CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1999

MATHEMATICS

3/4 UNIT

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER:
4 UNIT COURSE – FIRST PAPER)

Afternoon session

Friday 13 August 1999

Time allowed – two hours

EXAMINERS
Graham Arnold, Terra Sancta College, Nirimba
Sandra Hayes, Aquinas College, Menai
Frank Reid, St Ursula's College, Kingsgrove

DIRECTIONS TO CANDIDATES:

- **ALL** questions may be attempted.
- **ALL** questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Standard integrals are printed at the end of the exam paper.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination. However, the committees responsible for the preparation of these 'Trial Examinations' do hope that they will provide a positive contribution to your preparation for the final examinations.

HSCFocus.com

HSCFocus.com

Marks

Question 1 Begin a new page

(a) Find the number of ways in which 6 books can be arranged in a row so that the shortest book and the longest book are at the ends. **1**

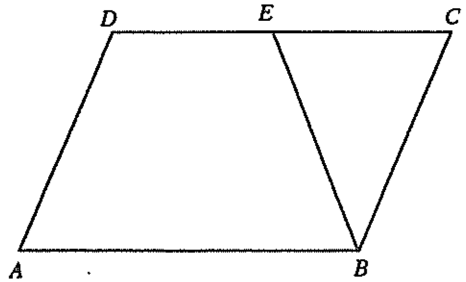
(b) The acute angle between the line $x-2y+3=0$ and the line $y=mx$ is $45°$. **3**

(i) Show that $\left|\frac{2m-1}{m+2}\right|=1$

(ii) Find the possible values of m.

(c) Solve the equation $\ln(x^3+19)=3\ln(x+1)$. **3**

(d) **5**



$ABCD$ is a parallelogram. E is the point on CD such that $BE=BC$.

(i) Copy the diagram showing the above information.

(ii) Show that $ABED$ is a cyclic quadrilateral.

Marks

Question 2 Begin a new page

(a) Find $\lim_{x\to 0} \frac{\sin 2x}{x}$ **1**

(b) Solve the inequality $\frac{x^2+9}{x} \le 6$ **3**

(c) (i) Factorise $3x^3+3x^2-x-1$ **3**

(ii) Solve the equation $3\tan^3\theta+3\tan^2\theta-\tan\theta-1=0$ for $0\le\theta\le\pi$

(d) $P(2t, t^2)$ is a point on the parabola $x^2=4y$ with focus F. The point M divides the interval FP externally in the ratio $3:1$. **5**

(i) Show that as P moves on the parabola $x^2=4y$, then M moves on the parabola $x^2=6y+3$.

(ii) Find the coordinates of the focus and the equation of the directrix of the locus of M.

Question 3 Begin a new page

(a) Find the gradient of the tangent to the curve $y=\tan^{-1}\frac{1}{x}$ at the point on the curve where $x=1$. **2**

(b) A function is given by the rule $f(x)=\frac{x+1}{x+2}$. Find the rule for the inverse function $f^{-1}(x)$. **2**

(c) At any point on the curve $y=f(x)$ the gradient function is given by $\frac{dy}{dx}=2\cos^2 x+1$. If $y=\pi$ when $x=\pi$, find the value of y when $x=2\pi$. **4**

(d) Use the substitution $x=u^2$, $u>0$, to express the value of $\int_1^{100}\frac{1}{x+2\sqrt{x}}\,dx$ in the form $\ln a$ for some constant $a>0$. **4**

HSCFocus.com

UESTION 1

) $2!$ (shortest and longest) $\times 4!$ (others)

$= 2 \times 24 = \underline{48 \text{ ways}}$.

) $x - 2y + 3 = 0$, $2y = x + 3$, $y = \frac{x}{2} + \frac{3}{2}$ gradient $\frac{1}{2}$

$y = mx$ gradient m

(i) $\left| \dfrac{m - (\frac{1}{2})}{1 + m(\frac{1}{2})} \right| = \tan 45°$ $\therefore \left| \dfrac{(2m-1)/2}{(2+m)/2} \right| = 1$

$\therefore \left| \dfrac{2m-1}{m+2} \right| = 1$.

(ii) $\dfrac{2m-1}{m+2} = -1$, $2m - 1 = -m - 2$, $3m = -1$, $\underline{m = -\frac{1}{3}}$

or $\dfrac{2m-1}{m+2} = 1$, $2m - 1 = m + 2$, $\underline{m = 3}$

$\ln(x^3 + 19) = 3 \ln(x+1)$

$\ln(x^3 + 19) = \ln(x+1)^3$

$\ln(x^3 + 19) = \ln(x^3 + 3x^2 + 3x + 1)$

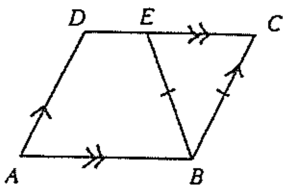
$\therefore x^3 + 19 = x^3 + 3x^2 + 3x + 1$

$\therefore 3x^2 + 3x - 18 = 0$ $\therefore x^2 + x - 6 = 0$

$\therefore (x+3)(x-2) = 0$ $\therefore \cancel{x = -3}$ or $\underline{x = 2}$

($x \neq -3$ since $\ln((-3)^3 + 19)$, $\ln((-3)+1)$ are not defined)

(i)



(ii) $\angle BEC = \angle BCE$ (in $\triangle BEC$ equal angles are opposite equal sides BC and BE)

$\angle BCE = \angle BAD$ (opposite angles are equal in parallelogram ABCD)

$\therefore \angle BEC = \angle BAD$

$\therefore$ ABED is a cyclic quadrilateral (exterior angle $\angle BEC$ is equal to interior opposite angle $\angle BAD$)

UESTION 2

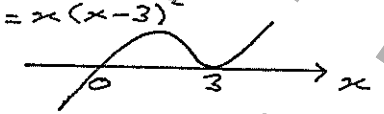
) $\lim_{x\to 0} \frac{\sin 2x}{x} = 2 \cdot \lim_{x\to 0} \frac{\sin 2x}{2x} = 2 \times 1 = \underline{\underline{2}}$.

) $\frac{x^2+9}{x} \le 6 \quad \therefore \frac{x^2+9}{x} \times x^2 \le 6 \times x^2$

$\therefore x^3 + 9x \le 6x^2 \quad \therefore x^3 - 6x^2 + 9x \le 0$

$\therefore x(x^2 - 6x + 9) \le 0 \quad \therefore x(x-3)^2 \le 0$

$y = x(x-3)^2$



$\therefore \underline{\underline{x < 0 \text{ or } x = 3}}$

($x \ne 0$ since $\{(0)^2+9\}/(0)$ is not defined)

(i) $3x^3 + 3x^2 - x - 1 = (x+1)(3x^2-1)$

(ii) $3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0 \quad (0 \le \theta \le \pi)$

$\therefore (\tan\theta + 1)(3\tan^2\theta - 1) = 0$

$\therefore \tan\theta + 1 = 0$, $\tan\theta = -1$, $\underline{\underline{\theta = 3\pi/4}}$

or $3\tan^2\theta - 1 = 0$, $3\tan^2\theta = 1$, $\tan^2\theta = 1/3$,

$\tan\theta = -1/\sqrt{3}$, $\underline{\underline{\theta = 5\pi/6}}$ or $\tan\theta = 1/\sqrt{3}$, $\underline{\underline{\theta = \pi/6}}$.

$x^2 = 4y$. $F(0,1)$, $P(2t, t^2)$

i) $M(x,y)$ divides FP externally in the ratio $3:1$.

$\therefore x = \frac{3(2t)-1(0)}{3-1} = 3t$, $y = \frac{3(t^2)-1(1)}{3-1} = \frac{3t^2-1}{2}$.

$x = 3t \quad \therefore t = x/3$

$y = (3t^2-1)/2 \quad \therefore 2y = 3t^2 - 1$

$\therefore 2y = 3(x/3)^2 - 1 \quad \therefore 2y = \cancel{3}(x^2/\cancel{9}_3) - 1$

$\therefore 6y = x^2 - 3 \quad \therefore x^2 = 6y + 3$.

$\therefore M$ moves on the parabola $x^2 = 6y + 3$.

i) $x^2 = 6y + 3 \quad \therefore x^2 = 6(y + 1/2)$

$\therefore x^2 = 4(3/2)(y + 1/2) \quad \therefore (x-0)^2 = 4(3/2)(y - (-1/2))$

$\therefore$ vertex $V(0, -1/2)$, focal length $a = 3/2$

$\therefore$ focus $(0, -1/2 + 3/2)$ ie. $\underline{\underline{(0,1)}}$

$\therefore$ directrix $y = -1/2 - 3/2$ ie. $\underline{\underline{y = -2}}$

QUESTION 3

(a) $y = \tan^{-1} 1/x \quad \therefore \frac{dy}{dx} = \frac{1}{1+(1/x)^2} \cdot (-1/x^2)$

$\therefore$ when $x = 1$, $dy/dx = 1/(1+1) \cdot (-1) = -1/2$

$\therefore$ the gradient of the tangent is $\underline{\underline{-1/2}}$.

(b) $f(x) = \frac{x+1}{x+2} \quad \therefore y = \frac{x+1}{x+2}$

interchange x and $y \quad \therefore x = \frac{y+1}{y+2}$

$\therefore x(y+2) = y + 1 \quad \therefore xy + 2x = y + 1$

$\therefore xy - y = 1 - 2x \quad \therefore y(x-1) = 1 - 2x$

$\therefore y = \frac{1-2x}{x-1} \quad \therefore \underline{\underline{f^{-1}(x) = \frac{1-2x}{x-1}}}$.

(c) $\frac{dy}{dx} = 2\cos^2 x + 1 \quad \therefore y = \int (2\cos^2 x + 1)\,dx$

$\therefore y = \int (\cos 2x + 1 + 1)\,dx \quad \therefore y = \int (2 + \cos 2x)\,dx$

$\therefore y = 2x + 1/2 \sin 2x + c$

When $x = \pi$, $y = \pi \quad \therefore \pi = 2\pi + 1/2 \sin 2\pi + c$

$\therefore \pi = 2\pi + 0 + c \quad \therefore c = -\pi$

$\therefore y = 2x + 1/2 \sin 2x - \pi$

$\therefore$ when $x = 2\pi$, $y = 4\pi + 1/2 \sin 4\pi - \pi$

$\therefore y = 4\pi + 0 - \pi \quad \therefore \underline{\underline{y = 3\pi}}$.

(d) $\int_1^{100} \frac{1}{x + 2\sqrt{x}}\,dx \qquad x = u^2 \ (u > 0)$

$\therefore dx/du = 2u$, $dx = 2u\,du$

$= \int_1^{10} \frac{1}{u^2 + 2u} \cdot 2u\,du \qquad$ when $x = 1$, $u = 1$

when $x = 100$, $u = 10$

$= \int_1^{10} \frac{1}{\cancel{u}(u+2)} \cdot 2\cancel{u}\,du = 2\int_1^{10} \frac{1}{u+2}\,du$

$= 2\,[\ln(u+2)]_1^{10} \qquad = 2(\ln 12 - \ln 3)$

$= 2 \ln 12/3 \qquad = 2 \ln 4$

$= \ln 4^2 \qquad = \underline{\underline{\ln 16}}$

HSCFocus.com

QUESTION 4

a) $\int_{\sqrt2}^{\sqrt3} \frac{1}{\sqrt{4-x^2}}\,dx = \int_{\sqrt2}^{\sqrt3} \frac{1}{\sqrt{2^2-x^2}}\,dx$

$= \left[\sin^{-1} x/2\right]_{\sqrt2}^{\sqrt3} = \sin^{-1}\sqrt3/2 - \sin^{-1}\sqrt2/2$

$= \pi/3 - \pi/4 = \underline{\pi/12}$

b) (i) $t = x^2 - 3x + 2$ $\therefore dt/dx = 2x-3$

$v = dx/dt = 1/\,dt/dx$ $\therefore \underline{v = 1/(2x-3)}$

(ii) $a = d/dx\,(\frac12 v^2) = d/dx\,(\frac12 \cdot 1/(2x-3)^2)$

$\therefore a = \frac12 \cdot -2/(2x-3)^3 \cdot 2$ $\therefore \underline{a = -2/(2x-3)^3}$

c) $y = 2\cos^{-1}(1-x)$

(i) domain : $-1 \le 1-x \le 1$

$\therefore -2 \le -x \le 0$ $\therefore \underline{0 \le x \le 2}$

range : $0 \le \cos^{-1}(1-x) \le \pi$

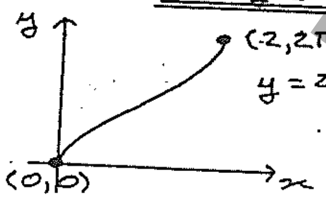
$\therefore 0 \le 2\cos^{-1}(1-x) \le 2\pi$ $\therefore \underline{0 \le y \le 2\pi}$

(ii) when $x = 0$

$y = 2\cos^{-1}(1) = 0$

when $x = 2$

$y = 2\cos^{-1}(-1) = 2\pi$



d) $r = \frac{1+3t}{1+t}$ $\therefore \frac{dr}{dt} = \frac{(1+t)(3)-(1+3t)(1)}{(1+t)^2} = \frac{2}{(1+t)^2}$

when $r = 2$, $\frac{1+3t}{1+t} = 2$

$\therefore 1+3t = 2+2t$ $\therefore t = 1$.

$A = \pi r^2$ $\therefore \frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = 2\pi r \cdot \frac{2}{(1+t)^2}$

when $r = 2$, $t = 1$

$\frac{dA}{dt} = 2\pi(2)\cdot\frac{2}{(1+(1))^2} = 2\pi$

$\therefore$ the area of the oil spill is increasing at a rate of $\underline{2\pi \text{ kilometres}^2/\text{hour}}$.

QUESTION 5

(a) $f(x) = \ln x/x$ $(x>0)$

(i) $f'(x) = \{(x)(1/x) - (\ln x)(1)\}/x^2 = (1-\ln x)/x^2$

when $f'(x) = 0$, $(1-\ln x)/x^2 = 0$, $1-\ln x = 0$, $\ln x = 1$, $x = e$

when $x = e$, $y = \ln e/e = 1/e$

when $0 < x < e$, $\ln x < 1$ and $f'(x) = (1-\ln x)/x^2 > 0$

when $x > e$, $\ln x > 1$ and $f'(x) = (1-\ln x)/x^2 < 0$

$\therefore \underline{(e, 1/e) \text{ is a maximum turning point}}$.

(ii) $f(e)$ is a local maximum and $f'(x) < 0$ for $x > e$ so that $f(x)$ is a decreasing function for $x > e$. $\therefore f(\pi) < f(e)$

$\therefore \ln\pi/\pi < \ln e/e$ $\therefore e\ln\pi < \pi\ln e$

$\therefore \ln\pi^e < \ln e^\pi$ $\therefore \underline{\pi^e < e^\pi}$.

(iii) $\ln x/x = -2$ $\therefore \ln x = -2x$ $\therefore \ln x + 2x = 0$

$P(x) = \ln x + 2x$, $P'(x) = 1/x + 2$

$\therefore$ with initial approximation of $X = 0.5$, improved approximation

$= 0.5 - \frac{P(0.5)}{P'(0.5)} = 0.5 - \frac{\ln 0.5 + 2(0.5)}{1/(0.5) + 2} = \underline{0.42}$ (to 2 d.p.)

(b) (i) $dM/dt < 0$, $dM/dt \propto (M-1000)$

$\therefore dM/dt = k(M-1000)$ $(k<0)$ or $dM/dt = -k(M-1000)$ $(k>0)$

If $M = 1000 + Ae^{-kt}$

$dM/dt = 0 + A(-ke^{-kt}) = -k(Ae^{-kt}) = -k(M-1000)$

$\therefore M = 1000 + Ae^{-kt}$ is a solution of $dM/dt = -k(M-1000)$.

(ii) when $t = 0$, $M = 49000$. $\therefore 49000 = 1000 + Ae^{-k(0)}$

$\therefore 49000 = 1000 + A$ $\therefore \underline{A = 48000}$

when $t = 2$, $M = 25000$ $\therefore 25000 = 1000 + 48000e^{-k(2)}$

$\therefore 24000 = 48000e^{-2k}$ $\therefore e^{2k} = 2$ $\therefore \underline{k = \frac12\ln 2}$

(iii) $dM/dt = -k(M-1000)$; when $t = 0$, $M = 49000$

$\therefore$ initial rate of losing value $= -k(49000-1000) = -48000k$

when $dM/dt = \frac14(-48000k) = -12000k$,

$-k(M-1000) = -12000k$, $M - 1000 = 12000$, $\underline{M = \$13000}$

when $M = 13000$, $13000 = 1000 + 48000e^{-kt}$

$\therefore 12000 = 48000e^{-kt}$ $\therefore e^{kt} = 4$ $\therefore kt = \ln 4$

$\therefore t = \frac{\ln 4}{k} = \frac{\ln 4}{\frac12\ln 2} = \underline{4 \text{ years}}$

HSCFocus.com

QUESTION 6

(a) In the expansion of $(1+x)^{14}$ the coefficients of x^4, x^5, x^6 are ${}^{14}C_4 = 1001$, ${}^{14}C_5 = 2002$, ${}^{14}C_6 = 3003$ which are consecutive terms in an arithmetic sequence with common difference 1001.

(b) In any one throw $P(\text{heads}) = p$.

$P(3 \text{ heads in } 6 \text{ throws}) = 2P(2 \text{ heads in } 6 \text{ throws})$

$\therefore {}^6C_3\, p^3(1-p)^3 = 2\, {}^6C_2\, p^2(1-p)^4$

$\therefore 20p^3(1-p)^3 = 2.15p^2(1-p)^4$

$\therefore 20p^3(1-p)^3 = 30p^2(1-p)^4 \quad \therefore 2p = 3(1-p)$

$\therefore 2p = 3 - 3p \quad \therefore 5p = 3 \quad \therefore \underline{\underline{p = 3/5}}$

(c) $x = 2\sin 3t - 2\sqrt{3}\cos 3t$

(i) $2\sin 3t - 2\sqrt{3}\cos 3t \equiv R\sin(3t-\alpha)$

$\equiv R(\sin 3t\cos\alpha - \cos 3t\sin\alpha)$

$\equiv (R\cos\alpha)\sin 3t - (R\sin\alpha)\cos 3t$

$\therefore R\cos\alpha = 2 \quad (1) \qquad R\sin\alpha = 2\sqrt{3} \quad (2)$

$(1)^2 \quad R^2\cos^2\alpha = 4 \qquad R^2(\cos^2\alpha + \sin^2\alpha) = 16$

$(2)^2 \quad R^2\sin^2\alpha = 12 \quad + \qquad R^2 = 16 \ , \ R = 4$

$\frac{(2)}{(1)} \quad \frac{R\sin\alpha}{R\cos\alpha} = \frac{2\sqrt{3}}{2} \qquad \tan\alpha = \sqrt{3} \ , \ \alpha = \pi/3$

$\therefore \underline{\underline{x = 4\sin(3t - \pi/3)}}$

(ii) $x = 4\sin(3t-\pi/3)$; when $t=0$, $x = -2\sqrt{3}$

$v = \dot{x} = 12\cos(3t-\pi/3)$; when $t=0$, $v = 6$

$a = \ddot{x} = -36\sin(3t-\pi/3)$; when $t=0$, $a = 18\sqrt{3}$

$\therefore$ initially the particle is $2\sqrt{3}$ metres to the left of O, moving to the right at a speed of $6\ ms^{-1}$ and speeding up at a rate of $18\sqrt{3}\ ms^{-2}$.

(iii) When $x = -2$, $4\sin(3t-\pi/3) = -2$

$\therefore \sin(3t-\pi/3) = -1/2 \quad \therefore 3t - \pi/3 = -\pi/6, \ldots$

$\therefore 3t = -\pi/6 + \pi/3, \ldots \quad \therefore 3t = \pi/6, \ldots$

$\therefore t = \pi/18, \ldots \qquad$ When $t = \pi/18$, $v = 6\sqrt{3} > 0$

$\therefore$ the first time is after $\underline{\underline{\pi/18 \text{ seconds}}}$

QUESTION 7

(a) $S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} \quad (n = 1, 2, 3, \ldots)$

(i) $S(n): \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

When $n = 1$, $LHS = 1/2! = 1/2$

$RHS = 1 - 1/(1+1)! = 1 - 1/2! = 1 - 1/2 = 1/2$

$\therefore$ statement is true when $n = 1$.

If the statement is true when $n = k$ for some $k \geq 1$

then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{\{(k+2) - (k+1)\}}{(k+2)!} = 1 - \frac{1}{(k+2)!}$

and the statement is also true when $n = k+1$.

Since $S(1)$ is true and if $S(k)$ is true for some $k \geq 1$ then $S(k+1)$ is also true it follows that $S(1+1) = S(2)$ is true, $S(2+1) = S(3)$ is true, ... and so $S(n)$ is true for all positive integers $n \geq 1$.

(ii) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{(n+1)!} = 1 - 0 = \underline{\underline{1}}$

(iii) $|S_n - 1| < 10^{-6}$

$\therefore \left| 1 - \frac{1}{(n+1)!} - 1 \right| < \frac{1}{10^6} \quad \therefore \left| \frac{-1}{(n+1)!} \right| < \frac{1}{10^6}$

$\therefore \frac{1}{(n+1)!} < \frac{1}{10^6} \qquad \therefore (n+1)! > 10^6$

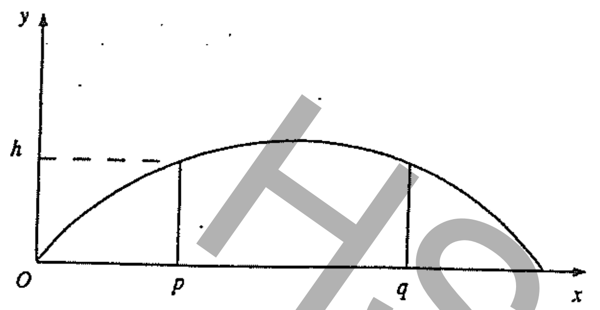
Now $9! = 362880$, $10! = 3628800$

$\therefore n+1 \geq 10 \quad \therefore n \geq 9$

$\therefore$ the smallest positive integer is $\underline{\underline{n = 9}}$.

QUESTION 7

(b)



(i) $x = (V\cos\alpha)t$ (1), $y = (V\sin\alpha)t - \frac{1}{2}gt^2$

$y = (V\sin\alpha)t - 5t^2$ (2).

(ii) When $x = p$, $y = h$

$\therefore$ in (1) $p = (V\cos\alpha)t$ $\therefore t = p/V\cos\alpha$

$\therefore$ in (2) $h = (V\sin\alpha)t - 5t^2$

$\therefore h = (V\sin\alpha)(p/V\cos\alpha) - 5(p/V\cos\alpha)^2$

$\therefore h = p\tan\alpha - 5p^2/V^2\cos^2\alpha$

$\therefore h = p\tan\alpha - 5p^2/V^2 \sec^2\alpha$

$\therefore h = p\tan\alpha - 5p^2/V^2 (1+\tan^2\alpha)$

$\therefore 5p^2/V^2 (1+\tan^2\alpha) = p\tan\alpha - h$

$\therefore V^2 = \dfrac{5p^2(1+\tan^2\alpha)}{p\tan\alpha - h}$

(iii) When $x = q$, $y = h$

$\therefore$ similarly $V^2 = \dfrac{5q^2(1+\tan^2\alpha)}{q\tan\alpha - h}$

$\therefore \dfrac{5p^2\cancel{(1+\tan^2\alpha)}}{p\tan\alpha - h} = \dfrac{5q^2\cancel{(1+\tan^2\alpha)}}{q\tan\alpha - h}$

$\therefore p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$

$\therefore p^2q\tan\alpha - p^2h = pq^2\tan\alpha - q^2h$

$\therefore p^2h - q^2h = p^2q\tan\alpha - pq^2\tan\alpha$

$\therefore (p^2 - q^2)h = (p^2q - pq^2)\tan\alpha$

$\therefore (p+q)\cancel{(p-q)}h = pq\cancel{(p-q)}\tan\alpha$

$\therefore \tan\alpha = \dfrac{h(p+q)}{pq}$

National Educational Advancement Programs

HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

1999

MATHEMATICS

3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON)

Time Allowed: Three hours
(Plus 5 minutes' reading time)

This paper must be kept under strict security and may only be used on or after the afternoon of Friday 13 August, 1999, as specified in the NEAP Examination Timetable.

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 9.
- Approved calculators may be used.
- Each question is to be returned in a separate Writing Booklet clearly labelled, showing your Student Name or Number.
- You may ask for extra Writing Booklets if you need them.

Students are reminded that this is a trial examination only and cannot in any way guarantee the content or the format of the 1999 Mathematics 3 Unit (Additional) and 3/4 Unit (Common) Higher School Certificate Examination.

This paper is issued by National Educational Advancement Programs (NEAP) to individual schools copyright free for the restricted use of students of that school only.

COPYRIGHT © 1999 NEAP

P.O. Box 214 St Leonards NSW 2065 (100 Christie St.) Tel: (02) 9438 1386 Fax: (02) 9438 1385

HSCFocus.com

QUESTION 1. Use a *separate* Writing Booklet.

Marks

(a) Solve $\frac{x+1}{x} \geq 2$. 3

(b) Find the acute angle between the lines $x+3y = 4$ and $2x-5y = 0$. 3
Give the answer to the nearest degree.

(c) Use the "t" results to solve the equation $2\cos\theta - \sin\theta = -1$ for $0 \leq \theta \leq 2\pi$. 4
Give any answers correct to three significant figures.

(d) Point A is $(2, -7)$ and B is $(-6, 9)$. Find the coordinates of point P which divides AB internally in the ratio $3:5$. 2

COPYRIGHT © 1999 NEAP

QUESTION 2. Use a *separate* Writing Booklet.

Marks

(a) Use the substitution $u = \ln x$ to find $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$. 2

(b) Consider the geometric series $1 - \tan^2 x + \tan^4 x - \ldots$, where $0 < x < \frac{\pi}{2}$. 3

(i) For what values of x does this series have a limiting sum?

(ii) Find the limiting sum in terms of $\cos x$.

(c) It is given that x^2+x-2 is a factor of x^3+rx^2-4x+s, where r and s are constants. 3

(i) Show that $r+s = 3$.

(ii) Evaluate r and s.

(d) On a certain railway line, there are eleven stations at which a train can stop. The rail authority wishes to print tickets for travel between every possible pair of stations on the line. 1

How many different one-way tickets must be printed if the ticket specifies which direction the passenger is travelling?

(e) (i) Five men and five women are arranged in a straight line. How many arrangements are there in which men and women alternate? 3

(ii) Five men and five women are arranged in a circle. How many arrangements are there in which men and women alternate?

(iii) Find the probability that men and women alternate if five men and five women are arranged at random in a circle.

COPYRIGHT © 1999 NEAP

HSCFocus.com

QUESTION 3. Use a *separate* Writing Booklet. **Marks**

(a) (i) Sketch $y = 3\sin x$ and $y = x$, for $0 \le x \le 2\pi$, on the same set of axes. **4**

(ii) By substitution, show that a solution for $3\sin x - x = 0$ lies between $x = 2.2$ and $x = 2.4$.

(iii) Taking $x = 2.3$ as an approximation to a solution of $3\sin x - x = 0$, apply Newton's method once to find a better approximation correct to three decimal places.

(b) (i) Find $\dfrac{d}{dx}(x\tan^{-1}x)$. **4**

(ii) Hence find the exact value of $\displaystyle\int_0^1 \tan^{-1}x \, dx$.

(c) Use mathematical induction to show **4**

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1} \quad \text{for } n \ge 1.$$

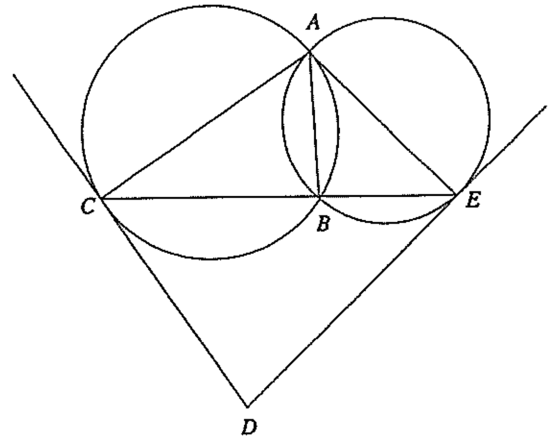
COPYRIGHT © 1999 NEAP

QUESTION 4. Use a *separate* Writing Booklet. **Marks**

(a) Assume that Antarctica can be approximately represented as a circle, and that its area is decreasing at the rate of 3000 square kilometres per year. **2**

Find the rate at which the radius of the circle is decreasing when the length of the radius is 2100 kilometres.

(b) **4**



In the diagram, two circles intersect at A and B. Points C and E lie on the circles and C, B and E are collinear. Tangents at C and E meet at D.

Copy the diagram into your booklet and show that quadrilateral $AEDC$ is cyclic.

(c) A particle moves in a straight line and at time t seconds, its velocity, v metres per second, is related to its displacement, x metres, by **6**

$$v^2 = 108 + 36x - 9x^2.$$

(i) By deriving an expression for acceleration, show that the motion is simple harmonic.

(ii) Find the period of the motion.

(iii) Find the amplitude of the motion.

(iv) For what value of x does the particle have maximum speed?

COPYRIGHT © 1999 NEAP

QUESTION 5. Use a *separate* Writing Booklet. **Marks**

(a) Consider the function $f(x) = \dfrac{1}{1+x^2}$ for $x \le 0$. **4**

(i) Sketch $y = f(x)$. (It is not necessary to show working.)

(ii) Find the inverse function $f^{-1}(x)$.

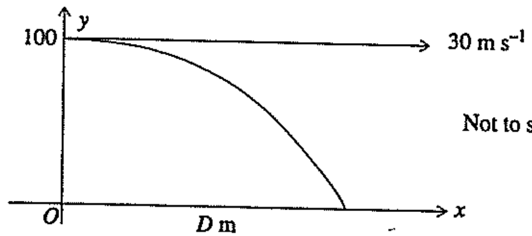
(iii) State the domain of $f^{-1}(x)$.

(b) Fish are taken at random from a large number of fish in a dam. For this particluar kind of fish, one quarter of the population is male. **4**

(i) Find the probability that if ten fish are taken, then exactly four are male.

(ii) How many fish have to be taken from the dam for the probability of there being at least one male to be greater than 99%?

(c) A plane designed to drop water "bombs" flies in a horizontal line towards a spot fire. It maintains a steady speed of 30 metres per second, and an altitude of 100 metres. The plane releases its water when at a horizontal distance D metres from the fire. **4**



Let the origin O be as shown in the diagram, and let x and y metres be the horizontal and vertical displacements of the centre of the water bomb at time t seconds after its release. Then $\dot{x}$ and $\dot{y}$ are the horizontal and vertical components of its velocity in metres per second. Let $g = 10$ metres per second per second, and assume that air resistance has negligible effect.

(i) Use calculus to derive expressions for $\dot{x}$, $\dot{y}$, x and y in terms of t.

(ii) Hence determine the value of D correct to the nearest metre.

COPYRIGHT © 1999 NEAP

QUESTION 6. Use a *separate* Writing Booklet. **Marks**

(a) Solve $|2x-1| < 3x+2$. **3**

(b) (i) On the same set of axes, sketch $y = \sin^{-1}x$ and $y = \cos^{-1}x$, showing all essential information. **4**

(ii) Let $f(x) = \sin^{-1}x + \cos^{-1}x$. By referring to the graphs in part (i), or otherwise, explain why $f(x)$ is a constant function, and find its value.

(iii) Hence evaluate $\displaystyle\int_{-1}^{1} (\sin^{-1}x + \cos^{-1}x)\,dx$.

(c) At time t seconds after the parachute has opened, a parachutist falls with speed v metres per second and with acceleration given by **5**

$$\frac{dv}{dt} = k(4-v)$$

where k is constant.

(i) Show that $v = 4 + Ae^{-kt}$, where A is constant, satisfies the above equation.

(ii) At the instant the parachute opens, the parachutist is falling at 25 metres per second. Find the value of A.

(iii) Two seconds after the parachute has opened, the parachutist's speed is 12 metres per second. Evaluate k correct to three significant figures.

(iv) As t increases indefinitely, v approaches a fixed value. Show that after 20 seconds v differs from this fixed value by less than 0.1%.

COPYRIGHT © 1999 NEAP

HSCFocus.com

QUESTION 7. Use a *separate* Writing Booklet. **Marks**

(a) An expression of the form $(1-ax)^n$, where a is a constant and n a positive integer, is expanded. The first three terms of the expansion are **4**

$$1-4x+\frac{20}{3}x^2.$$

Find the values of a and n.

(b) (i) Show that $\cos 3x = 4\cos^3 x - 3\cos x$ by using the expansion of $\cos(A+B)$. **8**

(ii) Show that the solution of $\cos 3x - \sin 2x = 0$ for $0 < x < \frac{\pi}{2}$ is given by

$$\sin x = \frac{\sqrt{5}-1}{4}.$$

(iii) Use a trigonometric identity to explain why $x = \frac{\pi}{10}$ is a solution to $\cos 3x = \sin 2x$.

(iv) Hence, using the results referred to in parts (ii) and (iii), prove $\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \frac{\sqrt{5}}{4}$.

COPYRIGHT © 1999 NEAP

The following list of standard integrals may be used:

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq 1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right).$$

Note: $\ln x = \log_e x, \quad x > 0$

Printed with kind permission of the Board of Studies.

COPYRIGHT © 1999 NEAP

HSCFocus.com

National Educational Advancement Programs

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

1999

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

SOLUTIONS AND SUGGESTED MARKING SCHEME

HSCFocus.com

Students are reminded that this is a trial examination only and cannot in any way guarantee the content or the format of the 1999 Mathematics 3 Unit (Additional) and 3/4 Unit (Common) Higher School Certificate Examination.

This paper is issued by National Educational Advancement Programs (NEAP) to individual schools copyright free for the restricted use of students of that school only.

COPYRIGHT © 1999 NEAP

P.O. Box 214 St Leonards NSW 2065 (100 Christie St.) Tel: (02) 9438 1386 Fax: (02) 9438 1385

HSCFocus.com

QUESTION 1.

(a) $\dfrac{x+1}{x} \geq 2 \quad x \neq 0$

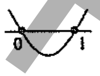
$\therefore \dfrac{(x^2)(x+1)}{x} \geq 2(x^2)$ ✓

$\therefore x(x+1) \geq 2x^2$

$2x^2 - x^2 - x \leq 0$

$x^2 - x \leq 0$

$x(x-1) \leq 0$ ✓



but $x \neq 0$

$\therefore 0 < x \leq 1$ ✓

(b) $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$x + 3y = 4$

$2x - 5y = 0$

$m_1 = -\dfrac{1}{3}$

$m_2 = \dfrac{2}{5}$

$\tan\theta = \left|\dfrac{\left(-\frac{1}{3}\right)-\left(\frac{2}{5}\right)}{1+\left(-\frac{1}{3}\right)\left(\frac{2}{5}\right)}\right|$ ✓

$= \left|\dfrac{\frac{-5-6}{15}}{1-\frac{2}{15}}\right|$

$= \left|\dfrac{-11}{13}\right|$

$= \dfrac{11}{13}$ ✓

$\therefore \theta = \tan^{-1}\left(\dfrac{11}{13}\right)$

$= 40.236\ldots°$ (calculator)

$\therefore$ the angle between the lines is 40° (to the nearest degree). ✓

(c) $2\cos\theta - \sin\theta = -1$

$2\left(\dfrac{1-t^2}{1+t^2}\right) - \left(\dfrac{2t}{1+t^2}\right) = -1$ ✓

$2 - 2t^2 - 2t = -1 - t^2$

$t^2 + 2t - 3 = 0$

$(t+3)(t-1) = 0$

$\therefore t = -3$ or $t = 1$ ✓

$\tan\dfrac{\theta}{2} = -3$ or $\tan\dfrac{\theta}{2} = 1$

$\dfrac{\theta}{2} = \pi - 1.2490\ldots \quad \dfrac{\theta}{2} = \dfrac{\pi}{4}$ ✓

$\therefore \theta = 3.79$ (3SF) or $\theta = 1.57$ (3SF) ✓

(d) $x = \dfrac{x_1 r_2 + x_2 r_1}{r_1 + r_2}$

$x_1 = 2 \quad x_2 = -6$

$y_1 = -7 \quad y_2 = 9$

$r_1 : r_2 = 3:5$

$= \dfrac{(2)(5)+(-6)(3)}{3+5}$

$= \dfrac{10-18}{8}$

$= -1$ ✓

$y = \dfrac{y_1 r_2 + y_2 r_1}{r_1 + r_2}$

$= \dfrac{(-7)(5)+(9)(3)}{3+5}$

$= \dfrac{-35+27}{8}$

$= -1$ ✓

$\therefore$ point P is $(-1,-1)$.

QUESTION 2.

(a) $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-u^2}}$ ✓

let $u = \ln x$

$du = \dfrac{1}{x}dx$

$= \sin^{-1}u + c$

$= \sin^{-1}(\ln x) + c$ ✓

(b) $r = \dfrac{T_2}{T_1} = -\tan^2 x$

(i) $-1 < r < 1$

$\therefore -1 < -\tan^2 x < 1$

$\therefore -1 < \tan^2 x < 1$

But $\tan^2 x \geq 0$, and $x > 0$, given.

$\therefore 0 < \tan^2 x < 1$

$0 < \tan x < 1 \quad \left(\text{since } 0 < x < \dfrac{\pi}{2}\right)$

$0 < x < \dfrac{\pi}{4}$ ✓

(ii) $S_\infty = \dfrac{a}{1-r}$

$= \dfrac{1}{1-(-\tan^2 x)}$

$= \dfrac{1}{1+\tan^2 x}$ ✓

$= \dfrac{1}{\sec^2 x}$

$= \cos^2 x$ ✓

(c) Let $P(x) = x^3 + rx^2 - 4x + s$

and note $x^2 + x - 2 = (x+2)(x-1)$

If $x^2 + x - 2$ is a factor of $P(x)$

then $P(1) = 0$ and $P(-2) = 0$

(i) $P(1) = (1)^3 + r(1)^2 - 4(1) + s = 0$

$\therefore (1 + r - 4 + s) = 0$

$r + s = 3$ ① ✓

(ii) $P(-2) = (-2)^3 + (-2)^2 r - 4(-2) + s = 0$

$-8 + 4r + 8 + s = 0$

$\therefore 4r + s = 0$ ② ✓

from (i) $r + s = 3$ ①

② − ① $3r = -3$

$r = -1$

$s = 4$ ✓

(d) No. of tickets with direction $= 2 \times {}^{11}C_2$ (OR ${}^{11}P_2$)

$= 110$ ✓

(e) (i) No. of arrangements in a straight line $= 2! \times 5! \times 5!$ ✓

$= 28800$

(ii) No. of arrangements in a circle $= 4! \times 5!$ ✓

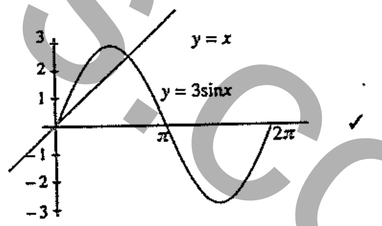
$= 2880$

(iii) P(m and w alternate) $= \dfrac{4! \times 5!}{9!}$

$= \dfrac{1}{126}$ ✓

QUESTION 3.

(a) (i)



✓

(ii) At $x = 2.2 \quad 3\sin(2.2) - (2.2) = 0.2254\ldots$

At $x = 2.4 \quad 3\sin(2.4) - (2.4) = -0.3736\ldots$

Since the sign of $3\sin x - x$ changes between $x = 2.2$ and $x = 2.4$ then a solution lies between 2.2 and 2.4. ✓

(iii) $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$f(x) = 3\sin x - x$

$f'(x) = 3\cos x - 1$

$x_2 = (2.3) - \dfrac{3\sin(2.3) - 2.3}{3\cos(2.3) - 1}$ ✓

$= 2.27903\ldots$(calculator)

$= 2.279$ (correct to 3 dec. places) ✓

(b) (i) $\dfrac{d}{dx}(x\tan^{-1}x) = (\tan^{-1}x)(1) + (x)\left(\dfrac{1}{1+x^2}\right)$

$= \tan^{-1}x + \dfrac{x}{1+x^2}$ ✓

(ii) $\displaystyle\int_0^1 \tan^{-1}x\,dx = \int_0^1\left(\frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}\right)dx$ ✓

$= \left[x\tan^{-1}x - \dfrac{1}{2}\log_e(1+x^2)\right]_0^1$ ✓

$= \left[(1)\tan^{-1}(1) - \dfrac{1}{2}\log_e(1+(1)^2)\right]$

$-\left[0 - \dfrac{1}{2}\log_e 1\right]$

$= \dfrac{\pi}{4} - \dfrac{1}{2}\log_e 2$ ✓

(c) To prove $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1},\ n \geq 1.$

When $n = 1$

$\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \sum_{r=1}^{1}\frac{1}{r(r+1)}$

$= \dfrac{1}{1(2)}$

$= \dfrac{1}{2}$

and $\dfrac{n}{n+1} = \dfrac{1}{1+1}$

$= \dfrac{1}{2}$

$\therefore \displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 1$. ✓

Assume $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k$.

i.e. assume $\displaystyle\sum_{r=1}^{k}\frac{1}{r(r+1)} = \frac{k}{k+1}$.

When $n = k + 1$

$$\sum_{r=1}^{n}\frac{1}{r(r+1)} = \sum_{r=1}^{k+1}\left(\frac{1}{r(r+1)}\right)$$

$$= \sum_{r=1}^{k}\frac{1}{r(r+1)} + \frac{1}{(k+1)(k+1+1)} \quad ✓$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{(from assumption)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{(k+1)}{(k+1)+1} \quad ✓$$

$\therefore$ if $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k$

then $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = k+1$.

Conclusion:

Since $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 1$

then $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ when $n = 2, 3, \ldots\ldots$

$\therefore \displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$. ✓

QUESTION 4.

(a) $A = \pi r^2$

$$\frac{dA}{dt} = \frac{dA}{dr}\times\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

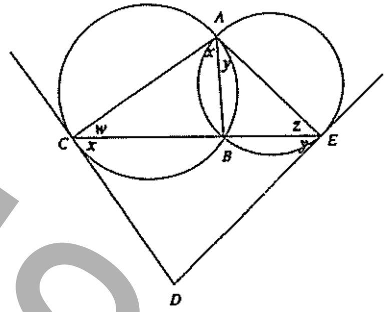
$$\frac{dr}{dt} = \frac{dA}{dt} \div 2\pi r$$

$$= -3000 \times \frac{1}{2\pi \times 2100} \quad ✓$$

$$= -0.22736\ldots \text{(calculator)}$$

$\therefore$ radius is decreasing at 227 metres per year (to the nearest metre) ✓

(b)



Let $x = \angle DCB$.

$\angle DCB = \angle CAB$ (Alternate segment theorem)

$\therefore \angle CAB = x$. ✓

Similarly, if $y = \angle DEB$,

then $\angle EAB = y$.

Let $\angle ACB = w$ and let $\angle AEB = z$.

In ΔAEC, $w + x + y + z = 180°$ (Angle sum of triangle) ✓

In quadrilateral $AEDC$,

$ACD + DEA = w + x + y + z$

$= 180$ as shown above ✓

$\therefore$ opposite angles are supplementary

$\therefore$ quadrilateral $AEDC$ is cyclic. ✓

OR

Proof:

$\angle BCD = \angle CAB$ (angle between tangent & chord = angle in alternate segement) ✓

$\angle BED = \angle EAB$ (same reason)

$\angle BCD + \angle BED + \angle D = 180°$ (angle sum of Δ) ✓

$\therefore \angle CAB + \angle EAB + \angle D = 180°$

$\therefore \angle CAE + \angle D = 180°$ ✓

$\therefore$ sum of opposite angles of $ACDE = 180°$ ✓

$\therefore$ $ACDE$ is cyclic.

(c) (i)

$$v^2 = 108 + 36x - 9x^2$$

$$\frac{1}{2}v^2 = \frac{1}{2}(108 + 36x - 9x^2)$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 18 - 9x \quad ✓$$

$$\therefore \ddot{x} = -9x + 18$$

$$\ddot{x} = -9(x-2)$$

$\therefore$ motion is simple harmonic of the form

$\ddot{x} = -n^2(x-2)$ ✓

(ii) Period of motion $= \dfrac{2\pi}{n} = \dfrac{2\pi}{3}$ seconds ✓

(iii) To find the amplitude let $v = 0$ i.e. $v^2 = 0$.

$$\therefore 9x^2 - 36x - 108 = 0$$

$$9(x^2 - 4x - 12) = 0 \quad ✓$$

$$9(x-6)(x+2) = 0$$

$$\therefore x = 6 \text{ or } x = -2$$

$$\therefore 2a = 6 - (-2)$$

$$= 8$$

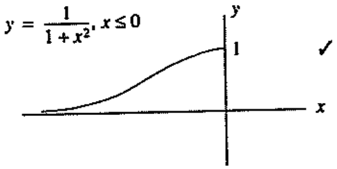
$$a = 4 \quad ✓$$

$\therefore$ amplitude is 4 metres.

(iv) Maximum speed occurs at centre of motion, at $x = 2$. ✓

QUESTION 5.

(a) (i) $y = \dfrac{1}{1+x^2}, x \le 0$ ✓



(ii) Inverse $\quad x = \dfrac{1}{1+y^2} \quad y \le 0$

$$1 + y^2 = \frac{1}{x} \quad ✓$$

$$y = \pm\sqrt{\frac{1}{x} - 1}, \; y \le 0$$

$$\therefore f^{-1}(x) = -\sqrt{\frac{1}{x} - 1} \quad ✓$$

(iii) Domain of $f^{-1}(x)$: $0 < x \le 1$. ✓

(b) Let p be the probability of selecting a male and q be the probability of selecting a female.

(i) Binomial expansion for ten fish:

$$(p+q)^{10} = {}^{10}C_0p^{10} + {}^{10}C_1p^9q^1 + {}^{10}C_2p^8q^2\ldots$$

$$P(4 \text{ males}) = {}^{10}C_6p^4q^6$$

$$= \frac{10\times9\times8\times7}{4\times3\times2\times1}\times\left(\frac{1}{4}\right)^4\times\left(\frac{3}{4}\right)^6$$

$$= \frac{10\times3\times7\times3^6}{4^{10}}$$

$$= \frac{76545}{524288}$$

$$= 0.146 \text{ (3 dp)} \quad ✓$$

(ii) $1 - P(\text{no males}) > \dfrac{99}{100}$ ✓

$$1 - \left(\frac{3}{4}\right)^n > \frac{99}{100}$$

$$\left(\frac{3}{4}\right)^n < 1 - \frac{99}{100}$$

$$(0.75)^n < 0.01$$

$$n\log 0.75 < \log 0.01$$

$$n > \frac{\log 0.01}{\log 0.75} \quad ✓$$

(N.B. $\log 0.75$ is negative)

$> 16.0078\ldots$ (calculator)

$n = 17$

$\therefore$ 17 fish have to be taken. ✓

(c) (i)

$$\ddot{x} = 0$$

$$\boxed{\dot{x} = 30}$$

$$x = \int 30\,dt$$

$x = 30t + c$ (where c is a constant)

when $t = 0, x = 0 \;\therefore\; c = 0$

$$\boxed{x = 30t} \quad ✓$$

$$\ddot{y} = -10$$

$$\dot{y} = \int -10\,dt$$

$$= -10t + c$$

when $t = 0, \dot{y} = 0 \;\therefore\; c = 0$

$$\boxed{\dot{y} = -10t}$$

$$y = \int -10t\,dt$$

$$= -5t^2 + c$$

when $t = 0, y = 100 \;\therefore\; c = 100$

$$\boxed{y = -5t^2 + 100} \quad ✓$$

(ii) Water bomb hits the ground when $y = 0$.

$$\therefore 5t^2 = 100$$

$$t^2 = 20$$

$$t = 2\sqrt{5} \text{ sec.} \quad ✓$$

$$\therefore x = 30(2\sqrt{5})$$

$$= 134.16\ldots \text{ (calculator)} \quad ✓$$

$\therefore D = 134$ metres (to the nearest metre)

HSCFocus.com

QUESTION 6.

(a) $|2x-1| < 3x+2$

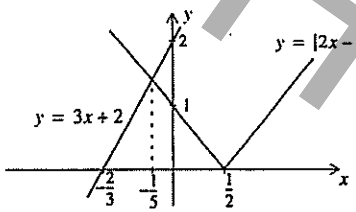
Solve $2x - 1 = 3x + 2$ and $2x - 1 = -3x - 2$

$x = -3$ $\qquad x = -\frac{1}{5}$

But if $x = -3$, $3x + 2 < 0$ ✓

$\therefore x = -3$ is not a solution.

Now consider the graphs



✓

From graph $|2x-1| < 3x+2$ for $x > -\frac{1}{5}$. ✓

OR

$|2x-1| < 3x+2$

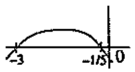
$(2x-1)^2 < (3x+2)^2$ (since $|2x-1| \geq 0$) ✓

$(2x-1)^2 - (3x+2)^2 < 0$

$(2x-1-(3x+2))(2x-1+(3x+2)) < 0$

$(-x-3)(5x+1) < 0$

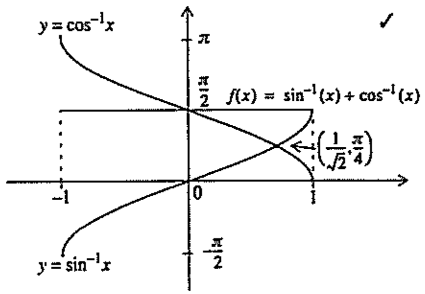
$\therefore x < -3$ or $x > -\frac{1}{5}$ ✓



But if $x < -3$, $3x + 2 < 0$. ✓

$\therefore x > -\frac{1}{5}$

(b) (i)



✓

(ii) By adding ordinates at some key points on the graphs, and by noting the symmetry of the graphs, it can be seen that

$f(x) = \sin^{-1}x + \cos^{-1}x$ ✓

$= \text{constant}$

$= \frac{\pi}{2}$ ✓

OR

$f(x) = \sin^{-1}x + \cos^{-1}x$

$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$

$= 0$

But $f(0) = \frac{\pi}{2}$, and from graph $f(x)$ has one value only.

$\therefore f(x) = \text{constant function}$

$= \frac{\pi}{2}$

(iii) $\int_{-1}^{1} (\sin^{-1}x + \cos^{-1}x)\ dx$

$= \int_{-1}^{1} \left(\frac{\pi}{2}\right) dx$

$= \left[\frac{\pi}{2}x\right]_{-1}^{1}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

$= \pi$ ✓

OR

From graph, value of integral is given by area of rectangle width 2 and height $\frac{\pi}{2}$.

Area $= 2 \times \frac{\pi}{2}$

$= \pi$

$\therefore \int_{-1}^{1} (\sin^{-1}x + \cos^{-1}x)\ dx = \pi$

(c) (i) $v = 4 + Ae^{-kt}$ (given)

$\therefore \frac{dv}{dt} = -k \times Ae^{-kt}$.

Now $k(4 - v) = k(4 - (4 + Ae^{-kt}))$

$= k \times (-Ae^{-kt})$

$= -k \times Ae^{-kt}$ ✓

$\therefore \frac{dv}{dt} = k(4 - v)$,

$\therefore v = 4 + Ae^{-kt}$ satisfies $\frac{dv}{dt} = k(4-v)$

(ii) $v = 4 + Ae^{-kt}$

At $t = 0$, $v = 25$

$25 = 4 + Ae^{0}$

$\therefore A = 21$ ✓

(iii) $v = 4 + 21e^{-kt}$

At $t = 2$, $v = 12$

$12 = 4 + 21e^{-2k}$

$8 = 21e^{-2k}$

$\frac{8}{21} = e^{-2k}$

$-2k = \ln\left(\frac{8}{21}\right)$

$k = 0.4825\ldots$ ✓

$= 0.483$ (3 significant figures)

(iv) $v = 4 + 21e^{-0.483t}$

$= 4 + \frac{21}{e^{0.483t}}$

$\to 4$ as $t \to \infty$ ✓

At $t = 20$

$v = 4 + 21e^{-0.483 \times 20}$

$= 4.0013$ (4 decimal places)

% difference $= \frac{0.0013}{4} \times \frac{100}{1}\%$

$\doteqdot 0.03\% < 0.1\%$ ✓

$\therefore$ speed differs from 4 by less than 0.1%.

QUESTION 7.

(a) $(1-ax)^n = 1 - nax + \frac{n(n-1)}{2 \times 1}(ax)^2 - \ldots$

$= 1 - 4x + \frac{20}{3}x^2 - \ldots$(given)

$\therefore na = 4$ ①

and $\frac{n(n-1)}{2}a^2 = \frac{20}{3}$ ② ✓

From ① $a = \frac{4}{n}$ ③

Substituting in ② :

$\frac{n(n-1)}{2} \times \left(\frac{4}{n}\right)^2 = \frac{20}{3}$ ✓

$48n(n-1) = 40n^2$

$6n^2 - 6n = 5n^2$

$n^2 - 6n = 0$

$n(n-6) = 0$

$n = 0$ or 6.

But $n \neq 0$, since n is a positive integer.

$\therefore n = 6$ ✓

From ③ $a = \frac{4}{6}$

$= \frac{2}{3}$ ✓

$\therefore a = \frac{2}{3}$, $n = 6$.

(b) (i) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\therefore \cos 3x = \cos(2x + x)$

$= \cos 2x\cos x - \sin 2x\sin x$ ✓

$= (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x$

$= (2\cos^2 x - 1)\cos x - 2(1 - \cos^2 x)\cos x$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$ ✓

$= 4\cos^3 x - 3\cos x$

$\therefore \cos 3x = 4\cos^3 x - 3\cos x$.

(ii) $\cos 3x - \sin 2x = 0 \qquad 0 < x < \frac{\pi}{2}$

$4\cos^3 x - 3\cos x - 2\sin x\cos x = 0$

$\cos x(4\cos^2 x - 3 - 2\sin x) = 0$

$\cos x(4(1 - \sin^2 x) - 3 - 2\sin x) = 0$

$\cos x(4 - 4\sin^2 x - 3 - 2\sin x) = 0$

$\cos x(4\sin^2 x + 2\sin x - 1) = 0$ ✓

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ which is not in required range.

$\therefore 4\sin^2 x + 2\sin x - 1 = 0$

$\sin x = \frac{-2 \pm \sqrt{4+16}}{8}$

$= \frac{\pm\sqrt{5} - 1}{4}$

$\sin x = \frac{-\sqrt{5} - 1}{4} \Rightarrow x$ is outside required range. ✓

$\therefore \sin x = \frac{\sqrt{5} - 1}{4}$.

(iii) $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ ✓

Now if $\theta = 3 \times \frac{\pi}{10}$,

then $\frac{\pi}{2} - \theta = \frac{\pi}{2} - 3 \times \frac{\pi}{10} = 2 \times \frac{\pi}{10}$.

$\therefore x = \frac{\pi}{10}$ is a solution of $\cos 3x = \sin 2x$.

OR $3x + 2x = \frac{\pi}{2}$

$5x = \frac{\pi}{2}$

$\therefore x = \frac{\pi}{10}$.

(iv) $\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \sin\left(2 \times \frac{\pi}{10}\right)\cos\frac{\pi}{10}$

$= \left(2\sin\frac{\pi}{10}\cos\frac{\pi}{10}\right)\cos\frac{\pi}{10}$ ✓

$= 2\sin\frac{\pi}{10}\cos^2\frac{\pi}{10}$

$= 2\sin\frac{\pi}{10}\left(1 - \sin^2\frac{\pi}{10}\right)$

$= 2 \times \frac{\sqrt{5}-1}{4} \times \left(1 - \left(\frac{\sqrt{5}-1}{4}\right)^2\right)$ ✓

$= \frac{\sqrt{5}-1}{2} \times \left(\frac{16 - (5 - 2\sqrt{5} + 1)}{16}\right)$

$= \frac{\sqrt{5}-1}{2} \times \frac{10 + 2\sqrt{5}}{16}$

$= \frac{\sqrt{5}-1}{2} \times \frac{5 + \sqrt{5}}{8}$

$= \frac{5\sqrt{5} + 5 - 5 - \sqrt{5}}{16}$ ✓

$= \frac{4\sqrt{5}}{16}$

$= \frac{\sqrt{5}}{4}$.

HSCFocus.com

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time Allowed - Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Write your student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the seven questions are to be handed in separately clearly marked Question 1, Question 2, etc..
- *The question paper must be handed to the supervisor at the end of the examination.*

STUDENT NUMBER / NAME..

HSCFocus.com

Question 1 (Start a new page) **Marks**

a. Two dice are rolled. If you know that at least one of the dice is a 5, what is the probability of getting a total of 8? **2**

b. At an election, 30% of the voters favoured candidate A. If 7 voters are selected at random, what is the probability that 4 of them favour A? **2**

c. The point C(−1, −4) divides the interval AB externally in the ratio 3:1. If the coordinates of A are (3, 2), find the coordinates of B. **2**

d. Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^3 x \, dx$ using the substitution $u = \cos x$ **3**

e. Find the exact value of $\int_0^{\frac{\pi}{4}} \cos^2 \frac{1}{2}x \, dx$ **3**

Question 2 (Start a new page)

a. Solve $\frac{1}{x+1} \geq 1 - x$ **3**

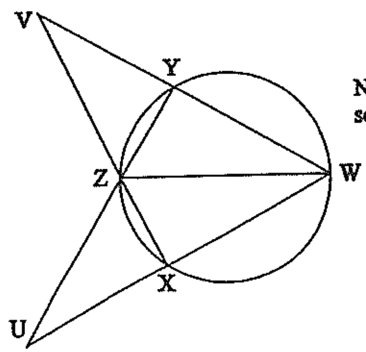
b. Find $\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{16 - 25x^2}}$ **3**

c. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x = 2at, y = at^2$. **3**

i. Find M, the midpoint of PQ.

ii. Show that, if the gradient of PQ is constant, the locus of M is a line parallel to the y–axis.

d. In the diagram, UZY, XZV, VYW and UXW are all straight lines. Given ZW bisects ∠XWY and ∠WUZ = ∠WVZ, prove that XW = YW. **3**



Question 3 (Start a new page) **Marks**

a. Show that $\frac{2x+1}{x+2} = 2 - \frac{3}{x+2}$ **3**

Hence or otherwise, find the exact value of $\int_0^1 \frac{2x+1}{x+2} \, dx$

b. Solve $\cos x - \sqrt{3}\sin x + 1 = 0$ for $0 \leq x \leq 2\pi$ **3**

c. i. Show that the solution of $x \ln x - 1 = 0$ lies between $x = 1$ and $x = 2$. **3**

ii. Using $x = 2$ as a first approximation, apply Newton's method once to obtain a better approximation. Give your answer to one decimal place.

d. A mixed tennis team consisting of 2 men and 2 women is to be chosen from 5 men and 7 women. **3**

i. Find the probability that a particular woman is in the selected team.

ii. If one of the original 5 men is selected as the captain of the team, find the probability that his brother, who was one of the original 5 men, is also in the team.

Question 4 (Start a new page)

a. Two circles, C_1 and C_2, are members of the set of circles defined by the equation $x^2 + y^2 - 6x + 2ky + 3k = 0$, where k is real. **4**

The centre of C_1 lies on the line $x - 3y = 0$ and C_2 touches the x–axis.

Find the equations of C_1 and C_2.

b. The acceleration, a, of a particle is given in terms of its position, x, by the equation $a = 2x^3 + 2x$. **4**

i. If $v = 2$ when $x = 1$, show that $v^2 = (1 + x^2)^2$

ii. Show that, if $x = \frac{1}{\sqrt{3}}$ when $t = 0$, then $t = \frac{\pi}{6}$ when $x = \sqrt{3}$

c. Prove by Mathematical Induction that $5^{2n} - 1$ is divisible by 6 when n is a positive integer **4**

Question 5 (Start a new page) — Marks

a. At 9 am, an ultralight aircraft flies directly over Daryl's head at 500 metres. It maintains a constant speed of 20 ms^{-1} and a constant altitude. **5**

If x is the horizontal distance travelled by the plane and θ is the angle of elevation from Daryl to the plane,

i. show that $\frac{dx}{d\theta} = -500\operatorname{cosec}^2\theta$.

ii. Hence show that $\frac{d\theta}{dt} = -\frac{1}{25}\sin^2\theta$.

iii. Find the rate of change of the angle of elevation at 9:01 am.

b. Two groups of terrorists are 150 metres from their target. **7**

The first group, Group A, is on the same horizontal level as the target and can fire their missiles in any direction at a speed of 50 ms^{-1}.

i. Show that Group A can hit the target and calculate the angle(s) at which their missiles are to be fired. [Use $g = 10$ ms^{-2}]

The second group, Group B, is positioned in a building 30 metres above the horizontal level of the target and can fire their missile only horizontally through a small window and at 55 ms^{-1}.

ii. Determine whether Group B can hit their target. [Use $g = 10$ ms^{-2}]

Question 6 (Start a new page) — Marks

a. The displacement, x cm, of an object from the origin is given by **5**

$$x = 2\sin t - 3\cos t, \quad t \geq 0$$

where time, t, is measured in seconds.

i. Show that the object is moving in Simple Harmonic Motion.

ii. Find the amplitude of the motion.

iii. At what time does the object first reach its maximum speed?

b. A cup of soup at temperature T°C loses heat when placed in the lounge room. It cools according to the law: **7**

$$\frac{dT}{dt} = k(T - T_0)$$

where t is the elapsed time in minutes and T_0 is the temperature of the room in degrees centigrade.

i. Show that the equation $T = T_0 + Ae^{kt}$ satisfies the above law of cooling.

ii. A cup of soup at 95°C is placed in the freezer at −10°C for 5 minutes and cools to 65°C. Find the exact value of k

iii. The same cup, at 65°C, is then taken into the lounge room where the surrounding temperature is 26°C. Assuming k remains the same, find, to the nearest degree, the temperature of the soup after another 5 minutes.

HSCFocus.com

STUDENT NUMBER / NAME......................................

Question 7 (Start a new page) **Marks**

a. Find the constant term in the expansion of $\left(3x - \frac{1}{x^2}\right)^6$ **3**

b. i. Solve the equation $x^4 + x^2 - 1 = 0$, giving your answer(s) to two decimal places. **9**

ii. On the same axes, draw the graphs of $y = \tan^{-1} x$ and $y = \cos^{-1} x$, showing all important features. Mark the point, P, where the curves intersect.

iii. Show that, if $\tan^{-1} x = \cos^{-1} x$, then $x^4 + x^2 - 1 = 0$. Hence find the coordinates of P.

iv. Find to two decimal places the area enclosed by the curves and the y-axis.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

HSCFocus.com

MATHEMATICS
3 UNIT TRIAL 1999

SUGGESTED ANSWERS

Q1 a. Possibilities are

1,5
2,5
3,5
4,5
5,1 5,2 5,3 5,4 5,5 5,6
6,5

$\therefore$ Probability of total of 8 $= \frac{2}{11}$

b. Let $p =$ prob of supporting A $= 3/10$
$q =$ prob of supporting other $= 7/10$
$n =$ no. of A supporters

Then $P(X=r) = {}^7C_r \left(\frac{3}{10}\right)^r \left(\frac{7}{10}\right)^{7-r}$

$\therefore P(X=4) = {}^7C_4 \left(\frac{3}{10}\right)^4 \left(\frac{7}{10}\right)^3$

$= 0.0972405$

$\doteqdot 0.1$

c. $x = \frac{kx_2 + lx_1}{k+l}$ $\qquad y = \frac{ky_2 + ly_1}{k+l}$

$-1 = \frac{-3 \times x_2 + 1 \times 3}{-3+1}$ $\qquad -4 = \frac{-3 \times y_2 + 1 \times 2}{-3+1}$

$2 = -3x_2 + 3$ $\qquad 8 = -3y_2 + 2$

$x_2 = 1/3$ $\qquad y_2 = -2$

$\therefore B\left(\frac{1}{3}, -2\right)$

d. $u = \cos x$

$du = -\sin x \, dx$

if $x = \pi/2$, $u = 0$

if $x = \pi/3$, $u = 1/2$

$\therefore I = \int_{.5}^{0} -u^3 \, du$

$= \left[\frac{u^4}{4}\right]_0^{.5}$

$= \frac{.5^4}{4} - 0 = \frac{1}{64}$

e. $\int_0^{\pi/4} \cos^2 \frac{1}{2}x \, dx$

$= \frac{1}{2} \int_0^{\frac{\pi}{4}} 1 + \cos x \, dx$

$= \frac{1}{2} \left[x + \sin x\right]_0^{\pi/4}$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$

These suggested answers/marking schemes are issued as a guide only -offerred as an assistance in constructing your own marking format (individual teachers/schools find many other acceptable responses)

HSCFocus.com

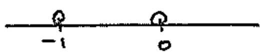
a. $\frac{1}{x+1} \geq 1-x$

Critical points at $x=-1$ and

$\frac{1}{x+1} = 1-x$

$1 = 1-x^2$

$\Rightarrow x = 0$



Test $x=-2$ False

Test $x=-\frac{1}{2} \Rightarrow 2 \geq 1\frac{1}{2}$ $\therefore$ True

$x = 1 \Rightarrow \frac{1}{2} \geq 0$ $\therefore$ True

Solution: $x > -1$

b. $\int_0^{2/5} \frac{dx}{\sqrt{16-25x^2}}$

$= \int_0^{2/5} \frac{dx}{5\sqrt{\frac{16}{25} - x^2}}$

$= \frac{1}{5}\left[\sin^{-1}\frac{x}{4/5}\right]_0^{2/5}$

$= \frac{1}{5}\left[\sin^{-1}\frac{5x}{4}\right]_0^{2/5}$

$= \frac{1}{5}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}0\right)$

$= \frac{1}{5}\cdot\frac{\pi}{6} = \frac{\pi}{30}$

c. (i) $M\left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

(ii) $M_{PQ} = \frac{p+q}{2} = k$, a constant

Then, for the point M,

$x = a(p+q)$

$= a \cdot 2k$

$x = 2ak$

Since a and k are constant, the locus of M is a line parallel to the y-axis

d. $\angle U = \angle V$ (given)

$\angle UZX = \angle VZY$ (vertically opposite)

Now $\angle ZXW = \angle UZX + \angle U$ (exterior angle of triangle)

and $\angle ZYW = \angle VZY + \angle V$ (ditto)

$\therefore$ $\angle ZXW = \angle ZYW$ (equal to sum of equal angles)

In ΔXZW & ΔYZW,

ZW is common

$\angle ZXW = \angle ZYW$ (above)

$\angle XWZ = \angle YWZ$ (given ZW bisects $\angle YWX$)

$\therefore \Delta XZW \equiv \Delta YZW$ (AAS)

and XW = YW

Q3.(a) $2 - \frac{3}{x+2} = \frac{2(x+2)-3}{x+2}$

$= \frac{2x+1}{x+2}$

$\therefore \int_0^1 \frac{2x+1}{x+2}\,dx$

$= \int_0^1 2 - \frac{3}{x+2}\,dx$

$= \left[2x - 3\ln(x+2)\right]_0^1$

$= (2 - 3\ln 3) - (0 - 3\ln 2)$

$= 2 + 3\ln(2/3)$

(b) Let $\cos x - \sqrt{3}\sin x = A\cos(x+\theta)$

$= A\cos x\cos\theta - A\sin x\sin\theta$

Then $A\cos\theta = 1$

$A\sin\theta = \sqrt{3}$

$\Rightarrow \tan\theta = \sqrt{3}$ and $\theta = \pi/3$

and $A = 2$

$\therefore 2\cos\left(x+\frac{\pi}{3}\right) + 1 = 0$

$\cos\left(x+\frac{\pi}{3}\right) = -1/2$

$x + \frac{\pi}{3} = \ldots \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$

$x = \pi, \frac{4\pi}{3}$ in given domain

(c)(i) Let $f(x) = x\ln x - 1$

$f(1) = 1.\ln 1 - 1 < 0$

$f(2) = 2\ln 2 - 1 > 0$

$\therefore$ a solution exists between $x=1$ & $x=2$ (assuming $f(x)$ is continuous)

(ii) $f'(x) = x.\frac{1}{x} + \ln x = \ln x + 1$

By Newton's method,

$x_1 = x - \frac{f(x)}{f'(x)}$

$= x - \frac{x\ln x - 1}{\ln x + 1}$

If $x=2$, $x_1 = 2 - \frac{2\ln 2 - 1}{\ln 2 + 1}$

$= +1.77184832$

$= 1.8$

(d)(i) Total no. of possible teams

$= {}^7C_2 \times {}^5C_2 = 210$

Teams with a particular woman

$= {}^6C_1 \times {}^5C_2 = 60$

$\therefore$ Probability of a particular woman

$= \frac{60}{210} = \frac{2}{7}$

(ii) Captain is specified, so the number of possible teams is ${}^4C_1 \times {}^7C_2 = 84$

No. of teams with his brother is ${}^7C_2 = 21$

$\therefore$ Probability of captain and brother

$= \frac{21}{84} = \frac{1}{4}$

OR with the captain as specified, the probability that his brother is chosen from the remaining four men is $\frac{1}{4}$

HSCFocus.com

MATHS 3U ANSWERS - 1999

4(a) $x^2+y^2-6x+2ky+3k=0$

Completing the squares:

$(x-3)^2+(y+k)^2=k^2-3k+9$

If the centre $(3,-k)$ is on the line $x-3y=0$, then

$3-3x-k=0 \Rightarrow k=-1$

$\therefore C_1: (x-3)^2+(y-1)^2=13$

If C_2 touches the x-axis, the radius is k

$\therefore \sqrt{k^2-3k+9}=k$

$k^2-3k+9=k^2$

$\Rightarrow k=3$

$\therefore C_2: (x-3)^2+(y+3)^2=9$

(b)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=2x^3+2x$

$\therefore \frac{1}{2}v^2=\frac{1}{2}x^4+x^2+C$

If $v=2$, $x=1$

$\therefore \frac{1}{2}.2^2=\frac{1}{2}.1+1+C \Rightarrow C=\frac{1}{2}$

& $\frac{1}{2}v^2=\frac{1}{2}x^4+x^2+\frac{1}{2}$

$v^2=x^4+2x^2+1$

$v^2=(x^2+1)^2$

(ii) so $v=\pm(x^2+1)$

but $v=2\ (>0)$ when $x=1$

$\therefore v=+(x^2+1)$

$\frac{dt}{dx}=\frac{1}{x^2+1}$

so $t=\tan^{-1}x+C$

Now $x=\frac{1}{\sqrt{3}}$ when $t=0$

$\therefore C=-\tan^{-1}\frac{1}{\sqrt{3}}=-\frac{\pi}{6}$

so $t=\tan^{-1}x-\frac{\pi}{6}$

when $x=\sqrt{3}$, $t=\tan^{-1}\sqrt{3}-\frac{\pi}{6}$

$=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$

(c) Let $S(n): 5^{2n}-1=6I$, where I is an integer.

$S(1)$: LHS $=5^2-1=24=6\times 4$

$\therefore S(1)$ is true

Assume $S(k): 5^{2k}-1=6I$ (I, integer)

Consider $S(k+1)$:

LHS $=5^{2k+2}-1$

$=5^{2k}.5^2-1$

$=25(5^{2k}-1)-1+25$

$=25.6I+24$ by $S(k)$

$=6[25I+4]$

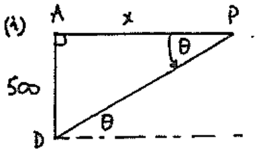
Now I is integer, $\therefore 25I+4$ is integer.

Hence, if $S(k)$ is true, $S(k+1)$ is true

But $S(1)$ is true, so $S(2)$ is true, and then $S(3)$ is true and so on for all integer values of n.

MATHS 3U ANSWERS - 1999

Q5(a)(i)



$\tan\theta=\frac{500}{x}$

$\therefore x=500\cot\theta$

$\frac{dx}{d\theta}=-500\operatorname{cosec}^2\theta$

(ii) $\frac{d\theta}{dt}=\frac{d\theta}{dx}\times\frac{dx}{dt}$

$=\frac{1}{-500\operatorname{cosec}^2\theta}\times 20$

$=-\frac{1}{25}\sin^2\theta$

(iii) At 9:01, $t=60$, $x=1200$

Then $PD=1300$ (Pythagoras' Theorem)

so $\sin\theta=\frac{500}{1300}=\frac{5}{13}$

$\therefore \frac{d\theta}{dt}=-\frac{1}{25}\times\left(\frac{5}{13}\right)^2$

$=-\frac{1}{169}$ degrees/sec.

(b)(i) $\ddot{x}=0$ $\quad \ddot{y}=-10$

$\dot{x}=c_1$ $\quad \dot{y}=-10t+c_2$

Initially $\dot{x}=50\cos\alpha$ $\quad \therefore \dot{x}=50\cos\alpha$

and $\dot{y}=50\sin\alpha$ $\quad \therefore \dot{y}=-10t+50\sin\alpha$

$x=50t\cos\alpha+C_3$ $\quad y=-5t^2+50t\sin\alpha+C_4$

& since $x=0$ when $t=0$, $\quad$ and $y=0$ when $t=0$

$C_3=0$ $\quad C_4=0$

$\therefore x=50t\cos\alpha$ $\quad \therefore y=-5t^2+50t\sin\alpha$

when $x=150$, $150=50t\cos\alpha$

so $3=t\cos\alpha$ --- (1)

when $y=0$, $0=-5t^2+50t\sin\alpha$

$=-5t(t-10\sin\alpha)$

$\Rightarrow t=10\sin\alpha$ (2)

Solving (1) + (2):

$3=10\sin\alpha\cos\alpha$

$=5\sin 2\alpha$

$\therefore \sin 2\alpha=3/5$

$2\alpha=36°52', 143°08'$

$\therefore \alpha=18°26'$ or $71°29'$

(ii) $\ddot{x}=0$ $\quad \ddot{y}=-10$

$\dot{x}=C_1$ $\quad \dot{y}=-10t+C_2$

Initially, $\dot{x}=55\cos\alpha$, $\dot{y}=55\sin\alpha$

$\therefore \dot{x}=55\cos\alpha$ $\quad \dot{y}=-10t+55\sin\alpha$

$\dot{x}=55$ $\quad \dot{y}=-10t$ since $\alpha=0$

Then $x=55t+C_3$ $\quad y=-5t^2+C_4$

When $t=0$, $x=0$ and $y=30$

$\Rightarrow C_3=0$ $\quad C_4=30$

$\therefore x=55t$ $\quad y=-5t^2+30$

Now when $y=0$, $-5t^2+30=0$

$\therefore t^2=6$

& $t=\sqrt{6}$

At $t=\sqrt{6}$, $x=55\sqrt{6}$

$\simeq 135$ m

$\therefore$ Group B cannot reach the target

HSCFocus.com

MATHS 3U ANSWERS - 1999

(a)(i) $x = 2\sin t - 3\cos t$

$\dot{x} = 2\cos t + 3\sin t$

$\ddot{x} = -2\sin t + 3\cos t$

$= -(2\sin t + 3\cos t)$

$= -x$

$\therefore$ motion is simple harmonic.

(ii) Amplitude $= \sqrt{2^2+3^2}$

$= \sqrt{13}$ cm

(iii) $\dot{x} = 2\cos t + 3\sin t$

$\ddot{x} = -2\sin t + 3\cos t$

Max velocity when $\ddot{x} = 0$

$-2\sin t + 3\cos t = 0$

$3\cos t = 2\sin t$

$3/2 = \tan t$

$t = 0.983, 4.1243 \ldots$ etc

$\therefore$ reaches maximum velocity when $t = 0.983$

b)(i) $T = T_0 + Ae^{kt}$

$\frac{dT}{dt} = k.Ae^{kt}$

$= k(T - T_0)$

(ii) When $t=0$, $T=95$, $T_0=-10$

$\Rightarrow A = 105$

when $t=5$, $T=65$

$\therefore 65 = -10 + 105e^{5k}$

$e^{5k} = 75/105 = 5/7$

$\ln e^{5k} = \ln 5/7$

$\therefore k = \frac{1}{5}\ln\frac{5}{7}$

(iii) When $t=0$, $T_0=26$ & $T=65$

$\therefore 65 = 26 + Be^{k.0}$

$\therefore B = 39$

Therefore, at $t=5$,

$T = 26 + 39e^{5k}$ with $k = \frac{1}{5}\ln\frac{5}{7}$

so $T = 53.86°$

$= 54°$ (to the nearest degree)

MATHS 3U ANSWERS - 1999

Q7(a) $\left(3x - \frac{1}{x^2}\right)^6 = \sum_{r=0}^{6} {}^6C_r (3x)^{6-r}\left(-\frac{1}{x^2}\right)^r$

Typical term, T_r,

$T_r = {}^6C_r\, 3^{6-r} . x^{6-r} . (-1)^r . (x^{-2})^r$

$= {}^6C_r\, 3^{6-r} (-1)^r x^{6-3r}$

Constant term when $6 - 3r = 0$

$r = 2$

then $T_2 = {}^6C_2\, 3^4 (-1)^2$

$= 1215$

b)(i) $x^4 + x^2 - 1 = 0$

$x^2 = \frac{-1 \pm \sqrt{1 - 4\times1\times-1}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2}$

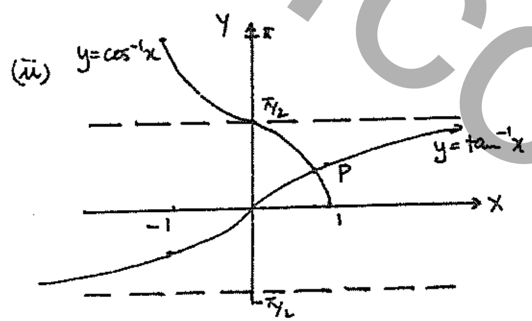
$\therefore x^2 = \cancel{\frac{-1-\sqrt{5}}{2}}$ or $\frac{-1+\sqrt{5}}{2}$

$x^2 = 0.618033988$

$x = \pm 0.786151377$

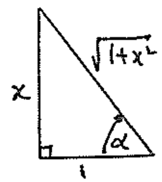
$= \pm 0.79$

(ii)



(iii) let $\tan^{-1}x = \alpha$

$\therefore x = \tan\alpha$



At P, $\cos^{-1}x = \tan^{-1}x = \alpha$

$\therefore$ at P $\cos^{-1}x = \alpha$ & $x = \cos\alpha$

But $\cos\alpha = \frac{1}{\sqrt{1+x^2}}$ (from diagram)

$\therefore x = \frac{1}{\sqrt{1+x^2}}$

Squaring, $x^2 = \frac{1}{1+x^2}$

& $x^4 + x^2 = 1$

$x^4 + x^2 - 1 = 0$

$\therefore x = 0.79$ (from (i))

and $y = \tan^{-1}0.79 = 0.6686$

so $P(0.79, 0.67)$

(iv) $A = \int_0^{0.67} \tan y\, dy + \int_{0.67}^{\pi/2} \cos y\, dy$

$= \left[-\ln|\cos y|\right]_0^{0.67} + \left[\sin y\right]_{0.67}^{\pi/2}$

$= -\ln|\cos 0.67| + \sin\frac{\pi}{2} - \sin 0.67$

$= 0.62258$

$= 0.62$ (to 2-decimal places)

WESTERN REGION

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

3 Unit (Additional)
and
3/4 Unit (Common)

Time allowed - TWO hours
(Plus 5 minutes reading time)

HSCFocus.com

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on the last page. These may be removed for your convenience.
- Board-approved calculators may be used.
- Each question should be started on a new page.

Question One. *Start a new page* **Marks**

(a) Find a general solution for x if $\tan x = \frac{1}{\sqrt{3}}$ (in terms of π). **2**

(b) If α, β and χ are the roots of

$$3x^3 + 5x^2 - 7x + 4 = 0.$$

Find the values of $\alpha\beta + \alpha\chi + \beta\chi$ and $\alpha + \beta + \chi$. **2**

(c) Solve $\sec^2 x + \tan x - 7 = 0$ for $0° \le x \le 360°$ (to the nearest minute). **3**

(d) Given that **2**

$$x = \cos\theta + 1 \text{ and } y = \sin\theta - 2$$

By eliminating θ determine a relationship between x and y only.

(e) Solve $\frac{2x}{x-3} \le 1$ if $x \ne 3$. **3**

Question Two. *Start a new page* **Marks**

(a) Given $\frac{x^3 + 3x^2 + 9x - 1}{x^2 + 9} = Q(x) + \frac{R(x)}{x^2 + 9}$ **4**

(i) By performing the division or otherwise find $Q(x)$ and $R(x)$.

(ii) Hence find $\int \frac{x^3 + 3x^2 + 9x - 1}{x^2 + 9}\, dx$

(b) Using Mathematical Induction prove that $7^n - 5^n$ is even for all positive integer n. **3**

(c) Eight people are seated about a round table. **3**

(i) How many different seating arrangements are possible ?

(ii) If the eight people consist of four couples find the probability that each person is seated adjacent to their partner.

(d) Given that the quadrilateral ABCD is cyclic show that the sum of the tangents of the angles is zero. **2**

That is, $\tan A + \tan B + \tan C + \tan D = 0$

HSCFocus.com

Question Three. *Start a new page* — **Marks**

(a) Evaluate $\int_0^1 \frac{2x}{(2x+1)^2}\,dx$ — **3**

using the substitution $u = 2x + 1$.

(b) By making suitable substitutions for A and B in the expansion of $\cos(A+B)$ find the exact value of $\cos 75°$. — **2**

(c) If $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$ determine the values of R and α. — **3**

(d) Find the constant term in the expansion of — **2**

$$\left(2x - \frac{1}{x^3}\right)^{20}$$

(Do not evaluate).

(e) If $f(x) = \sin^{-1} x + \cos^{-1} x$ and $1 \ge x \ge 0$, find — **2**

(i) $f'(x)$

(ii) $\int_0^1 f(x)\,dx$

Question Four. *Start a new page* — **Marks**

(a) A Horticulturalist knows that the probability of grafting a particular rose successfully is 0·4. — **5**

(i) If ten grafts are made find the probability of six successful grafts. Give your answer correct to 3 significant figures.

(ii) How many grafts must be made to ensure that the probability of at least one success is at least 99·9 %.

(b) A spherical balloon is being inflated at the rate of $1000\ \text{cm}^3\ \text{s}^{-1}$ and given that — **5**

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad A = 4\pi r^2$$

find :

(i) An expression for the instantaneous rate of change of the radius in terms of r. $\left(\text{Find } \frac{dr}{dt}\right)$.

(ii) The rate of change of the surface area of the balloon when the radius is 10 cm.

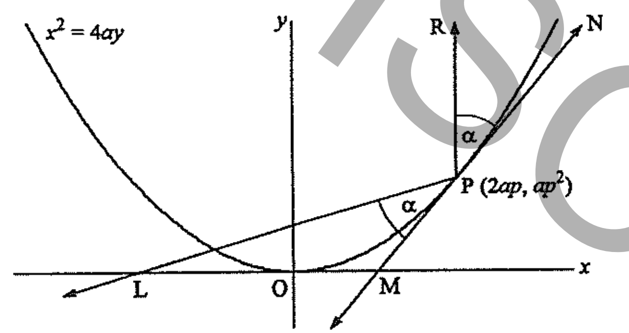
(c) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ — **2**

HSCFocus.com

Question Five. *Start a new page* **Marks**

(a) It is known that $f(x)=\frac{1}{2}x-\sin x$ has a root of $f(x)=0$ near $x=2$.
By using one application of Newton's method find a better approximation to two decimal places. **2**

(b) **7**



The parabola $x^2=4ay$ is shown in the sketch above.
The tangent at $P(2ap, ap^2)$ cuts the x axis at M and passes through the point N.
PR is parallel to the axis of the parabola and makes $\angle RPN=\alpha°$.
PL is such that it cuts the x axis at L and $\angle LPM=\alpha°$.

(i) Show that $\tan\alpha=\frac{1}{p}$.

(ii) Show that the gradient of LP is $\frac{p^2-1}{2p}$.

(iii) Show that the line LP passes through the focus of the parabola.

(c) A particle is projected such that at any time 't', the equation of the trajectory is given by **3**

$$x=36t \quad\text{and}\quad y=15t-\frac{1}{2}gt^2$$

find the angle of projection to the nearest minute.

Question Six. *Start a new page* **Marks**

(a) (i) State the domain and range of **3**

$$y=2\sin^{-1}(3x).$$

(ii) Sketch $y=2\sin^{-1}(3x)$.

(b) The rate at which a body cools is proportional to the difference between its temperature (T) and the constant temperature of the surrounding air (S). **7**
That is

$$\frac{dT}{dt}=k(T-S)$$

where t is the time elapsed and k is a constant.

(i) Show that $T=S+Be^{kt}$, where B is a constant, is a solution of the above differential equation.

(ii) A body cools from 150° C to 90° C in three hours. If the air temperature is 30° C find the value of B and hence the value of k to 3 decimal places.

(iii) Using the values of B and k found in (ii) determine the temperature of the body after a further three hours.

(c) Find the coordinates of the point P which divides the interval joining A (−1 , 2) and B (5 , −3) internally such that AP : PB = 3 : 2. **2**

HSCFocus.com

Question Seven. *Start a new page* **Marks**

(a) Two circles touch internally at P. A line through P cuts the smaller circle at A and the larger circle at B.
A second line through P cuts the smaller and larger circles at C and D respectively. **4**

(i) Sketch this information.

(ii) Prove that the line joining A and C is parallel to the line joining B and D.

(b) You are given that x, v, a and t represent displacement, velocity, acceleration and time elapsed respectively. **2**
Show that

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

(c) A particle moves in a straight line such that its acceleration is given by **6**

$$a = -e^{-2x}$$

If $v = 1$ and $x = 0$ when $t = 0$

(i) Express v in terms of x. (Use the result proven in (b))

(ii) Express x in terms of t.

(iii) Express v in terms of t.

HSCFocus.com

WESTERN REGION

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

3 Unit (Additional)
and
3/4 Unit (Common)

Solutions and Marking Scheme

PLEASE NOTE :

- These are suggested solutions. They are not intended to specify the amount of working required or the method to be applied.
- Teachers should accept any valid method of solution providing adequate working is shown.

HSCFocus.com

Course : 3 UNIT

Solutions	Marks	Comments
Question 1		
(a) $\tan x = \frac{1}{\sqrt{3}}$		
$\therefore x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$ for $2\pi \geq x \geq 0$		
$\therefore$ General solution		
$x = \frac{\pi}{6} \pm 2n\pi, \pi + \frac{\pi}{6} \pm 2n\pi$	1	
$= \frac{\pi}{6} + n\pi$ for all integral n	1	Accept $\frac{\pi}{6} \pm n\pi$
(b) $3x^3 + 5x^2 - 7x + 4 = 0$		
$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{7}{3}$	1	
$\alpha + \beta + \gamma = -\frac{5}{3}$	1	
(c) $\sec^2 x + \tan x - 7 = 0$		
$1 + \tan^2 x + \tan x - 7 = 0$	1	
$\tan^2 x + \tan x - 6 = 0$		
$(\tan x + 3)(\tan x - 2) = 0$		
$\therefore \tan x = -3, 2$	1	
$\therefore x = 108°26', 288°26', 63°26', 243°26'$	1	

HSCFocus.com

Solutions	Marks	Comments
Question 1 (continued)		
(d) $x = \cos\theta + 1$		
$\therefore x - 1 = \cos\theta$ — ①		
$y = \sin\theta - 2$		
$y + 2 = \sin\theta$ — ②	1	
$①^2 + ②^2$		
$(x-1)^2 + (y+2)^2 = \sin^2\theta + \cos^2\theta$		
$\therefore (x-1)^2 + (y+2)^2 = 1$ is the required equation	1	
(e) $\frac{2x}{x-3} \le 1 \qquad x \ne 3$		Some may use another method. If solution is correct give full marks
If $x - 3 > 0$		
$x > 3$		
$2x \le x - 3$		
$x \le -3$		
No solution	1	
If $x - 3 < 0$		
$x < 3$		
$\therefore 2x \ge x - 3$		
$x \ge -3$	1	
$\therefore$ Solution is $3 > x \ge -3$	1	

Solutions	Marks	Comments
Question 2		
(a) (i) $x^2+9 \overline{)x^3+3x^2+9x-1}$ quotient $x+3$; $x^3 \quad +9x$; $3x^2 \quad -1$; $3x^2 \quad +27$; -28		
$\therefore Q(x) = x + 3 \quad R(x) = -28$	2	
(ii) $\int \frac{x^3+3x^2+9x-1}{x^2+9}\,dx$		
$= \int \left(x + 3 - \frac{28}{x^2+9}\right) dx$		
$= \frac{1}{2}x^2 + 3x - \frac{28}{3}\tan^{-1}\frac{x}{3} + C$	2	Do not worry about C
(b) Let $f(n) = 7^n - 5^n$		
$\therefore f(1) = 7 - 5$		
$= 2$		
$\therefore$ True for $n = 1$		
Assume true for $n = k$	1	
$\therefore f(k) = 7^k - 5^k = 2A$		
where A is an integer		
when $n = k+1$		
$f(k+1) = 7^{k+1} - 5^{k+1}$		Some may test $f(k+1) - f(k)$ even
$= 7 \times 7^k - 5 \times 5^k$		
$= 7(7^k - 5^k) + 2 \times 5^k$		
$= 7 \times 2A + 2 \times 5^k$	1	
$= 2(7A + 5^k)$		
$\therefore$ True as $7A + 5^k$ is an integer		
$\therefore$ If true for $n=1$ then true $n=2$		
" " " $n=2$ " " $n=3$ etc.		
$\therefore$ True for all $n \ge 0$ integers	1	

Solutions	Marks	Comments
Question 2 (continued)		
(c)(i) Number of arrangements $= (8-1)!$ $= 5040$	1	
(ii) Number of couple arrangements $= 2^4 \times (4-1)!$ $= 96$	1	
$\therefore$ Probability of pairs together $= \frac{96}{5040}$ $= \frac{2}{105}$	1	
(d) Quadrilateral is cyclic $\therefore$ opposite angles are supplementary $\therefore \angle C = 180° - \angle A$, $\angle D = 180° - \angle B$	1	
$\therefore \tan A + \tan B + \tan C + \tan D$ $= \tan A + \tan B + \tan(180-A) + \tan(180-B)$ $= \tan A + \tan B - \tan A - \tan B$ $= 0$	1	

Solutions	Marks	Comments
Question 3		
(a) $\int_0^1 \frac{2x\,dx}{(2x+1)^2}$; $u = 2x+1$, $\frac{du}{dx} = 2$, $dx = \frac{du}{2}$, $x=1$ $u=3$, $x=0$ $u=1$		
$\therefore \int_1^3 \frac{u-1}{u^2}\frac{du}{2}$	1	
$= \frac{1}{2}\int_1^3 \left(\frac{1}{u} - \frac{1}{u^2}\right)du$ $= \frac{1}{2}\left[\ln u + \frac{1}{u}\right]_1^3$	1	
$= \frac{1}{2}\left(\ln 3 + \frac{1}{3}\right) - \frac{1}{2}(\ln 1 + 1)$ $= \frac{1}{2}\left(\ln 3 - \frac{2}{3}\right)$	1	
(b) $\cos(A+B) = \cos A\cos B - \sin A \sin B$; $\cos 75° = \cos(45° + 30°)$	1	
$\therefore \cos 75° = \cos 45° \cos 30° - \sin 45° \sin 30°$ $= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$ $= \frac{\sqrt{3}-1}{2\sqrt{2}}$	1	
(c) $\sqrt{3}\cos x - \sin x$ $= R\cos x\cos\alpha - R\sin x\sin\alpha$ $\therefore R\cos\alpha = \sqrt{3}$, $R\sin\alpha = 1$	1	
$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$, $\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = 30°$; $R^2\sin^2\alpha + R^2\cos^2\alpha = 1+3$, $R^2 = 4$, $\therefore R = 2$	2	1each

Solutions	Marks	Comments
Question 3 (continued)		
(d) General Term $= {}^{20}C_r (2x)^{20-r}\left(-\frac{1}{x^3}\right)^r$		
Ignoring coefficient to find r		
$x^{20-r} \cdot (x^{-3})^r = x^0$		
$x^{20-4r} = x^0$	1	
$\therefore 4r = 20$		
$r = 5$		
$\therefore$ Term is ${}^{20}C_5 \cdot 2^{15} \cdot (-1)^5$		
$= -{}^{20}C_5 \cdot 2^{15}$	1	
-508035072		
(e) $f(x) = \sin^{-1}x + \cos^{-1}x$		
$= \frac{\pi}{2}$		
(i) $f'(x) = 0$	1	
(ii) $\int_0^1 f(x)\,dx = \int_0^1 \frac{\pi}{2}\,dx$		
$= \left[\frac{\pi}{2}x\right]_0^1$		
$= \frac{\pi}{2} - 0$		
$= \frac{\pi}{2}$	1	

Solutions	Marks	Comments
Question 4		
(a)(i) Let $P(S) = 0.4$ and $P(F) = 0.6$		
$\therefore P(S=6) = {}^{10}C_6 (0.6)^4 (0.4)^6$		
$= 0.111$ (3sig figs)	2	
(ii) Let there be n grafts		
$\therefore P(F=n) = (0.6)^n$		
$\therefore P(S \geq 1) = 1 - (0.6)^n \geq 0.999$	1	
$\therefore 0.001 \geq (0.6)^n$		
$\frac{\ln(0.001)}{\ln(0.6)} \leq n$	1	
$13.523 \leq n$		
$\therefore$ 14 or more grafts are required.	1	
(b)(i) Given $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dt} = 1000$		
$\frac{dV}{dr} = 4\pi r^2$	1	
$\therefore \frac{dr}{dt} = \frac{dV}{dt} \div \frac{dV}{dr}$		
$= \frac{1000}{4\pi r^2}$		
$= \frac{250}{\pi r^2}$	1	
Radius is changing at $\frac{250}{\pi r^2}$ cm s^{-1}		

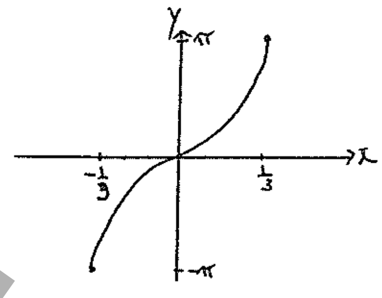
HSCFocus.com

Solutions	Marks	Comments
Question 4 (continued)		
(b) (ii) Given $A = 4\pi r^2$		
$\frac{dA}{dr} = 8\pi r$		
$\therefore \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$	1	
$= 8\pi r \times \frac{250}{\pi r^2}$		
$= \frac{2000}{r}$	1	
when $r = 10$ $\frac{dA}{dt} = \frac{2000}{10} = 200$		
Area is changing at $200\ cm^2 s^{-1}$	1	
(c) $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)\,dx$		
$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$	1	
$= \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right) - 0\right]$		
$= \frac{\pi}{4}$	1	

Solutions	Marks	Comments
Question 5		
(a) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$	1	
$f(x) = \frac{1}{2}x - \sin x$		
$f'(x) = \frac{1}{2} - \cos x$		
$x_1 = 2$ $\therefore x_2 = 2 - \frac{\frac{1}{2}\cdot 2 - \sin 2}{\frac{1}{2} - \cos 2}$	1	
$= 1\cdot 90$		
(b)(i) Gradient of tangent $= p$		
$\angle PMX = 90 - \alpha$		
$\therefore$ Gradient of $MN = \tan(90-\alpha)$	1	
$\therefore$ $p = \tan(90-\alpha)$		
$= \cot\alpha$		
$\frac{1}{p} = \tan\alpha$	1	
(ii) $\angle PLM + \angle LPM = \angle PMX$ (exterior $\angle$ $\triangle PLM$)	1	
$\therefore \angle PLM + \alpha = 90 - \alpha$		
$\angle PLM = 90 - 2\alpha$ ✓	1	
$\therefore$ Gradient $LP = \tan(90-2x)$	1	
$= \cot 2x$		
$= \frac{1-\tan^2 x}{2\tan x}$		
$= \frac{1-\frac{1}{p^2}}{2\cdot\frac{1}{p}}$		
$= \frac{p^2-1}{2p}$	1	

HSCFocus.com

Solutions	Marks	Comments
Question 5 (continued)		
(b)(iii) Coords. of S are $(0, a)$		
Gradient PS $= \frac{ap^2 - a}{2ap}$		Other ways are O.K.
$= \frac{p^2-1}{2p}$	1	
Gradient PS = Gradient LP		
$\therefore$ LP passes through S		
(c) $x = 36t$ $y = 15t - \frac{1}{2}gt^2$		
$\frac{dx}{dt} = 36$ $\frac{dy}{dt} = 15 - gt$		
$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$		Give marks for correct method
$= \frac{15 - gt}{36}$	2	
If θ is the angle of projection		
$\tan\theta = \frac{15 - gt}{36}$ when $t = 0$		
$= \frac{15}{36}$		
$\therefore \theta = \tan^{-1}\frac{15}{36}$	1	
$= 22°37'$		

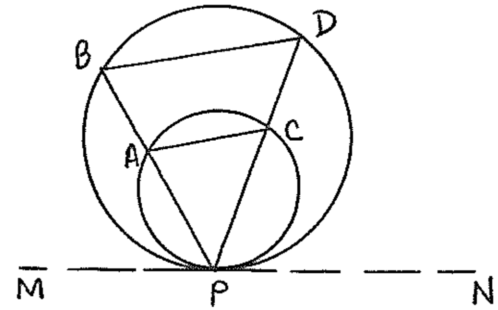
Solutions	Marks	Comments
Question 6		
(a)(i) $y = 2\sin^{-1}(3x)$.		
$1 \geq 3x \geq -1$		
$\therefore$ Domain $\frac{1}{3} \geq x \geq -\frac{1}{3}$	1	
Range $\pi \geq y \geq -\pi$	1	
(ii) 	1	
(b)(i) $T = S + Be^{kt}$ —— ①		
$\frac{dT}{dt} = Bke^{kt}$	1	
Now $Be^{kt} = T - S$ from ①		
$\therefore Bke^{kt} = k(T - S)$		
$\therefore$ It is a solution.	1	

Course : 3UNIT Page No : 12

Solutions	Marks	Comments
Question 6 (continued)		
(ii) $T = S + Be^{kt}$		
$S = 30°$, $T = 150°$ when $t = 0$		
$150° = 30° + Be^{0}$		
$120° = B$	2	
when $t = 3$ $T = 90°$		
$\therefore 90 = 30 + 120e^{3k}$		
$60 = 120e^{3k}$		
$0.5 = e^{3k}$		
$\therefore K = \frac{1}{3}\ln 0.5$		
$= -0.231$ (3 d.p.)	2	
(iii) when $t = 6$		
$T = 30° + 120 \times e^{-0.231 \times 6}$		
$= 60°$		
Temperature is 60°	1	
(c) $x = \frac{kx_2 + lx_1}{k+l}$ $y = \frac{ky_2 + ly_1}{k+l}$		
$= \frac{3\times5 + 2\times-1}{3+2}$ $= \frac{3\times-3 + 2\times2}{3+2}$		
$= 2\frac{3}{5}$ $= -1$		
$\therefore$ Coordinates of P are		
$(2\frac{3}{5}, -1)$		

1998 Western Region Trial HSC Marking Scheme

Course : 3UNIT Page No : 13

Solutions	Marks	Comments
Question 7		
(a)(i) 	1	
(ii) Draw tangent MN through P as shown		
$\angle ACP = \angle APM$ (Angle in alternate segment.)	1	
$\angle BDP = \angle BPM$ (" " ")		
$\angle BPM = \angle APM$ (Same angle)	1	
$\therefore \angle ACP = \angle BDP$ ($= \angle APM$)	1	
$\therefore BD \parallel CA$ corresponding angles equal	1	
(b) $\frac{d}{dx}(\frac{1}{2}v^2) = \frac{d}{dv}(\frac{1}{2}v^2) \cdot \frac{dv}{dx}$		
$= v \cdot \frac{dv}{dx}$	1	
$= \frac{dx}{dt} \times \frac{dv}{dx}$		
$= \frac{dv}{dt}$		
$= a$	1	

Solutions	Marks	Comments
Question 7 (Continued)		
(c)(i) $\frac{d}{dx}\left(\frac{1}{2}V^2\right) = -e^{-2x}$		
$\frac{1}{2}V^2 = \frac{1}{2}e^{-2x} + C$	1	
Given $V=1$ when $x=0$		
$\frac{1}{2} = \frac{1}{2}e^0 + C$		
$\therefore C = 0$		
$\frac{1}{2}V^2 = \frac{1}{2}e^{-2x}$		
$V^2 = e^{-2x}$		
$V = e^{-x}$	1	strictly speaking $V = \pm e^{-x}$
(ii) $\frac{dx}{dt} = e^{-x}$		
$\therefore \frac{dt}{dx} = e^{x}$	1	
$t = e^{x} + D$		
When $t=0$ $x=0$		
$\therefore D = -1$	1	
$\therefore t = e^{x} - 1$		
$t+1 = e^{x}$		
$\therefore x = \ln(t+1)$	1	
(iii) $V = \frac{dx}{dt} = \frac{1}{t+1}$	1	

HSCFocus.com

Student Number: ....

1999
TRIAL EXAMINATION

MATHEMATICS
3 Unit (Additional)
and
3/4 Unit (Common)

Time Allowed - Two (2) hours
(plus 5 minutes reading time)

HSCFocus.com

Directions to Candidates

- Attempt ALL questions
- Show all necessary working, marks may be deducted for careless or untidy work
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- Additional Answer Booklets are available

Directions to School or College

To ensure maximum confidentiality and security, examination papers must NOT be removed from the examination room. Examination papers may not be returned to students or used for revision purposes till September 1999. These examination papers are supplied Copyright Free, as such the purchaser may photocopy and/or cut and paste same for educational purposes within the confines of the school or college.

All care has been taken to ensure that this examination paper is error free and that it follows the style, format and material content of previous Higher School Certificate Examination. Candidates are advised that authors of this examination paper cannot in any way guarantee that the 1999 HSC Examination will have a similar content or format.

Question One

(a) For the function $f(x) = e^{x+1}$ find the inverse function $f^{-1}(x)$ and hence show that $f[f^{-1}(x)] = f^{-1}[f(x)] = x$ *(3 marks)*

(b) Solve the inequality $\frac{1}{x+2} \geq \frac{2}{x-3}$ and represent the solution on a number line. *(3 marks)*

(c) If $\sum_{r=1}^{n} \frac{3}{2}(2)^{r-1} = 766\frac{1}{2}$, find n *(3 marks)*

(d) The word **EQUATION** contains all five vowels. How many 3-letter 'words' consisting of at least 1 vowel and 1 consonant can be made from the letters of **EQUATION**? *(3 marks)*

Question Two

(a) Prove that $\frac{2\cos A}{\operatorname{cosec} A - 2\sin A} = \tan 2A$ *(3 marks)*

(b) Show that $\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}}$ *(3 marks)*

(c) Write down the expansion for $\sin(\alpha - \beta)$ and hence prove that $\sin(-\theta) = -\sin\theta$ *(3 marks)*

(d) Find $\frac{d}{dx}\left[\frac{\ln x}{x}\right]$ and hence find the primitive function of $\frac{2 - \ln x}{x^2}$ *(4 marks)*

Question Three

(a) The sides of a square sheet of cardboard are each 12m long. At each corner a square of $x^2\text{m}^2$ is cut away. The sides of the sheet are then turned up to form a box. Calculate:

(i) the values of x so that the box has a volume of 108m^3

(ii) the value of x so that the box has a maximum volume *(5 marks)*

(c) Use mathematical induction to show that for all positive integers n, $\sum_{r=1}^{n} a^{-r} = \frac{a^n - 1}{(a-1)a^n}$ *(4 marks)*

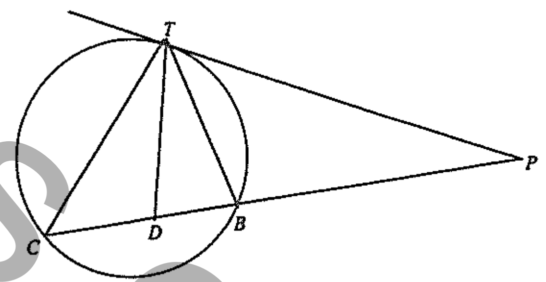
(d) A polynomial $P(x) = ax^3 + bx^2 + cx + d$ has zeroes at -2, 2 and $\frac{3}{2}$.
It leaves a remainder of 12 when divided by x - 1.
Find the values of a, b, c and d. *(3 marks)*

Question Four

(a) The tangent at the point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ cuts the y-axis at T. The line through the focus S parallel to this tangent cuts the directrix at V. M is the mid-point of TV. Find the locus of M as P moves on the parabola. *(5 marks)*

(b) If $3^x = 2^y = 6^z$, prove that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ *(3 marks)*

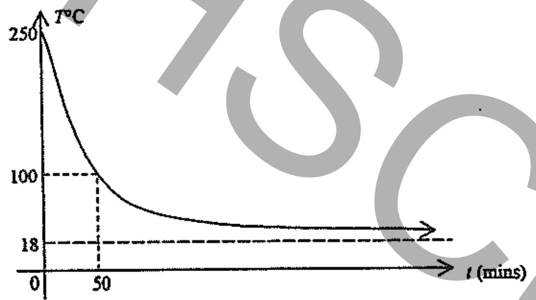
(c)



PT is a tangent to the circle and PBC is a secant. D is a point on PBC such that $TD = TB$.
Prove that $\angle CTD = \angle P$. *(4 marks)*

Question Five

(a) The graph shown below shows the cooling curve for a container of paraffin oil which as been heated to a temperature of 250° then allowed to cool in air whose temperature is 18°C.



It is known that the rate at which the temperature T of the oil is changing is given by $\frac{dT}{dt} = k(T - M)$
where M is the temperature of the surrounding air and t is the time elapsed after cooling begins, in minutes.

(i) Show that $T = M + Ae^{kt}$ is a solution to the given equation.

(ii) Use the graph to write down the values of M and A.

(iii) Find the value of k to one significant figure if the temperature of the oil drops to 103.3°C in 50 minutes of cooling. *(6 marks)*

(b) Write the equation $2\tan\theta - 3\sec\theta = -2\sqrt{3}$ in the form $a\sin\theta + b\cos\theta = c$, and then by expressing the left hand side as sine of a compound angle, solve the equation for $0 \le \theta \le 360°$ *(6 marks)*

Question Six

(a) A particle moving in simple harmonic motion, passes through the centre of oscillation O with a velocity of 5cm/s. If it has a period of π seconds, find

(i) the value of n

(ii) the amplitude of the motion

(iii) the time taken for the particle to first reach $x = 1.5$cm *(5 marks)*

(b) Express $\sec(\sin^{-1} x)$ in terms of x and hence write down $\int \sec(\sin^{-1} x)dx$ $(-1<x<1)$ *(2 marks)*

(c) The acceleration of a body moving in a straight line is given by

$$\frac{d^2x}{dt^2} = -\frac{24}{x^2}$$

where x is the displacement from the origin after t seconds. When $t = 0$ the body is 3 metres to the right of the origin with a velocity of 4m/s.

(i) Show that the velocity v of the body in terms of x, is

$$v = \frac{4\sqrt{3}}{\sqrt{x}}$$

(ii) Find an expression for t in terms of x

(iii) How long does it take for the body to reach a point 10m to the right of the origin? *(5 marks)*

Question Seven

(a) (i) Show that the range of flight of a projectile fired at an angle of α to the horizontal and at a velocity v is

$$\frac{v^2 \sin 2\alpha}{g}$$

where g is the acceleration due to gravity

(ii) A cannon fires a shell at an angle of 45° to the horizontal and strikes a point 50m beyond its target. When fired with the same velocity at an angle of 30° it hits a point 20m in front of the target. Calculate

(I) the distance of the target from the cannon

(II) the correct angles required to hit the target *(7 marks)*

(b) By considering the value of $(1+x)^{2n}$ when $x = 1$, prove that

$$\sum_{k=0}^{n} \binom{2n}{k} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$$ *(5 marks)*

SUGGESTED SOLUTIONS for 3 UNIT MATHS - 1999

Question One

(a)

$f(x) = y = e^{x+1}$
$x = e^{y+1}$

$y + 1 = \ln x$
$y = \ln x - 1$ the inverse function $f^{-1}(x)$

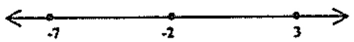
$$f[f^{-1}(x)] = e^{\ln x - 1 + 1} = e^{\ln x} = x$$

$$f^{-1}[f(x)] = \ln[e^{x+1}] - 1 = (x+1)\ln e - 1 = x + 1 - 1 = x$$

(b)

$$\frac{1}{x+2} \geq \frac{2}{x-3}$$
$(x-3)^2(x+2) \geq 2(x-3)(x+2)^2$
$(x-3)^2(x+2) - 2(x-3)(x+2)^2 \geq 0$
$(x-3)(x+2)[x-3-2(x+2)] \geq 0$
$(x-3)(x+2)[x-3-2x-4] \geq 0$
$(x-3)(x+2)(-x-7) \geq 0$ ....(1)

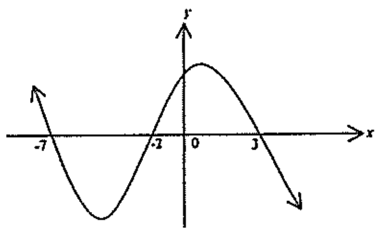
Consider $(x-3)(x+2)(-x-7) = 0$



Test x = -10, inequality (1) holds true
Test x = -5, inequality (1) does not hold true
Test x = 0, inequality (1) holds true
Test x = 4, inequality (1) does not hold true

$\therefore$ Solution is $x \leq -7$ or $-2 < x < 3$
$(x \neq -2), (x \neq 3)$

From the graph of $y = (x-3)(x+2)(-x-7)$



Solution is $x \le -7$ or $-2 < x < 3$

(c) $\sum_{r=1}^{n} \frac{3}{2}(2)^{r-1} = \frac{3}{2} + \frac{3}{2}(2) + \frac{3}{2}(2)^2 + \ldots + \frac{3}{2}(2)^n$

$a = \frac{3}{2}; r = 2$

$S_n = \frac{a(r^n - 1)}{r-1}$

$\therefore \quad \frac{\frac{3}{2}(2^n - 1)}{2-1} = 766\frac{1}{2}$

$\frac{3}{2}(2^n - 1) = \frac{1533}{2}$

$3(2^n - 1) = 1533$

$2^n - 1 = 511$

$2^n = 512$

$n = 9$

(d) There are 5 vowels and 3 consonants. For words with 2 vowels and 1 consonant the number of selections is ${}^5C_2 \times {}^3C_1 = 30$. For words with 1 vowel and 2 consonants the number of selections is ${}^5C_1 \times {}^3C_2 = 15$.

$\therefore$ The total number of selections = 45

But each selection may be arranged 3! ways.

$\therefore$ The total number of words $= 45 \times 3!$

$= 270$

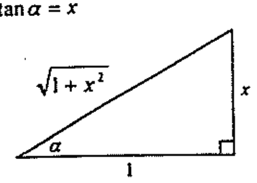
Question Two

(a)

$$\text{LHS} = \frac{2\cos A}{\operatorname{cosec} A - 2\sin A}$$
$$= \frac{2\cos A}{\frac{1}{\sin A} - 2\sin A}$$
$$= \frac{2\cos A}{\frac{1 - 2\sin^2 A}{\sin A}}$$
$$= \frac{2\sin A\cos A}{1 - 2\sin^2 A}$$
$$= \frac{\sin 2A}{\cos 2A}$$
$$= \tan 2A$$
$$= \text{RHS}$$

(b) Let $\tan^{-1} x = \alpha$

$\therefore \tan\alpha = x$



From the triangle $\quad \sin\alpha = \frac{x}{\sqrt{1+x^2}}$

$\therefore \alpha = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\therefore \tan^{-1} x = \frac{\sin^{-1} x}{\sqrt{1+x^2}}$

(c) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Put $\alpha = 0$ and $\beta = \theta$

$\therefore$

$$\sin(-\theta) = \sin 0\cos\theta - \cos 0\sin\theta$$
$$= 0 - (1)\sin\theta$$
$$= -\sin\theta$$

(d)

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x \cdot \frac{1}{x} - \ln x(1)}{x^2}$$
$$= \frac{1 - \ln x}{x^2}$$

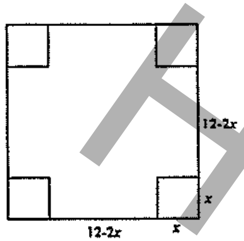
$\therefore$

$$\int\left(\frac{2 - \ln x}{x^2}\right)dx = \int\left(\frac{1 + 1 - \ln x}{x^2}\right)dx$$
$$= \int\left(\frac{1}{x^2} + \frac{1 - \ln x}{x^2}\right)dx$$
$$= \int x^{-2}dx + \int\left(\frac{1 - \ln x}{x^2}\right)dx$$
$$= \frac{x^{-1}}{-1} + \frac{\ln x}{x} + c$$
$$= \frac{\ln x}{x} - \frac{1}{x} + c$$

HSCFocus.com

Question Three

(a) (i)



$V = x(12-2x)^2$

$= 144x - 48x^2 + 4x^3$

$\therefore 4x^3 - 48x^2 + 144x = 108$

$x^3 - 12x^2 + 36x - 27 = 0$

Let $f(x) = x^3 - 12x^2 + 36x - 27$

$f(3) = 27 - 108 + 108 - 27$

$= 0$

$\therefore x-3$ is a factor

$$\begin{array}{r|l} & x^2 - 9x + 9 \\ x-3 & x^3 - 12x^2 + 36x - 27 \\ & \underline{x^3 - 3x^2} \\ & -9x^2 + 36x \\ & \underline{-9x^2 + 27x} \\ & 9x - 27 \\ & \underline{9x - 27} \\ & 0 \end{array}$$

$\therefore (x-3)(x^2 - 9x + 9) = 0$

x=3 or

$x = \dfrac{9 \pm \sqrt{81 - 4(1)(9)}}{2}$

$= \dfrac{9 \pm \sqrt{81-36}}{2}$

$= \dfrac{9 \pm \sqrt{45}}{2}$

$= \dfrac{9 \pm 3\sqrt{5}}{2}$

$x = \dfrac{9+3\sqrt{5}}{2}$ is rejected since it is >12

$\therefore$ Values of x so that box has a volume of 108m^3 are

3m and $\dfrac{9-3\sqrt{5}}{2}$ m (≈ 1.15m)

(ii) $V = 4x^3 - 48x^2 + 144x$

$dV/dx = 12x^2 - 96x + 144 = 0$

for maximum/minimum

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

reject $x = 6$; $x = 2$

$d^2V/dx^2 = 24x - 96$

When $x = 2$, $d^2V/dx^2 = 48 - 96$

$= -48 < 0$

$\therefore$ maximum

$\therefore$ the box has a maximum volume when $x = 2$m

(b) $\displaystyle\sum_{r=1}^{n} a^{-r} = \frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^n}$

We are required to prove that

$\dfrac{1}{a} + \dfrac{1}{a^2} + \ldots + \dfrac{1}{a^n} = \dfrac{a^n - 1}{(a-1)a^n}$ for all positive integers n

Test $n = 1$:

$LHS = \dfrac{1}{a}$

$RHS = \dfrac{a-1}{(a-1)a}$

$= \dfrac{1}{a}$

$\therefore$ The result is true for $n = 1$

Assume that it is true for $n = k$

$\dfrac{1}{a} + \dfrac{1}{a^2} + \ldots + \dfrac{1}{a^k} = \dfrac{a^k - 1}{(a-1)a^k}$

Put $n = k + 1$

HSCFocus.com

$$LHS = \frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^k} + \frac{1}{a^{k+1}}$$
$$= \frac{a^k - 1}{(a-1)a^k} + \frac{1}{a^{k+1}}$$
$$= \frac{a(a^k - 1) + (a-1)}{(a-1)a^{k+1}}$$
$$= \frac{a^{k+1} - a + a - 1}{(a-1)a^{k+1}}$$
$$= \frac{a^{k+1} - 1}{(a-1)a^{k+1}}$$

Hence if the result is true for $n = k$, it is true for $n = k + 1$. But it is true for $n = 1$. Therefore it is true for $n = 2$ and so, by mathematical induction, for all positive integers n.

(c) Since $P(x)$ has zeros at -2, 2 and $\frac{3}{2}$, $P(x)$ may be expressed in the form

$P(x) = k(x+2)(x-2)(2x-3)$ where k is a real number.

Now $P(1) = k(3)(-1)(-1) = 12$

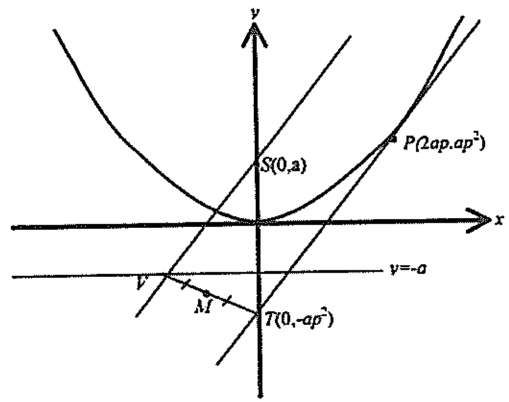
$\therefore k = 4$

$$\therefore P(x) = 4(x+2)(x-2)(2x-3)$$
$$= (x^2 - 4)(8x - 12)$$
$$= 8x^3 - 12x^2 - 32x + 48$$

$\therefore a = 8, b = -12, c = -32, d = 48$

Question Four

(a)



Gradient of tangent at $P = p$

Equation of tangent at P: $\quad y = px - ap^2$

$\therefore$ It cuts the y-axis at $-ap^2$

$\therefore$ $T(0,-ap^2)$

Equation of VS: $\quad y = px + a$

y-coordinate of $V = -a$

$$\therefore \quad -a = px_v + a$$
$$x_v = \frac{-2a}{p}$$
$$\therefore \quad V\left(\frac{-2a}{p}, -a\right)$$
$$\therefore \quad x_m = \frac{\frac{-2a}{p} + 0}{2} \qquad y_m = \frac{-ap^2 - a}{2}$$
$$M\left(\frac{-a}{p}, \frac{-ap^2 - a}{2}\right)$$
$$x = \frac{-a}{p} \qquad \therefore p = \frac{-a}{x}$$
$$y = \frac{-ap^2 - a}{2} = \frac{-a\left(\frac{-a}{x}\right)^2 - a}{2}$$
$$\therefore \quad y = \frac{-a^3}{2x^2} - \frac{a}{2} \text{ which is the locus of } M.$$

(b) $3^x = 6^z$

$\therefore x\log 3 = z\log 6$

Similarly $\quad y\log 2 = z\log 6$

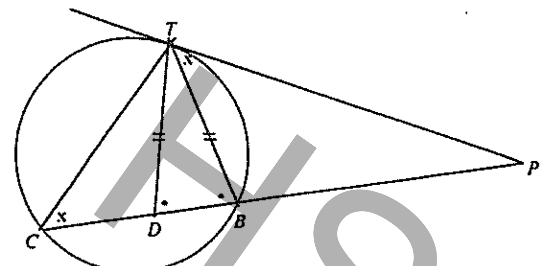
so that

$$x = \frac{z\log 6}{\log 3}$$
$$y = \frac{z\log 6}{\log 2}$$

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{\log 3}{z\log 6} + \frac{\log 2}{z\log 6}$$
$$= \frac{\log 3 + \log 2}{z\log 6}$$
$$= \frac{\log 6}{z\log 6}$$
$$= \frac{1}{z}$$

HSCFocus.com

(c)



$\angle TDB = \angle C + \angle CTD$ (exterior $\angle$ of ΔCTD)
and $\angle TBD = \angle P + \angle PTB$ (exterior $\angle$ of ΔPTB)
But $\angle TDB = \angle TBD$ (base $\angle$s of isosceles ΔTBD)
$\therefore\ \angle C + \angle CTD = \angle P + \angle PTB$
But $\angle PTB = \angle C$ ($\angle$ between tangent and chord)
$\therefore\ \angle CTD = \angle P$

Question Five

(a) (i) $T = M + Ae^{kt}$

$\frac{dT}{dt} = k \cdot Ae^{kt}$

$= k(T - M)$

$\therefore\ T = M + Ae^{kt}$ is a solution to the equation

(ii) $M = 18°C$

$A = 250 - 18 = 232°C$

(iii) $T = 18 + 232e^{kt}$

When $t = 50$ mins and $T = 103.3°C$

$103.3 = 18 + 232e^{50k}$

$\therefore 232e^{50k} = 85.3$

$e^{50k} = \frac{85.3}{232}$

$50k = \ln\frac{85.3}{232}$

$k = \frac{\ln\frac{85.3}{232}}{50}$

≈ -0.002 (one significant figure)

(b) $2\tan\theta - 3\sec\theta = -2\sqrt{3}$

$\frac{2\sin\theta}{\cos\theta} - \frac{3}{\cos\theta} = -2\sqrt{3}$

$2\sin\theta - 3 = -2\sqrt{3}\cos\theta$

ie. $2\sin\theta + 2\sqrt{3}\cos\theta = 3$

Let $2\sin\theta + 2\sqrt{3}\cos\theta = r\sin(\theta + \alpha)$

where $r = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$

and $\tan\alpha = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$\therefore\ 4\sin(\theta + \frac{\pi}{3}) = 3$

$\sin(\theta + \frac{\pi}{3}) = \frac{3}{4}$

$\therefore\ \theta + 60° = \underset{\text{reject}}{48°35'}, 180° - 48°35', 360° + 48°35'$

$\theta = 71°25'$ or $348°35'$

Question Six

(a) (i) $T = \frac{2\pi}{n}$

$\pi = \frac{2\pi}{n}$

$n = 2$

(ii) When $t = 0, x = 0$ and $v = 5$cm/s

$v^2 = n^2(a^2 - x^2)$

$25 = 4(a^2 - 0)$

$a^2 = \frac{25}{4}$

$a = \frac{5}{2}$cm

(iii) Let $x = a\sin(nt + \alpha)$ describe the SHM

When $t = 0, x = 0$ so that $\alpha = 0$

$\therefore\ x = a\sin nt$

$x = \frac{5}{2}\sin 2t$

When $x = 1.5$cm

$\frac{3}{2} = \frac{5}{2}\sin 2t$

$\sin 2t = 0.6$

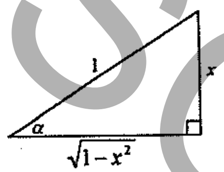
$2t = 0.64$

$t = 0.32\text{s}.$

(b) Let $\sec(\sin^{-1} x) = y$

Put $\sin^{-1} x = \alpha$

then $\sin\alpha = x$ $(-1 < x < 1)$



$\therefore y = \sec\alpha$

$= \frac{1}{\sqrt{1-x^2}}$

$\therefore \int \sec(\sin^{-1} x)dx = \int \frac{1}{1-x^2}dx$

$= \sin^{-1} x + c$

(c) (i) $v^2 = 2\int \ddot{x}dx$

$= 2\int \frac{-24}{x^2}dx$

$= -48\int x^{-2}dx$

$= 48x^{-1} + c$

$v^2 = \frac{48}{x} + c$

When $t = 0$, $v = 4$ and $x = 3$

$\therefore 16 = \frac{48}{3} + c$

$\therefore c = 0$

$\therefore v^2 = \frac{48}{x}$

$v = \sqrt{\frac{48}{x}}$

$= \frac{4\sqrt{3}}{\sqrt{x}}$

(ii) $v = \frac{dx}{dt}$

$= \frac{4\sqrt{3}}{\sqrt{x}}$

$dt/dx = \frac{\sqrt{x}}{4\sqrt{3}}$

$t = \int \frac{\sqrt{x}}{4\sqrt{3}}dx$

$= \frac{1}{4\sqrt{3}} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{6\sqrt{3}}x^{\frac{3}{2}} + c$

When $t = 0$, $x = 3\text{m}$

$\therefore 0 = \frac{3\sqrt{3}}{6\sqrt{3}} + c$

$\therefore c = -\frac{1}{2}$

$\therefore t = \frac{x^{\frac{3}{2}}}{6\sqrt{3}} - \frac{1}{2}$

(iii) When $x = 10$

$t = \frac{10^{\frac{3}{2}}}{6\sqrt{3}} - \frac{1}{2}$

$\approx 2.5\sec$

Question Seven

(a) (i) Starting with

$y = vt\sin\alpha - \frac{1}{2}gt^2$

$x = vt\cos\alpha$

When $y = 0$,

$vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t(v\sin\alpha - \frac{1}{2}gt) = 0$

$t = 0$ or $t = \frac{2v\sin\alpha}{g}$

HSCFocus.com

$\therefore \quad x = \dfrac{v(2v\sin\alpha)\cos\alpha}{g}$

$= \dfrac{2v^2\sin\alpha\cos\alpha}{g}$

I.e. $Range = \dfrac{v^2\sin 2\alpha}{g}$

(ii) (I) Let T = distance from cannon to target:

When $\alpha = 45°$, $T + 50 = \dfrac{v^2\sin 90°}{g}$

$\therefore\ T + 50 = \dfrac{v^2}{g}$ ....(1)

When $\alpha = 30°$, $T - 20 = \dfrac{v^2\sin 60°}{g}$

$T - 20 = \dfrac{v^2\sqrt{3}}{2g}$ ....(2)

(1) ÷ (2):

$$\frac{T+50}{T-20} = \frac{2}{\sqrt{3}}$$

$$\sqrt{3}(T+50) = 2(T-20)$$

$$\sqrt{3}T + 50\sqrt{3} = 2T - 40$$

$$T(2-\sqrt{3}) = 40 + 50\sqrt{3}$$

$$T = \frac{40+50\sqrt{3}}{2-\sqrt{3}}$$

$$= 472.5\text{m}$$

(II) $T + 50 = \dfrac{v^2}{g}$ ....(1)

$T = \dfrac{v^2\sin 2\alpha}{g}$ ....(2)

Substitute $T = 472.5$,

$522.5 = \dfrac{v^2}{g}$

$472.5 = \dfrac{v^2\sin 2\alpha}{g}$

$\dfrac{472.5}{522.5} = \sin 2\alpha$

$\therefore\ 2\alpha = 64°44'$ or $115°16'$

$\therefore\ \alpha = 32°22'$ or $57°38'$

(b)

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \ldots + \binom{2n}{n}x^n + \ldots + \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}$$

Putting $x = 1$,

$$(2)^{2n} = \binom{2n}{0} + \binom{2n}{1} + \ldots + \binom{2n}{n} + \ldots + \binom{2n}{2n-1} + \binom{2n}{2n}$$

$$(2)^{2n} = \binom{2n}{0} + \binom{2n}{2n} + \binom{2n}{1} + \binom{2n}{2n-1} \ldots + \binom{2n}{n} + \binom{2n}{n} - \binom{2n}{n}$$

$$= 2\left[\binom{2n}{0} + \binom{2n}{1} + \ldots + \binom{2n}{n}\right] - \binom{2n}{n}$$

$$= 2\sum_{k=0}^{n}\binom{2n}{k} - \binom{2n}{n}$$

$$\therefore\ \frac{2^{2n}}{2} = \sum_{k=0}^{n}\binom{2n}{k} - \tfrac{1}{2}\binom{2n}{n}$$

$$2^{2n-1} = \sum_{k=0}^{n}\binom{2n}{k} - \frac{1}{2}\frac{(2n)!}{n!(2n-n)!}$$

$$\therefore\ 2^{2n-1} = \sum_{k=0}^{n}\binom{2n}{k} - \frac{(2n)!}{2(n!)^2}$$

ie. $\displaystyle\sum_{k=0}^{n}\binom{2n}{k} = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$

HSCFocus.com