 1

which is divisible by 64

So induction starts.

Assume that

$9^k - 8k - 1 = 64A$ for $k \geq 2$ 1

and $A \in \mathbb{Z}$.

Then

$9^{k+1} - 8(k+1) - 1 = 9.9^k - 8k - 9$

$= 9(9^k - 1) - 8k$

$= 9(9^k - 8k - 1) + 64k$

$= 9 \times 64A + 64k$

$= 64(9A+k)$ 2

Thus if true for $n=k$, also true for $n=k+1$.

$\therefore$ By principle of M.I.,

$9^n - 8n - 1$ is divisible by 64. 1

for $n \geq 2$

c) i) $\frac{r\,{}^nC_r}{{}^nC_{r-1}} = \frac{r\,n!}{(n-r)!\,r!} \cdot \frac{(r-1)!\,(n-(r-1))!}{n!}$

$= \frac{r(r-1)!}{r!} \cdot \frac{(n-r+1)!}{(n-r)!}$

$= n-r+1$ 1

2 c ii) Using result from (i)

$n + (n-1) + (n-2) + \ldots + 2 + 1$ 1

This an AP with $a = n, d = -1$

Sum $= \frac{n}{2}\left(2n + (n-1)(-1)\right)$

$= \frac{n(n+1)}{2}$ 2

3 a) Let $I = \int_0^{1/\sqrt{2}} \frac{x\,dx}{\sqrt{1-x^4}}$

Let $u = x^2$ $\quad x = 1/\sqrt{2} \Rightarrow u = 1/2$

"$du = 2x\,dx$" $\quad x = 0 \Rightarrow u = 0$ 1

$\therefore I = \frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$ 1

$= \left[\frac{1}{2}\sin^{-1} u\right]_0^{1/2}$ 1

$= \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}$ 1

b) i) $y = px$ 1

ii) Crosses parabola where

$x^2 = 4apx$

$x = 4ap \quad (x \neq 0)$ 1

$\therefore y = 4ap^2$

Q is $(4ap, 4ap^2)$ 1

iii) Q is point with parameter "$2p$"

$\therefore$ Tangent is $y = (2p)x - a(2p)^2$

$y = 2px - 4ap^2$ 1

iv) Two tangents cross at R

$px - ap^2 = 2px - 4ap^2$

$\Rightarrow x = 3ap \quad (p \neq 0)$ 1

$\therefore y = 2ap^2$

R is $(3ap, 2ap^2)$ 1

v) Eliminate p from $\left.\begin{array}{l} x = 3ap \\ y = 2ap^2 \end{array}\right\}$ 1

$p = x/3a$, $\quad y = 2a\left(\frac{x}{3a}\right)^2$

$x^2 = \frac{9ay}{2}$ 1

4 i) $f(x) = 1 - \frac{1}{1+e^x} = 1 - (1+e^x)^{-1}$

$f'(x) = \frac{e^x}{(1+e^x)^2}$ 1

$e^x > 0 \quad (1+e^x)^2 > 0 \quad \therefore f'(x) > 0$

$\therefore f(x)$ increasing for all x 1

ii) Range $\{y : 0 < y < 1\}$ 1

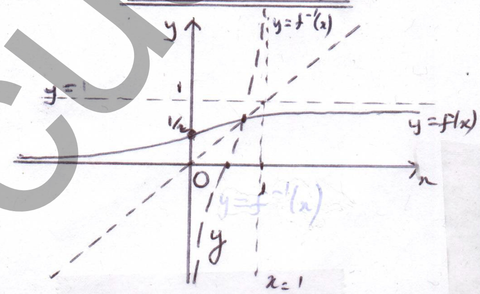
iii) Let $y = 1 - \frac{1}{1+e^x}$

$\frac{1}{1+e^x} = 1 - y$

$1 + e^x = \frac{1}{1-y}$

$e^x = \frac{1}{1-y} - 1 = \frac{y}{1-y}$

$f^{-1}(x) = \ln\left(\frac{x}{1-x}\right)$ 2



2

b) i) $\frac{d}{dx}\left(\frac{v^2}{2}\right) = 4x - 4$

$\therefore \frac{v^2}{2} = 2x^2 - 4x + k$ 1

When $x = 6$, $v^2 = 64$

$\therefore 32 = 72 - 24 + k \Rightarrow k = -16$

$\therefore v^2 = 4x^2 - 8x - 32$ 1

ii) $v^2 = 4(x^2 - 2x - 8)$

$= 4(x-4)(x+2)$

$v^2 \geq 0 \quad \therefore -2 \leq x \leq 4$ 1

iii) Particle moving to left from $x=6$

Stops at $x=4$ and immediately

... to right, accelerating 2

⑤ a) i) $R\sin(3t+\alpha) = R\sin 3t\cos\alpha + R\cos 3t\sin\alpha$

Equating coefficients:

$$\left.\begin{aligned} R\sin\alpha &= 2 \\ R\cos\alpha &= 1 \end{aligned}\right\} \qquad \begin{aligned} R &= \sqrt{5} \\ \tan\alpha &= 2 \end{aligned}$$

$$\therefore x = \underline{\sqrt{5}\sin(3t + \tan^{-1}(2))}$$

ii) $\dot{x} = 3\sqrt{5}\cos(3t + \tan^{-1}(2))$

$\ddot{x} = -9\sqrt{5}\sin(3t + \tan^{-1}(2))$

$\ddot{x} = -9x$

which is of the form $\ddot{x} = -n^2x$ which defines S.H.M.

iii) Period $\left(= \frac{2\pi}{n}\right) = \underline{\frac{2\pi}{3} \text{ secs.}}$

iv) When $x = 1$,

$\sin(3t + \tan^{-1}(2)) = 1/\sqrt{5}$

$3t + \tan^{-1}(2) = n\pi + (-1)^n\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

$n=1$ gives 1st +ve soln:

$3t = \pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \tan^{-1}(2)$

$t = \frac{1}{3}\left(\pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \tan^{-1}(2)\right)$

$= \underline{0.5 \text{ secs}}$ (to 1D)

b) i) After 1 withdrawal

$\$(20000 \times 1.005 - 50) = \underline{\$20050}$

ii) Let $P = \$20000$, $D = \$50$, $r = 0.005$

After 1 month $\quad P(1+r) - D$

.. 2 .. $\quad P(1+r)^2 - D(1+r) - D$

.. n .. $\quad P(1+r)^n - D(1+r)^{n-1} - D(1+r)^{n-2} - \dots - D$

$= P(1+r)^n - \frac{D((1+r)^n - 1)}{r}$

$= 20000(1.005)^n - 10000(1.005)^n + 10000$

$= \underline{10000(1.005)^n + 10000}$

iii) Solve

$10000 \times (1.005)^n \geq 40000$

$(1.005)^n \geq 4$

$n\log(1.005) \geq \log 4$

$n \geq \frac{\log 4}{\log 1.005}$

$\therefore \underline{n = 278} \text{ (months)}$

⑥ a) r^{th} term is ${}^9C_r\left(\frac{2x^2}{3}\right)^r\left(\frac{-3}{2x}\right)^{9-r}$

$= {}^9C_r\left(\frac{2}{3}\right)^r\left(\frac{-3}{2}\right)^{9-r}x^{3r-9}$

Required term has $3r - 9 = 0$
$r = 3$

$\therefore$ Coefficient is ${}^9C_3(-1)^6\left(\frac{3}{2}\right)^3$

$= \underline{{}^9C_3\left(\frac{3}{2}\right)^3\left(= \frac{567}{2} = 283\tfrac{1}{2}\right)}$

b) i) Intercept when x and y equal simultaneously

$240t\cos\theta = 3600$

$t\cos\theta = 15$

$t = 15\sec\theta$

$2000 + 240t\sin\theta - \frac{gt^2}{2} = 3200 - \frac{gt^2}{2}$

$240t\sin\theta = 1200$

$t\sin\theta = 5$

Solving $\tan\theta = 1/3$

$\underline{\theta = \tan^{-1}(1/3)\ (= 18^\circ 26')}$

At time $t = 15\sec\theta$

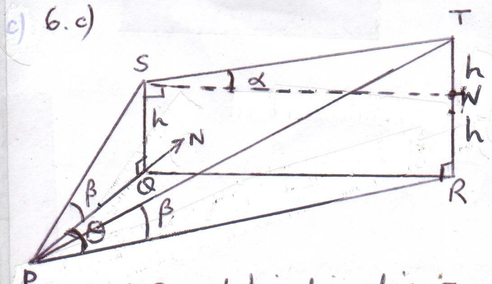
$= \underline{5\sqrt{10} \text{ secs.} (= 15.81 \text{ s})}$

ii) Putting $t = 5\sqrt{10}$ and $g = 10$ into last equation

height $= 3200 - \frac{10 \cdot 250}{2}$

$= 1950 \text{ m}$

6.c)



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WPQRST defined in diagram.

$\triangle PSQ \; ||| \; \triangle PTR$ (Equiangular)

$\therefore \dfrac{PR}{PQ} = \dfrac{RT}{QS} = 2$

But $PQ = h\cot\beta$ 1

$\therefore PR = 2h\cot\beta.$

$QR = SW = h\cot\alpha$ 1

Apply cosine rule to $\triangle PQR$ 1

$$\cos\theta = \frac{PQ^2 + PR^2 - QR^2}{2.PQ.PR}$$

$$= \frac{h^2\cot^2\beta + 4h^2\cot^2\beta - h^2\cot^2\alpha}{2.h\cot\beta.2h\cot\beta}$$ 1

$$\cos\theta = \frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}$$

7 a) i) Ways of choosing team of 4 $= {}^{13}C_4 \; (=715)$

Ways of choosing 3 boys and 1 girl $= {}^8C_3\,{}^5C_1 = 280$

$\therefore$ Probability $= \dfrac{{}^8C_3\,{}^5C_1}{{}^{13}C_4} \; (\doteqdot 0.39)$ 1 1

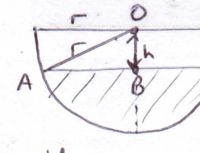
ii) Ways of choosing 4 girls $= {}^8C_4 = 70$ 1

Prob all girls ${}^8C_4/{}^{13}C_4$

Prob 3 or 4 girls

$= \dfrac{{}^8C_4 + {}^8C_3\,{}^5C_1}{{}^{13}C_4} \; (\doteqdot 0.49)$ 1

b)



i) Consider cross section of trough.

$OA = r \quad \therefore AB = \sqrt{r^2 - h^2}$ (Pythag)

$\therefore$ Surface area $= l \times 2AB$

$= 2l\sqrt{r^2 - h^2}$ 2

ii) Let $\angle AOB = \theta \; (= \cos^{-1}(h/r))$

Area of shaded segment is

$\dfrac{r^2}{2}(2\theta - \sin 2\theta)$

$= r^2(\theta - \sin\theta\cos\theta)$

$= r^2\left(\cos^{-1}(h/r) - \dfrac{\sqrt{r^2-h^2}}{r}\cdot\dfrac{h}{r}\right)$

$\therefore Vol = l\left(r^2\cos^{-1}(h/r) - h\sqrt{r^2-h^2}\right)$ 2

iii) $\dfrac{dV}{dt} = l\left(\dfrac{-r^2/r}{\sqrt{1-h^2/r^2}} - \sqrt{r^2-h^2} + \dfrac{h^2}{\sqrt{r^2-h^2}}\right)\dfrac{dh}{dt}$ 2

$= \left(\dfrac{-r^2-(r^2-h^2)+h^2}{\sqrt{r^2-h^2}}\right) l \dfrac{dh}{dt}$

$= \dfrac{-2l(r^2-h^2)}{\sqrt{r^2-h^2}}\dfrac{dh}{dt}$

$= -2l\sqrt{r^2-h^2}\,\dfrac{dh}{dt} = -A\dfrac{dh}{dt}$ 1

iv) $-\dfrac{dV}{dt} \propto A \quad \therefore -\dfrac{dV}{dt} = kA$

where k is constant of proportionality

$\therefore kA = A\dfrac{dh}{dt} \Rightarrow \dfrac{dh}{dt} = k$ 1

$\therefore h$ increasing at constant rate

(ie. water falling at constant rate)