CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NEW SOUTH WALES

PAST YEAR 12 FINAL TESTS

MATHEMATICS

3 UNIT

1989 TO 1997

PLEASE DO NOT WRITE IN THIS BOOKLET AS IT IS RETURNABLE WITH OTHER ISSUED TEXTS

Marks

Question 1

(a) Show that $\dfrac{d}{dx}\dfrac{\tan x}{e^{2x}} = \dfrac{(1-\tan x)^2}{e^{2x}}$ **3**

(b) Solve the equation $\sin 2x = \tan x$ for $0 \le x \le \pi$ **4**

(c) **5**

(i) Find the domain and range of the function $y = 4\cos^{-1}\left(\dfrac{x}{3}\right)$

(ii) Sketch the graph of the function $y = 4\cos^{-1}\left(\dfrac{x}{3}\right)$ showing clearly the intercepts on the coordinate axes and the coordinates of any endpoints.

(iii) Find the area of the region in the first quadrant bounded by the curve $y = 4\cos^{-1}\left(\dfrac{x}{3}\right)$ and the coordinate axes.

Question 2

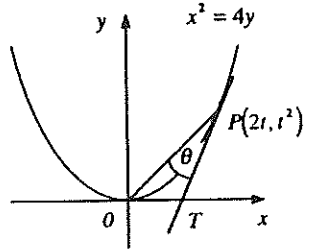
(a) Solve the inequality $\dfrac{1}{|x-3|} \ge \dfrac{1}{2}$ **3**

(b) **4**

$P(2t, t^2)$, where $t > 0$, is a point on the parabola $x^2 = 4y$.

The tangent to the parabola at P cuts the x axis at T.

$\angle OPT = \theta$.



(i) Find the gradients of OP and TP.

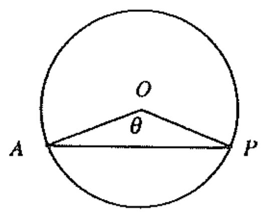
(ii) Show that $\tan\theta = \dfrac{t}{t^2+2}$

(c) **5**

A is a fixed point on a circle centre O, radius 1cm.

P is a variable point which moves around the circle with a constant speed of one revolution per second.

$\angle AOP = \theta$.



(i) Show that $\dfrac{d\theta}{dt} = 2\pi$ radians per second.

(ii) Find the rate at which the area of ΔAOP is changing when $\theta = \frac{2\pi}{3}$

Marks

Question 3

(a) Find $\displaystyle\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)}\,dx$ using the substitution $x = u^2$, $u > 0$. **3**

(b) The region bounded by the curve $y = \sin x$, the x axis and the lines $x = \frac{\pi}{12}$ and $x = \frac{\pi}{4}$ is rotated through one complete revolution about the x axis. Find the volume of the solid so formed. **4**

(c) A particle moving in Simple Harmonic Motion starts from rest at a distance 10 metres to the right of its centre of oscillation O. The period of the motion is 2 seconds. **5**

(i) Find the speed of the particle when it is 4 metres from its starting point.

(ii) Find the time taken by the particle to first reach the point 4 metres from its starting point, in seconds correct to 2 decimal places.

Question 4

(a) One of the roots of the equation $x^3 + ax^2 + 1 = 0$ is equal to the sum of the other two roots. **4**

(i) Show that $x = -\dfrac{a}{2}$ is a root of the equation.

(ii) Find the value of a.

(b) **8**

(i) Sketch the graph of the function $f(x) = e^x - 4$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes.

(ii) On the same diagram sketch the graph of the inverse function $f^{-1}(x)$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes.

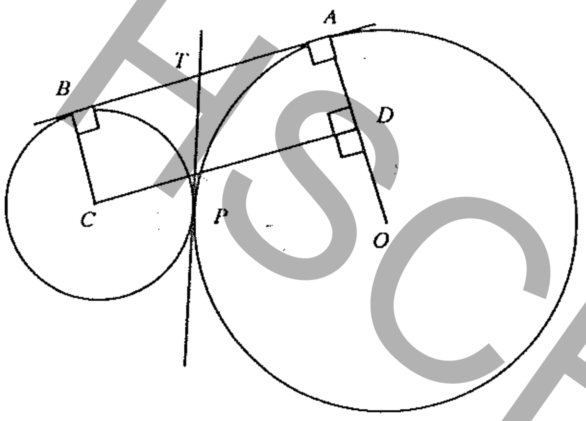
(iii) Explain why the x coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$.

(iv) Show the equation $e^x - x - 4 = 0$ has a root between $x = 0$ and $x = 2$, and use the 'halving the interval' method to find this root correct to the nearest whole number.

Question 5

Marks

(a) **6**



Two circles, one with centre C and radius 1cm and the other with centre O and radius 3cm, touch each other externally at P. AB is a common tangent to the two circles. CD is drawn perpendicular to OA to complete the rectangle $ABCD$. The common tangent to the two circles at P meets AB at T.

(i) Copy the diagram.

(ii) Find the exact length of AB giving reasons for your answer.

(iii) Find the exact length of PT giving reasons for your answer.

(b) A particle is moving on a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} and acceleration a ms^{-2}. The particle starts from O and at time t seconds $v = (1 - x)^2$. **6**

(i) Find an expression for a in terms of x.

(ii) Find an expression for x in terms of t.

(iii) Find the time taken for the particle to slow down to a speed of 1% of its initial speed.

Question 6

Marks

(a) **4**

(i) Write down the Binomial expansion of $(1 + x)^n$ in ascending powers of x.

(ii) Find the value of n such that the coefficient of x^4 is twice the coefficient of x^3.

(b) A motorway pay station has five toll gates, three of which are automatically operated and two of which are manually operated. Drivers with the exact money can use any one of the five gates, but drivers requiring change have to use one of the two manually operated gates. A Ford driver, a Holden driver and a Toyota driver use the motorway every day. **4**

(i) On one day the Ford driver requires change but the other two drivers have the exact money. Find the number of ways in which the three drivers can go through the pay station so that they all use different gates.

(ii) On another day, all three drivers have the exact money. Find the number of ways in which the three drivers can go through the pay station so that exactly one goes through a manually operated gate.

(c) It is known that 5% of men are colourblind. A random sample of 20 men is chosen. **4**

(i) Find the probability, correct to two decimal places, that the sample contains at most one colourblind man.

(ii) Find the probability, correct to two decimal places, that the sample contains at least two colourblind men.

Question 7

(a) When a particle is fired in the open from a point O at a speed of 40 ms^{-1} and at an angle θ above the horizontal, where $0 < \theta < \frac{\pi}{2}$, *you may assume without proof* that the horizontal displacement (x metres) and the vertical displacement (y metres) of the particle from O at time t seconds after firing are given by **6**

$$x = 40t\cos\theta \qquad \text{and} \qquad y = 40t\sin\theta - 5t^2$$

If the particle is fired with the same speed from a point O on the floor of a horizontal tunnel of height 20 metres, find the maximum horizontal range of the particle along the tunnel.

(b) **6**

(i) Use the method of mathematical induction to show that if x is a positive integer then $(1 + x)^n - 1$ is divisible by x for all positive integers $n \geq 1$.

(ii) Factorise $12^n - 4^n - 3^n + 1$. Without using the method of mathematical again, use the result of part (i) to deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers $n \geq 1$.

CSSA MATHEMATICS 3 UNIT SOLUTIONS 1997

QUESTION 1

(a) $\dfrac{d}{dx}\dfrac{\tan x}{e^{2x}} = \dfrac{e^{2x}.\sec^2 x - \tan x . 2e^{2x}}{(e^{2x})^2}$

$= \dfrac{\cancel{e^{2x}}(\sec^2 x - 2\tan x)}{(e^{2x})^{\cancel{2}}}$

$= \dfrac{1+\tan^2 x - 2\tan x}{e^{2x}} = \dfrac{(1-\tan x)^2}{e^{2x}}$

(b) $\sin 2x = \tan x \quad \therefore 2\sin x \cos x = \dfrac{\sin x}{\cos x}$

$\therefore 2\sin x \cos^2 x = \sin x \quad (\cos x \neq 0)$

$\therefore 2\sin x \cos^2 x - \sin x = 0$

$\therefore \sin x\,(2\cos^2 x - 1) = 0 \quad \therefore \sin x \cos 2x = 0$

$\therefore \sin x = 0$ or $\cos 2x = 0$

$\therefore x = 0$ or π or $2x = \pi/2$ or $3\pi/2 \quad (0 \le x \le \pi)$

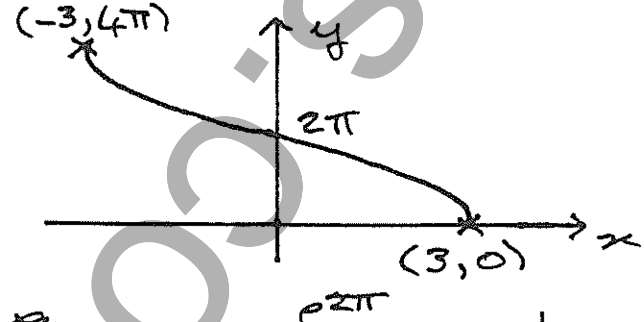
$x = \pi/4$ or $3\pi/4$

$\therefore x = 0$ or $\pi/4$ or $3\pi/4$ or $\pi \quad (0 \le x \le \pi)$

(c) (i) domain : $-1 \le x/3 \le 1 \quad \therefore -3 \le x \le 3$.

range : $0 \le \cos^{-1} x/3 \le \pi$

$\therefore 0 \le 4\cos^{-1} x/3 \le 4\pi \quad \therefore 0 \le y \le 4\pi$.

(ii) 

$y = 4\cos^{-1} x/3$.

(iii) Area $= \int_0^{2\pi} x\, dy$

$= \int_0^{2\pi} 3\cos y/4\, dy$

$= [12\sin y/4]_0^{2\pi}$

$= 12(1) - 12(0)$

$= 12$ units^2.

QUESTION 2

(a) $\dfrac{1}{|x-3|} \geq \dfrac{1}{2} \quad (\therefore x \neq 3)$

$\therefore |x-3| \leq 2 \quad (x \neq 3) \quad \therefore -2 \leq x-3 \leq 2 \; (x \neq 3)$

$\therefore 1 \leq x \leq 5 \quad (x \neq 3)$

$\therefore 1 \leq x < 3$ or $3 < x \leq 5$.

(b) (i) gradient OP : $m_{OP} = \dfrac{t^2-0}{2t-0} = \dfrac{t^2}{2t} = \dfrac{t}{2}$.

$x^2 = 4y \quad \therefore y = x^2/4$

$\therefore dy/dx = 2x/4 = x/2$

$\therefore$ at P, $dy/dx = 2t/2 = t$

$\therefore$ gradient of tangent at P $= t$

$\therefore$ gradient TP : $m_{TP} = t$.

(ii) $\tan\theta = \left|\dfrac{m_{TP} - m_{OP}}{1 + m_{TP}\, m_{OP}}\right|$

$= \left|\dfrac{t - t/2}{1 + t.t/2}\right| = \left|\dfrac{(2t-t)/2}{(2+t^2)/2}\right|$

$= \left|\dfrac{t}{t^2+2}\right| = \dfrac{t}{t^2+2} \quad (t > 0)$

(c) (i) P moves around the circle at a constant speed of one revolution per second.

$\therefore$ θ changes at a constant rate of 2π radians per second

$\therefore d\theta/dt = 2\pi$ radians per second.

(ii) area ΔAOP : $A = 1/2\,(1)(1)\sin\theta \quad \therefore A = 1/2 \sin\theta$

$\therefore \dfrac{dA}{dt} = \dfrac{dA}{d\theta} \cdot \dfrac{d\theta}{dt} = 1/2 \cos\theta \,.\, 2\pi = \pi\cos\theta$

$\therefore$ when $\theta = 2\pi/3$, $\dfrac{dA}{dt} = \pi\cos 2\pi/3 = \pi(-1/2) = -\pi/2$

$\therefore$ the area is decreasing at a rate of $\pi/2 \; cm^2 s^{-1}$.

QUESTION 3

(a) $\displaystyle\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$ $\qquad x = u^2 \quad (u = \sqrt{x})$

$\therefore \dfrac{dx}{du} = 2u$

$\therefore dx = 2u\,du$

$= \displaystyle\int \frac{1}{u\,(1+u)}\, 2u\,du$

$= 2\ln(1+u) + c$

$= 2\ln(1+\sqrt{x}) + c$.

(b) volume $= \pi \int_{\pi/12}^{\pi/4} y^2\,dx$

$= \pi \int_{\pi/12}^{\pi/4} \sin^2 x\,dx$

$= \pi \int_{\pi/12}^{\pi/4} 1/2(1-\cos 2x)\,dx$

$= \pi \left[x/2 - 1/4 \sin 2x\right]_{\pi/12}^{\pi/4}$

$= \pi \{(\pi/8 - 1/4(1)) - (\pi/24 - 1/4(1/2))\}$

$= \pi/24\,(3\pi - 6 - \pi + 3)$

$= \pi/24\,(2\pi - 3)$ units^3.

(c) $a = 10$

$T = 2\pi/n \quad \therefore 2 = 2\pi/n \quad \therefore n = 2\pi/2 \quad \therefore n = \pi$

(i) $v^2 = n^2(a^2 - x^2) \quad \therefore v^2 = \pi^2(100 - x^2)$

$\therefore$ when $x = 6$, $v^2 = \pi^2(100-36)$

$\therefore v^2 = 64\pi^2 \quad \therefore |v| = 8\pi$

$\therefore$ speed $= 8\pi \; ms^{-1}$.

(ii) $x = a\cos(nt+\alpha) \quad \therefore x = 10\cos(\pi t + \alpha)$

when $t = 0$, $x = 10 \quad \therefore \alpha = 0$

$\therefore x = 10\cos\pi t$

$\therefore$ when $x = 6$, $6 = 10\cos\pi t$

$\therefore \cos\pi t = 0.6 \quad \therefore t = 0.30$ (to 2 d.p.)

$\therefore$ time taken $= 0.30$ seconds (to 2 d.p.)

QUESTION 6

(a)(i) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

$= 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + {}^nC_4 x^4 + \dots + x^n$

(ii) coefficient of x^4 = 2 coefficient of x^3

$\therefore {}^nC_4 = 2\,{}^nC_3$

$\therefore \dfrac{\cancel{n(n-1)(n-2)}(n-3)}{\cancel{1\times2\times3}\times4} = 2\,\dfrac{\cancel{n(n-1)(n-2)}}{\cancel{1\times2\times3}}$

$\therefore n-3 = 2\times4 \quad \therefore n-3 = 8 \quad \therefore n = 11.$

(b) 2 manual gates M_1, M_2. 3 automatic gates A_1, A_2, A_3

(i) the Ford driver has 2 choices,
then the Holden driver has 4 choices,
then the Toyota driver has 3 choices.
$\therefore$ number of ways $= 2\times4\times3 = 24$.

(ii) there are ${}^3C_1 = 3$ ways of choosing the one driver to go through a manually operated gate,
this driver has 2 choices,
the other two drivers have 3×3 choices.
$\therefore$ number of ways $= 3\times2\times3\times3 = 54$.

(c)(i) P(at most one colourblind man)
$= P(\text{none}) + P(\text{one})$
$= (0.95)^{20} + {}^{20}C_1 (0.95)^{19}(0.05)$
$= 0.74$ (to 2 d.p.)

(ii) P(at least two colourblind men)
$= 1 -$ P(at most one colourblind man)
$= 1 - 0.74$ (to 2 d.p.)
$= 0.26$ (to 2 d.p.)

QUESTION 7

(a) $y = 40t\sin\theta - 5t^2 = 5t(8\sin\theta - t)$

$\therefore y = 0$ when $t = 0$ or $t = 8\sin\theta$ (time of flight)

when $t = 8\sin\theta$, $x = 40(8\sin\theta)\cos\theta$

$\therefore x = 320\sin\theta\cos\theta = 160\sin2\theta$ (horizontal range)

$\therefore v_y = dy/dt = 40\sin\theta - 10t = 10(4\sin\theta - t)$

$\therefore$ at maximum height $v_y = 0 \quad \therefore t = 4\sin\theta$

when $t = 4\sin\theta$, $y = 40(4\sin\theta)\sin\theta - 5(4\sin\theta)^2$

$\therefore y = 160\sin^2\theta - 80\sin^2\theta = 80\sin^2\theta$ (maximum height)

Now maximum height ≤ 20 (height of tunnel)

$\therefore 80\sin^2\theta \le 20 \quad \therefore \sin^2\theta \le 1/4$

$\therefore \sin\theta \le 1/2 \quad \therefore 0 < \theta \le \pi/6 \quad (0 < \theta < \pi/2)$

$\therefore$ when $\theta = \pi/6$, maximum horizontal range

$= 160\sin 2\pi/6 = 160.\sqrt{3}/2 = 80\sqrt{3}$ metres.

(b)(i) $S(n)$: $(1+x)^n - 1$ is divisible by x

When $n=1$, $(1+x)^1 - 1 = 1 + x - 1 = x = x(1)$

$\therefore S(1)$ is true.

If $S(k)$ is true for some $k \ge 1$, so that $(1+x)^k - 1 = x(M)$ for some integer M, then

$(1+x)^{k+1} - 1 = (1+x)^k(1+x) - 1$

$= (Mx+1)(1+x) - 1 = Mx + \cancel{1} + Mx^2 + x - \cancel{1}$

$= x(Mx + M + 1)$

$= x(N)$ for some integer N,

and so $S(k+1)$ is also true.

Since $S(1)$ is true and if $S(k)$ is true for $k \ge 1$ then $S(k+1)$ is also true, it follows that $S(1+1) = S(2)$ is true, $S(2+1) = S(3)$ is true etc.

$\therefore S(n)$ is true for all positive integers $n \ge 1$.

(ii) $12^n - 4^n - 3^n + 1 = (4^n - 1)(3^n - 1)$

From (i), $4^n - 1$ is divisible by 3 and $3^n - 1$ is divisible by 2 for all positive integers $n \ge 1$.

$\therefore (4^n-1)(3^n-1)$ is divisible by $3\times2 = 6$.

$\therefore 12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers $n \ge 1$.

QUESTION 4

(a) (i) Let the roots of $x^3 + ax^2 + 1 = 0$ be α, β and γ where $\gamma = \alpha + \beta$

sum of roots : $\alpha + \beta + \gamma = -{}^{(a)}/_{(1)}$

$\therefore 2\gamma = -a \quad \therefore \gamma = -a/2$

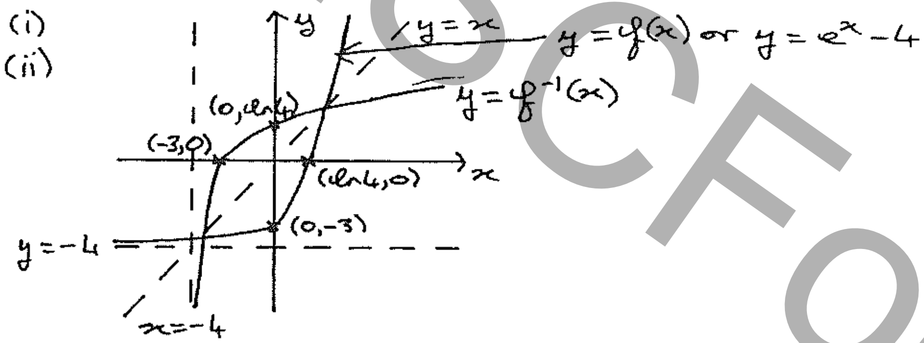
$\therefore x = -a/2$ is a root of the equation.

(ii) $(-a/2)^3 + a(-a/2)^2 + 1 = 0$

$\therefore -a^3/8 + a^3/4 + 1 = 0 \quad \therefore -a^3 + 2a^3 + 8 = 0$

$\therefore a^3 = -8 \quad \therefore a = -2$.

(b) (i)
(ii)



(iii) The x coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ lies on the line $y = x$, and satisfies the equation $f(x) = x$, ie. $e^x - 4 = x$, ie. $e^x - x - 4 = 0$.

(iv) Let $P(x) = e^x - x - 4$

$\therefore P(0) = -3 < 0$

$P(2) = e^2 - 6 = 1{\cdot}39$ (to 2d.p.) > 0

$\therefore P(x)$ is continuous and $P(0)$ and $P(2)$ have opposite signs. $\therefore P(x) = 0$ or $e^x - x - 4 = 0$ has a root between $x = 0$ and $x = 2$.

$P(1) = e - 5 = -2{\cdot}28$ (to 2d.p.) < 0

$\therefore$ root lies between $x = 1$ and $x = 2$.

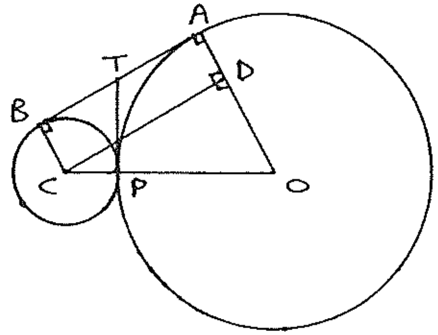
$P(1{\cdot}5) = e^{1{\cdot}5} - 5{\cdot}5 = -1{\cdot}02$ (to 2d.p.) < 0

$\therefore$ root lies between $x = 1{\cdot}5$ and $x = 2$

$\therefore$ root is $x = 2$ (correct to nearest whole number)

QUESTION 5

(a) (i)



(ii) Draw the line CPO (line of centres passes through point of contact)

$\therefore CO = CP + PO = 1 + 3 = 4$

$DO = AO - AD = 3 - 1 = 2$

$CO^2 = CD^2 + DO^2$

$\therefore CD^2 = 4^2 - 2^2 = 12$

$\therefore CD = \sqrt{12} = 2\sqrt{3}$

$\therefore AB = CD = 2\sqrt{3}$ cm

(iii) $TB = TP$ (tangents from T to circle centre C are equal)

$TA = TP$ (tangents from T to circle centre O are equal)

$\therefore TB = TA = \frac{1}{2}AB = \frac{1}{2}(2\sqrt{3}) = \sqrt{3}$

$\therefore PT = TB = TA = \sqrt{3}$ cm.

(b) (i) $v = (1-x)^2$

$\therefore a = \frac{d}{dx}(\frac{1}{2}v^2) = \frac{d}{dx}(\frac{1}{2}(1-x)^4) = -2(1-x)^3$.

(ii) $v = (1-x)^2$

$\therefore dx/dt = (1-x)^2 \quad \therefore dt/dx = 1/(1-x)^2$

$\therefore t = \int \frac{1}{(1-x)^2}\,dx \quad \therefore t = \frac{1}{1-x} + c$

When $t = 0$, $x = 0 \quad \therefore c = -1$

$\therefore t = \frac{1}{1-x} - 1 \quad \therefore \frac{1}{1-x} = t + 1$

$\therefore 1 - x = \frac{1}{t+1} \quad \therefore x = 1 - \frac{1}{t+1}$

$\therefore x = \frac{t+1-1}{t+1} \quad \therefore x = \frac{t}{t+1}$.

(iii) When $t = 0$, $x = 0$, $v = 1 \quad \therefore$ initial speed $v = 1$

$\therefore$ when $v = 1/100$, $(1-x)^2 = 1/100$

$\therefore 1 - x = 1/10 \quad \therefore x = 1 - 1/10 \quad \therefore x = 9/10$

$\therefore$ when $x = 9/10$, $t/(t+1) = 9/10$

$\therefore 10t = 9(t+1) \quad \therefore 10t = 9t + 9 \quad \therefore t = 9$

$\therefore$ the particle takes 9 seconds to slow down to a speed of 1% of its initial speed.

CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1996

MATHEMATICS

3/4 UNIT COMMON PAPER

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER:
4 UNIT COURSE – FIRST PAPER)

Afternoon session

Friday 9th August 1996.

Time Allowed — Two Hours

EXAMINERS

Graham Arnold, John Paul II Senior High, Marayong
Sandra Hayes, All Saints Catholic Senior High, Casula.
Frank Reid, School of Mathematics, University of NSW.

DIRECTIONS TO CANDIDATES :

ALL questions may be attempted.

ALL questions are of equal value.

All necessary working should be shown in every question.

Full marks may not be awarded for careless or badly arranged work.

Approved calculators may be used.

Standard integrals are printed on a separate page.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination. However, the committees responsible for the preparation of these 'Trial Examinations' do hope that they will provide a positive contribution to your preparation for the final examinations

Question 1

(a) (3 marks)

Find the acute angle between the lines $2x - y = 0$ and $x + 3y = 0$, giving the answer correct to the nearest minute.

(b) (4 marks)

Consider the function $y = x\ln x - x$.

(i) Solve the equation $y = 0$.

(ii) Find $\frac{d^2y}{dx^2}$ and hence show that the function is concave up for all values of x in its domain.

(c) (5 marks)

Consider the polynomial $P(x) = 6x^3 - 5x^2 - 2x + 1$

(i) Show that 1 is a zero of $P(x)$.

(ii) Express $P(x)$ as a product of 3 linear factors.

(iii) Solve the inequality $P(x) \le 0$.

Question 2

(a) (3 marks)

(i) Find $\frac{d}{dx}\left(e^{\tan x}\right)$

(ii) Hence find $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

(b) Use the substitution $u = 1 - x$ to evaluate $\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$. (4 marks)

(c) (5 marks)

(i) Find the value of x such that $\sin^{-1} x = \cos^{-1} x$.

(ii) On the same axes sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

(iii) On the same diagram as the graphs in (ii), draw the graph of $y = \sin^{-1} x + \cos^{-1} x$.

Question 3

(a) (8 marks)

$$f(x) = \frac{8}{4 + x^2}$$

(i) Show that f is an even function, and the x axis is a horizontal asymptote to the curve $y = f(x)$.

(ii) Find the coordinates and nature of the stationary point on the curve $y = f(x)$.

(iii) Sketch the graph of the curve showing the above features.

(iv) Find the exact area of the region in the first quadrant bounded by the curve $y = f(x)$ and the line $x = 2$.

(b) (4 marks)

A vertical tower of height h metres stands on horizontal ground. From a point P on the ground due east of the tower the angle of elevation of the top of the tower is 45°. From a point Q on the ground due south of the tower the angle of elevation of the top of the tower is 30°. If the distance PQ is 40 metres, find the exact height of the tower.

Question 4

(a) (8 marks)

N is the number of animals in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = -k(N - 1000)$, for some constant k.

(i) Verify by differentiation that $N = 1000 + Ae^{-kt}$, A constant, is a solution of the equation.

(ii) Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of A and k.

(iii) Find when the number of animals has fallen to 1300.

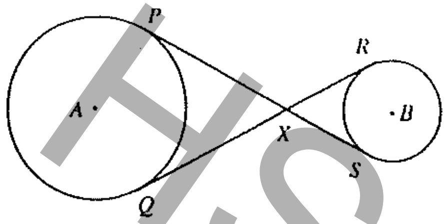
(iv) Sketch the graph of the population size against time.

(b) (4 marks)

Use Mathematical Induction to show that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \ge 1$.

Question 5

(a) (7 marks)



In the diagram PS and QR are tangents to each of the circles with centres A and B. The tangents intersect at X and A, X, B are collinear.

(i) Copy the diagram and show that $\Delta APX \parallel\!\mid \Delta BSX$.

(ii) Suppose that the diagram represents two circles of radii 5 cm and 3 cm that are placed in the same plane with their centres 16 cm apart. A taut string surrounds the circles and crosses itself between them. Find the exact length of the string.

(b) (5 marks)

The interior of a circle is divided into two segments with areas in the ratio 3 : 1 by a chord which subtends an angle θ radians at the centre of the circle.

(i) Show that $\theta - \sin\theta = \frac{\pi}{2}$.

(ii) Taking $\theta = 2\cdot 5$ as a first approximation, use Newton's method twice to find a better approximation to θ, giving the answer correct to 2 decimal places.

Question 6

(a) (7 marks)

A group consisting of 3 men and 6 women attends a prizegiving ceremony.

(i) If the members of the group sit down at random in a straight line, find the probability that the 3 men sit next to each other.

(ii) If 5 prizes are awarded at random to members of the group, find the probability that exactly 3 of the prizes are awarded to women if

(α) there is a restriction of at most one prize per person.

(β) there is no restriction on the number of prizes per person.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 6 (cont.)

(b)

A particle moving in a straight line is performing Simple Harmonic Motion about a fixed point O on the line. At time t seconds the displacement x metres of the particle from O is given by (5 marks)

$$x = a\cos nt \quad , \qquad \text{where } a > 0 \text{ and } 0 < n < \pi \ .$$

After 1 second the particle is 1 metre to the right of O, and after 2 seconds the particle is 1 metre to the left of O.

(i) Find the values of n and a.

(ii) Find the amplitude and period of the motion.

Question 7

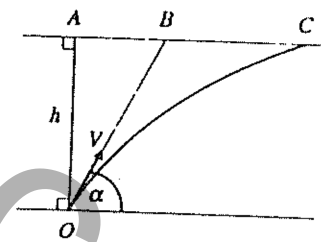
(a)

(i) Write down the Binomial expansion of $(1+x)^n$ in ascending powers of x. (4 marks)

Hence show that $\qquad {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$.

(ii) Find how many groups of 1 or more digits can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 where repetition is not allowed.

(b) (8 marks)



In the diagram an aircraft is flying with constant velocity U at a constant height h above horizontal ground. When the plane is at A it is directly over a gun at O. When the plane is at B a shell is fired from the gun at the aircraft along OB. The shell is fired with initial velocity V at an angle of elevation α .

(i) If x and y are the horizontal and vertical displacements of the shell from O at time t seconds, show that if g is the acceleration due to gravity,

$$x = Vt\cos\alpha \quad \text{and} \quad y = Vt\sin\alpha - \tfrac{1}{2}gt^2 \ .$$

(ii) Show that if the shell hits the aircraft at time T at point C, then

$$VT\cos\alpha = \frac{h}{\tan\alpha} + UT \ .$$

(iii) Show that if the shell hits the aircraft ... $2U$ ...

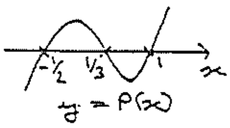
CSSA MATHEMATICS 3 UNIT SOLUTIONS 1996

QUESTION 1

(a) $2x-y=0$ has gradient $m_1=2$.
$x+3y=0$ has gradient $m_2=-1/3$.
Let θ be the acute angle between the lines.
$\therefore \tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{2-(-1/3)}{1+2(-1/3)}\right| = 7$
$\therefore \theta = 81°52'$ to the nearest minute.

(b) $y = x\ln x - x$ has domain $x>0$
(i) $x\ln x - x = 0$
$\therefore x(\ln x - 1) = 0$
$\therefore \cancel{x=0}$ (not in domain) or $\ln x = 1$
$\therefore x = e$.
(ii) $dy/dx = (x)(1/x) + (\ln x)(1) - (1)$
$= 1 + \ln x - 1$
$= \ln x$
$d^2y/dx^2 = 1/x$.
$\therefore$ for $x>0$, $d^2y/dx^2>0$ and the function is concave up for all x in its domain.

(c) $P(x) = 6x^3-5x^2-2x+1$
(i) $P(1) = 6-5-2+1 = 0$
$\therefore 1$ is a zero of $P(x)$.
(ii) $(x-1)$ is a factor of $P(x)$
$\therefore P(x) = 6x^3-5x^2-2x+1$
$= (x-1)(6x^2+x-1)$
$= (x-1)(3x-1)(2x+1)$
(iii)



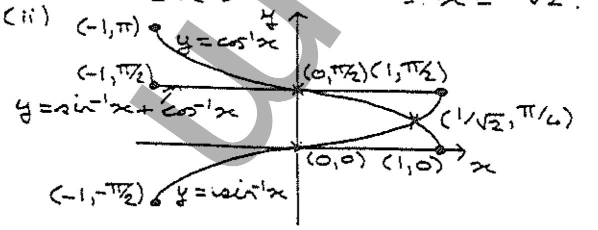
$P(x)\le 0$ when the graph lies on or below the x axis.
$\therefore x\le -1/2$ or $1/3\le x\le 1$.

QUESTION 2

(a) (i) $d/dx\, e^{\tan x} = e^{\tan x}.\sec^2 x$
$= \sec^2 x\, e^{\tan x}$
(ii) $\int \frac{e^{\tan x}}{\cos^2 x}\,dx = \int \sec^2 x\, e^{\tan x}\,dx$
$= e^{\tan x} + c$

(b) $\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$ $u = 1-x$, $x = 1-u$, $dx/du = -1$, $dx = -du$
$= \int_4^1 \frac{(1-u)}{\sqrt u}.(-du)$ when $x=-3, u=4$
$= \int_1^4 (u^{-1/2} - u^{1/2})\,du$ when $x=0, u=1$
$= \left[2\sqrt u - 2/3\, u\sqrt u\right]_1^4$
$= \{4 - 16/3\} - \{2 - 2/3\}$
$= -8/3$.

(c) (i) Let $\sin^{-1}x = a$ $(-1\le x\le 1, -\pi/2\le a\le \pi/2)$
Let $\cos^{-1}x = a$ $(-1\le x\le 1,\ 0\le a\le\pi)$
$\therefore \sin a = x$, $\cos a = x$ $\therefore \tan a = 1$, $\therefore a = \pi/4$
$\therefore x = 1/\sqrt2$.
(ii)



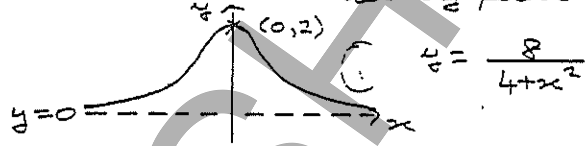
(iii) on diagram $y = \sin^{-1}x + \cos^{-1}x$
$\therefore y = \pi/2$ $(-1\le x\le 1)$

QUESTION 3

(a) (i) $f(-x) = \frac{8}{4+(-x)^2} = \frac{8}{4+x^2} = f(x)$
for all values of x $\therefore$ function is even.
$\lim_{x\to\pm\infty} \frac{8}{4+x^2} = 0$ $\therefore y=0$ is a horizontal asymptote.
(ii) $\frac{dy}{dx} = \frac{-16x}{(4+x^2)^2} = 0$ at stationary point $(0,2)$.

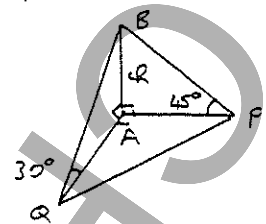
x	0^-	0	0^+
dy/dx	+ve	0	-ve

$\therefore (0,2)$ is a maximum turning point.
(iii)



(iv) area $= \int_0^2 \frac{8}{4+x^2}\,dx = \left[8.\tfrac12\tan^{-1}x/2\right]_0^2$
$= 4\tan^{-1}1 - \cancel{4\tan^{-1}0} = 4.\pi/4 = \pi$ units^2.

(b)

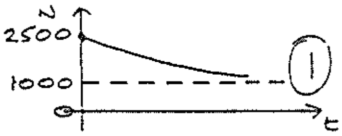


In $\triangle PAB$, $\tan 45° = h/PA$
$\therefore PA = \frac{h}{\tan 45°} = h$
In $\triangle QAB$, $\tan 30° = h/QA$
$\therefore QA = \frac{h}{\tan 30°} = h\sqrt3$

In $\triangle PAQ$, $PA^2 + QA^2 = PQ^2$
$\therefore (h)^2 + (h\sqrt3)^2 = 40^2$, $\therefore h^2 + 3h^2 = 1600$
$\therefore 4h^2 = 1600$, $\therefore h^2 = 400$, $\therefore h = 20$ $(h>0)$
$\therefore$ the height of the tower is 20 metres

QUESTION 4

(a) (i) LHS $= dN/dt = d/dt(1000 + Ae^{-kt})$
$= -kAe^{-kt} = -k(N-1000) =$ RHS
(ii) When $t=0$, $N = 2500$, $\therefore 2500 = 1000 + A$
$\therefore A = 1500$, $\therefore N = 1000 + 1500e^{-kt}$
When $t = 2$, $N = 2200$, $\therefore 2200 = 1000 + 1500e^{-2k}$
$\therefore e^{2k} = 5/4$, $\therefore k = 1/2\ln(5/4)$.
(iii) When $N = 1300$, $1300 = 1000 + 1500e^{-kt}$
$\therefore e^{kt} = 5$, $\therefore t = 1/k\ln(5)$
$\therefore t = \ln(5)/\tfrac12\ln(5/4) = 14.4$ years (to 1 d.p.)
(iv)



As $t\to\infty$, $N\to 1000$
$N = 1000$ is a horizontal asymptote

(b) $S(n)$: $\cos(x+n\pi) = (-1)^n\cos x$ $(n\ge 1)$
When $n=1$, LHS $= \cos(x+\pi) = -\cos x$
RHS $= (-1)^1\cos x = -\cos x$
$\therefore S(1)$ is true.
If $S(k)$ is true $(k\ge 1)$
i.e. if $\cos(x+k\pi) = (-1)^k\cos x$
then $\cos(x+(k+1)\pi)$
$= \cos((x+k\pi)+\pi)$
$= \cos(x+k\pi)\cos\pi - \sin(x+k\pi)\sin\pi$
$= (-1)^k\cos x.(-1) - \sin(x+k\pi).0$
$= (-1)^{k+1}\cos x$
and $S(k+1)$ is also true.
Since $S(1)$ is true and if $S(k)$ $(k\ge1)$ is true then $S(k+1)$ is also true it follows that $S(n)$ is true for all positive integers $n\ge 1$.
$\therefore \cos(x+n\pi) = (-1)^n\cos x$ for all $n\ge 1$.

QUESTION 5

(a) (i) In $\triangle APX$ and $\triangle BSX$

$\angle APX = \angle BSX (=90°)$ (tangent ⊥ radius) 1

$\angle PXA = \angle SXB$ (vertically opposite angles) 1

$\angle XAP = \angle XBS$ (alternate angles $AP \parallel SB$) 1

$\therefore \triangle APX ||| \triangle BSX$ (equiangular)

(ii) $\frac{AX}{BX} = \frac{AP}{BS} \left(= \frac{5}{3}\right)$ (corresponding sides are proportional) 1

But $AB = 16$ cm $\therefore AX = 10$ cm, $BX = 6$ cm.

Now $\cos P\hat{A}X = \cos S\hat{B}X = 1/2$

$\therefore P\hat{A}X = S\hat{B}X = \pi/3$, $P\hat{A}Q = S\hat{B}R = 2\pi/3$

$\therefore$ Length of string

= length of major arc $PQ + PX + QX$ 1

+ length of major arc $SR + SX + RX$

$= 5(4\pi/3) + 5\sqrt{3} + 5\sqrt{3} + 3(4\pi/3) + 3\sqrt{3} + 3\sqrt{3}$

$= (32\pi/3 + 16\sqrt{3})$ cm. 1

(b) (i) Area minor segment = 1/4 area circle

$\therefore \frac{1}{2} r^2(\theta - \sin\theta) = \frac{1}{4}\pi r^2$ 1

$\therefore \theta - \sin\theta = \pi/2$. 1

(ii) $P(\theta) = \theta - \sin\theta - \pi/2$

$\therefore P'(\theta) = 1 - \cos\theta$ 1

If $\theta = 2.5$ is a first approximation then a second approximation is

$2.5 - \frac{P(2.5)}{P'(2.5)} = 2.5 - \frac{2.5 - \sin 2.5 - \pi/2}{1 - \cos 2.5}$ 1

$= 2.32$ (to 2 d.p)

and a third approximation is

$2.32 - \frac{P(2.32)}{P'(2.32)} = 2.32 - \frac{2.32 - \sin 2.32 - \pi/2}{1 - \cos 2.32}$

$= 2.31$ (to 2 d.p.) 1

QUESTION 6

(a) (i) $(M_1, M_2, M_3)\ W_1\ W_2\ W_3\ W_4\ W_5\ W_6$

$\therefore$ P(the 3 men sit next to each other)

$= \frac{7! \times 3!}{9!} = \frac{7! \times 3 \times 2 \times 1}{9 \times 8 \times 7!} = \frac{1}{12}$ 1 1

(ii) (α) With repetition of people not allowed

P(exactly 3 of the prizes are awarded to women)

$= \frac{^6C_3 \times {^3C_2}}{^9C_5} = \frac{20 \times 3}{126} = \frac{10}{21}$ 1 1

(β) With repetition of people allowed (binomial probability distribution) 1

P(exactly 3 of the prizes are awarded to women)

$= {^5C_3}(6/9)^3(3/9)^2$ 1

$= 10(8/27)(1/9) = 80/243$. 1

(b) (i) $x = a\cos nt \quad (a>0,\ 0<n<\pi)$

When $t = 1, x = 1 \quad \therefore a\cos n = 1$ 1

When $t = 2, x = -1 \quad \therefore a\cos 2n = -1$

$\therefore a\cos n + a\cos 2n = 0$

$\therefore \cos n + \cos 2n = 0$

$\therefore \cos n + 2\cos^2 n - 1 = 0, \therefore 2\cos^2 n + \cos n - 1 = 0$.

$\therefore (2\cos n - 1)(\cos n + 1) = 0, \therefore \cos n = 1/2$ or $\cos n = -1$

$\therefore n = \pi/3, 5\pi/3$ or $n = \pi$

$\therefore n = \pi/3 \quad (0 < n < \pi)$ 1

$\therefore a\cos \pi/3 = 1, \quad \therefore a(1/2) = 1$

$\therefore a = 2$

(ii) Amplitude $= a = 2$ metres 1

Period $= \frac{2\pi}{n} = \frac{2\pi}{\pi/3} = 6$ seconds. 1

QUESTION 7

(a) (i) $(1+x)^n$

$= 1 + {^nC_1}x + {^nC_2}x^2 + \dots + {^nC_r}x^r + \dots + x^n$

$= {^nC_0} + {^nC_1}x + {^nC_2}x^2 + \dots + {^nC_r}x^r + \dots + {^nC_n}x^n$

$\therefore$ putting $x = 1$. 1

${^nC_0} + {^nC_1} + {^nC_2} + \dots + {^nC_n} = 2^n$ 1

(ii) Total number of groups of 1 or more digits

$= {^{10}C_1}$ (groups of 1 digit) $+ {^{10}C_2}$ (groups of 2 digits) $+ \dots + {^{10}C_{10}}$ (groups of 10 digits)

$= {^{10}C_1} + {^{10}C_2} + \dots + {^{10}C_{10}}$

$= 2^{10} - {^{10}C_0}$

$1024 - 1$

$= 1023$.

(b) (i) Horizontally $a_x = 0$

$\therefore v_x = \int 0\, dt$

$\therefore v_x = c_1$ 1

When $t = 0, v_x = V\cos\alpha \quad \therefore c_1 = V\cos\alpha$

$\therefore v_x = V\cos\alpha$

$\therefore x = \int V\cos\alpha\, dt$

$\therefore x = (V\cos\alpha)t + c_2$

When $t = 0, x = 0 \quad \therefore c_2 = 0$

$\therefore x = Vt\cos\alpha$. 1

Vertically $a_y = -g$

$\therefore v_y = \int -g\, dt$

$\therefore v_y = -gt - c_3$

When $t = 0, v_y = V\sin\alpha \quad \therefore c_3 = V\sin\alpha$

$\therefore v_y = V\sin\alpha - gt$

$\therefore y = \int (V\sin\alpha - gt)\, dt$

$\therefore y = (V\sin\alpha)t - \frac{1}{2}gt^2 + c_4$

When $t = 0, y = 0 \quad \therefore c_4 = 0$

$\therefore y = Vt\sin\alpha - \frac{1}{2}gt^2$.

(ii) If the shell hits the aircraft at t= then horizontal distance AC (for shell)

= horizontal distance AB

+ horizontal distance BC (for aircraft)

$\therefore VT\cos\alpha = \frac{h}{\tan\alpha} + UT$ (1)

(iii) If the shell hits the aircraft then also vertical distance (for shell)

= height h (for aircraft)

$\therefore VT\sin\alpha - \frac{1}{2}gT^2 = h$ (2)

From (1), $VT\cos\alpha - UT = \frac{h}{\tan\alpha}$

$\therefore T(V\cos\alpha - U) = \frac{h}{\tan\alpha}, \therefore T = \frac{h}{(V\cos\alpha - U)\tan\alpha}$

In (2),

$\frac{V\sin\alpha . h}{(V\cos\alpha - U)\tan\alpha} - \frac{gh^2}{2(V\cos\alpha - U)^2\tan^2\alpha} = h$

$\therefore V\cos\alpha . 2(V\cos\alpha - U)\tan^2\alpha - gh$

$= 2(V\cos\alpha - U)^2\tan^2\alpha$

$\therefore 2V\cos\alpha(V\cos\alpha - U)\tan^2\alpha$

$- 2(V\cos\alpha - U)^2\tan^2\alpha = gh$

$\therefore 2(V\cos\alpha - U)\tan^2\alpha[V\cos\alpha - (V\cos\alpha - U)] = gh$

$\therefore 2U(V\cos\alpha - U)\tan^2\alpha = gh$.

3UNIT CATHOLIC

Question 1

(a) (3 marks)

Find the acute angle between the lines $2x - y = 0$ and $x + 3y = 0$, giving the answer correct to the nearest minute.

(b) (4 marks)

Consider the function $y = x\ln x - x$.

(i) Solve the equation $y = 0$.

(ii) Find $\frac{d^2y}{dx^2}$ and hence show that the function is concave up for all values of x in its domain.

(c) (5 marks)

Consider the polynomial $P(x) = 6x^3 - 5x^2 - 2x + 1$

(i) Show that 1 is a zero of $P(x)$.

(ii) Express $P(x)$ as a product of 3 linear factors.

(iii) Solve the inequality $P(x) \le 0$.

Question 2

(a) (3 marks)

(i) Find $\frac{d}{dx}\left(e^{\tan x}\right)$

(ii) Hence find $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

(b) Use the substitution $u = 1 - x$ to evaluate $\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$. (4 marks)

(c) (5 marks)

(i) Find the value of x such that $\sin^{-1} x = \cos^{-1} x$.

(ii) On the same axes sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

(iii) On the same diagram as the graphs in (ii), draw the graph of $y = \sin^{-1} x + \cos^{-1} x$.

Question 3

(a) (8 marks)

$$f(x) = \frac{8}{4 + x^2}$$

(i) Show that f is an even function, and the x axis is a horizontal asymptote to the curve $y = f(x)$.

(ii) Find the coordinates and nature of the stationary point on the curve $y = f(x)$.

(iii) Sketch the graph of the curve showing the above features.

(iv) Find the exact area of the region in the first quadrant bounded by the curve $y = f(x)$ and the line $x = 2$.

(b) (4 marks)

A vertical tower of height h metres stands on horizontal ground. From a point P on the ground due east of the tower the angle of elevation of the top of the tower is $45°$. From a point Q on the ground due south of the tower the angle of elevation of the top of the tower is $30°$. If the distance PQ is 40 metres, find the exact height of the tower.

Question 4

(a) (8 marks)

N is the number of animals in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = -k(N - 1000)$, for some constant k.

(i) Verify by differentiation that $N = 1000 + Ae^{-kt}$, A constant, is a solution of the equation.

(ii) Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of A and k.

(iii) Find when the number of animals has fallen to 1300.

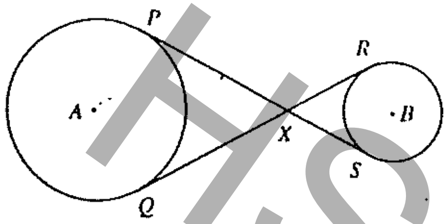
(iv) Sketch the graph of the population size against time.

(b) (4 marks)

Use Mathematical Induction to show that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \ge 1$.

Question 5

a) (7 marks)



In the diagram PS and QR are tangents to each of the circles with centres A and B. The tangents intersect at X and A, X, B are collinear.

(i) Copy the diagram and show that $\Delta APX \ ||| \ \Delta BSX$.

(ii) Suppose that the diagram represents two circles of radii 5 cm and 3 cm that are placed in the same plane with their centres 16 cm apart. A taut string surrounds the circles and crosses itself between them. Find the exact length of the string.

(b) (5 marks)

The interior of a circle is divided into two segments with areas in the ratio $3:1$ by a chord which subtends an angle θ radians at the centre of the circle.

(i) Show that $\theta - \sin\theta = \frac{\pi}{2}$.

(ii) Taking $\theta = 2 \cdot 5$ as a first approximation, use Newton's method twice to find a better approximation to θ, giving the answer correct to 2 decimal places.

Question 6

(a) (7 marks)

A group consisting of 3 men and 6 women attends a prizegiving ceremony.

(i) If the members of the group sit down at random in a straight line, find the probability that the 3 men sit next to each other.

(ii) If 5 prizes are awarded at random to members of the group, find the probability that exactly 3 of the prizes are awarded to women if

(α) there is a restriction of at most one prize per person.

(β) there is no restriction on the number of prizes per person.

Question 6 (cont.)

(b) (5 marks)

A particle moving in a straight line is performing Simple Harmonic Motion about a fixed point O on the line. At time t seconds the displacement x metres of the particle from O is given by

$$x = a\cos nt \quad , \quad \text{where } a > 0 \text{ and } 0 < n < \pi .$$

After 1 second the particle is 1 metre to the right of O, and after 2 seconds the particle is 1 metre to the left of O.

(i) Find the values of n and a.

(ii) Find the amplitude and period of the motion.

Question 7

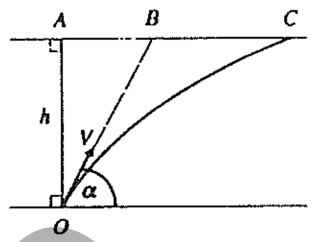
(a) (4 marks)

(i) Write down the Binomial expansion of $(1+x)^n$ in ascending powers of x.

Hence show that $\quad {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$.

(ii) Find how many groups of 1 or more digits can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 where repetition is not allowed.

(b) (8 marks)



In the diagram an aircraft is flying with constant velocity U at a constant height h above horizontal ground. When the plane is at A it is directly over a gun at O. When the plane is at B a shell is fired from the gun at the aircraft along OB. The shell is fired with initial velocity V at an angle of elevation α.

(i) If x and y are the horizontal and vertical displacements of the shell from O at time t seconds, show that if g is the acceleration due to gravity,

$$x = Vt\cos\alpha \quad \text{and} \quad y = Vt\sin\alpha - \tfrac{1}{2}gt^2 .$$

(ii) Show that if the shell hits the aircraft at time T at point C, then

$$VT\cos\alpha = \frac{h}{\tan\alpha} + UT .$$

(iii) Show that if the shell hits the aircraft then $2U(V\cos\alpha - U)\tan^2\alpha = gh$.

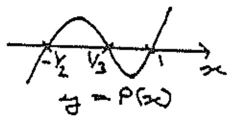
MATHEMATICS 3 UNIT SOLUTIONS 1996

QUESTION 1

(a) $2x - y = 0$ has gradient $m_1 = 2$.
$x + 3y = 0$ has gradient $m_2 = -1/3$.
Let θ be the acute angle between the lines.
$\therefore \tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{2 - (-1/3)}{1 + 2(-1/3)}\right| = 7$
$\theta = 81°52'$ to the nearest minute.

(b) $y = x\ln x - x$ has domain $x > 0$.
(i) $x \ln x - x = 0$
$\therefore x(\ln x - 1) = 0$
$\therefore \cancel{x = 0}$ (not in domain) or $\ln x = 1$
$\therefore x = e$.
(ii) $dy/dx = (x)(1/x) + (\ln x)(1) - (1)$
$= 1 + \ln x - 1$
$= \ln x$
$d^2y/dx^2 = 1/x$.
for $x > 0$, $d^2y/dx^2 > 0$ and the function is concave up for all x in its domain.

(c) $P(x) = 6x^3 - 5x^2 - 2x + 1$
(i) $P(1) = 6 - 5 - 2 + 1 = 0$
1 is a zero of $P(x)$.
(ii) $(x-1)$ is a factor of $P(x)$
$P(x) = 6x^3 - 5x^2 - 2x + 1$
$= (x-1)(6x^2 + x - 1)$
$= (x-1)(3x-1)(2x+1)$
(iii)



$P(x) \le 0$ when the graph lies on or below the x axis.
$\therefore x \le -1/2$ or $1/3 \le x \le 1$.

①

QUESTION 2

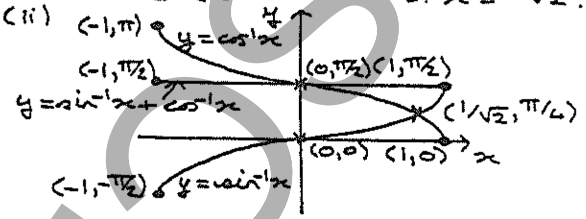
(a) (i) $d/dx\, e^{\tan x} = e^{\tan x} \cdot \sec^2 x$
$= \sec^2 x\, e^{\tan x}$
(ii) $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\,dx = \int \sec^2 x\, e^{\tan x}\,dx$
$= e^{\tan x} + c$

(b) $\displaystyle\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$
$= \displaystyle\int_4^1 \frac{(1-u)}{\sqrt{u}} \cdot (-du)$
$= \displaystyle\int_1^4 (u^{-1/2} - u^{1/2})\,du$
$= \left[2\sqrt{u} - 2/3\, u\sqrt{u}\right]_1^4$
$= \{4 - 16/3\} - \{2 - 2/3\}$
$= -8/3$.

$u = 1 - x$
$x = 1 - u$
$dx/du = -1$
$dx = -du$
when $x = -3$, $u = 4$
when $x = 0$, $u = 1$

(c) (i) Let $\sin^{-1} x = a$ $(-1 \le x \le 1,\ -\pi/2 \le a \le \pi/2)$
Let $\cos^{-1} x = a$ $(-1 \le x \le 1,\ 0 \le a \le \pi)$
$\therefore \left.\begin{array}{l}\sin a = x\\ \cos a = x\end{array}\right\}$ $\therefore \tan a = 1$, $\therefore a = \pi/4$
$\therefore x = 1/\sqrt{2}$.

(ii)



(iii) on diagram $y = \sin^{-1}x + \cos^{-1}x$
$\therefore y = \pi/2$ $(-1 \le x \le 1)$

②

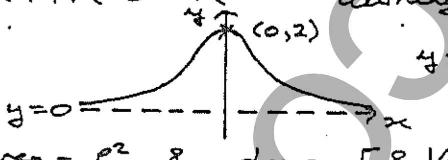
QUESTION 3

(a) (i) $f(-x) = \dfrac{8}{4 + (-x)^2} = \dfrac{8}{4 + x^2} = f(x)$
for all values of x. $\therefore$ function is even.
$\lim\limits_{x \to \pm\infty} \dfrac{8}{4 + x^2} = 0$ $\therefore y = 0$ is a horizontal asymptote.
(ii) $\dfrac{dy}{dx} = \dfrac{-16x}{(4 + x^2)^2} = 0$ at stationary point $(0, 2)$.

x : 0^- 0 0^+
dy/dx : $+ve$ 0 $-ve$

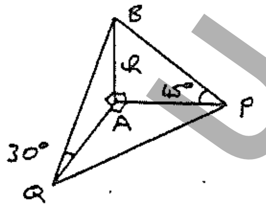
$\therefore (0,2)$ is a maximum turning point.

(iii)



(iv) Area $= \displaystyle\int_0^2 \frac{8}{4 + x^2}\,dx = \left[8 \cdot 1/2 \tan^{-1} x/2\right]_0^2$
$= 4\tan^{-1} 1 - \cancel{4\tan^{-1} 0} = 4 \cdot \pi/4 = \pi$ units^2.

(b)



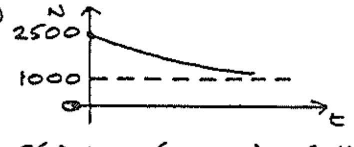
In $\triangle PAB$, $\tan 45° = h/PA$
$\therefore PA = \dfrac{h}{\tan 45°} = h$
In $\triangle QAB$, $\tan 30° = h/QA$
$\therefore QA = \dfrac{h}{\tan 30°} = h\sqrt{3}$.

In $\triangle PAQ$, $PA^2 + QA^2 = PQ^2$
$\therefore (h)^2 + (h\sqrt{3})^2 = 40^2$, $\therefore h^2 + 3h^2 = 1600$
$\therefore 4h^2 = 1600$, $\therefore h^2 = 400$, $\therefore h = 20$ $(h > 0)$
$\therefore$ the height of the tower is 20 metres.

③

QUESTION 4

(a) (i) LHS $= dN/dt = d/dt\,(1000 + Ae^{-kt})$
$= -kAe^{-kt} = -k(N - 1000) =$ RHS.
(ii) when $t = 0$, $N = 2500$, $\therefore 2500 = 1000 + A$
$\therefore A = 1500$, $\therefore N = 1000 + 1500e^{-kt}$
when $t = 2$, $N = 2200$, $\therefore 2200 = 1000 + 1500e^{-2k}$
$\therefore e^{2k} = 5/4$, $\therefore k = 1/2 \ln(5/4)$.
(iii) when $N = 1300$, $1300 = 1000 + 1500e^{-kt}$
$\therefore e^{kt} = 5$, $\therefore t = 1/k \ln(5)$
$\therefore t = \ln(5) / 1/2 \ln(5/4) = 14{\cdot}4$ years (to 1 d.p.)
(iv)



As $t \to \infty$, $N \to 1000$
$N = 1000$ is a horizontal asymptote

(b) $S(n)$: $\cos(x + n\pi) = (-1)^n \cos x$ $(n \ge 1)$
when $n = 1$, LHS $= \cos(x + \pi) = -\cos x$
RHS $= (-1)^1 \cos x = -\cos x$
$\therefore S(1)$ is true.
If $S(k)$ is true $(k \ge 1)$
ie. if $\cos(x + k\pi) = (-1)^k \cos x$
then $\cos(x + (k+1)\pi)$
$= \cos((x + k\pi) + \pi)$
$= \cos(x + k\pi)\cos\pi - \sin(x + k\pi)\sin\pi$
$= (-1)^k \cos x \cdot (-1) - \sin(x + k\pi) \cdot 0$
$= (-1)^{k+1} \cos x$
and $S(k+1)$ is also true.
Since $S(1)$ is true and if $S(k)$ $(k \ge 1)$ is true then $S(k+1)$ is also true it follows that $S(n)$ is true for all positive integers $n \ge 1$.
$\therefore \cos(x + n\pi) = (-1)^n \cos x$ for all $n \ge 1$.

④

QUESTION 5

(a)(i) In ΔAPX and ΔBSX

$\angle APX = \angle BSX (=90°)$ (tangent ⊥ radius)

$\angle PXA = \angle SXB$ (vertically opposite angles)

$\angle XAP = \angle XBS$ (alternate angles $AP \parallel SB$)

$\therefore \Delta APX \parallel\mid \Delta BSX$ (equiangular)

(ii) $\frac{AX}{BX} = \frac{AP}{BS} \left(=\frac{5}{3}\right)$ (corresponding sides are proportional)

But $AB = 16$ cm $\therefore AX = 10$ cm, $BX = 6$ cm.

Now $\cos P\hat{A}X = \cos S\hat{B}X = 1/2$

$\therefore P\hat{A}X = S\hat{B}X = \pi/3$, $P\hat{A}Q = S\hat{B}R = 2\pi/3$

$\therefore$ Length of string

= length of major arc $PQ + PX + QX$

+ length of major arc $SR + SX + RX$

$= 5(4\pi/3) + 5\sqrt{3} + 5\sqrt{3} + 3(4\pi/3) + 3\sqrt{3} + 3\sqrt{3}$

$= (32\pi/3 + 16\sqrt{3})$ cm.

(b)(i) Area minor segment = 1/4 area circle

$\therefore \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{4}\pi r^2$

$\therefore \theta - \sin\theta = \pi/2$

(ii) $P(\theta) = \theta - \sin\theta - \pi/2$

$\therefore P'(\theta) = 1 - \cos\theta$

If $\theta = 2.5$ is a first approximation then a second approximation is

$2.5 - \frac{P(2.5)}{P'(2.5)} = 2.5 - \frac{2.5 - \sin 2.5 - \pi/2}{1 - \cos 2.5}$

$= 2.32$ (to 2 d.p.)

and a third approximation is

$2.32 - \frac{P(2.32)}{P'(2.32)} = 2.32 - \frac{2.32 - \sin 2.32 - \pi/2}{1 - \cos 2.32}$

$= 2.31$ (to 2 d.p.)

QUESTION 6

(a) (i) $(M_1, M_2, M_3)\ W_1\ W_2\ W_3\ W_4\ W_5\ W_6$

$\therefore$ P(the 3 men sit next to each other)

$= \frac{7! \times 3!}{9!} = \frac{7! \times 3 \times 2 \times 1}{9 \times 8 \times 7!} = \frac{1}{12}$.

(ii) (α) With repetition of people not allowed

P(exactly 3 of the prizes are awarded to women)

$= \frac{{}^6C_3 \times {}^3C_2}{{}^9C_5} = \frac{20 \times 3}{126} = \frac{10}{21}$

(β) With repetition of people allowed (binomial probability distribution)

P(exactly 3 of the prizes are awarded to women)

$= {}^5C_3 (6/9)^3 (3/9)^2$

$= 10(8/27)(1/9) = 80/243$.

(b)(i) $x = a\cos nt$ $(a > 0,\ 0 < n < \pi)$

When $t = 1, x = 1$ $\therefore a\cos n = 1$

When $t = 2, x = -1$ $\therefore a\cos 2n = -1$

$\therefore a\cos n + a\cos 2n = 0$

$\therefore \cos n + \cos 2n = 0$

$\therefore \cos n + 2\cos^2 n - 1 = 0$, $\therefore 2\cos^2 n + \cos n - 1 =$

$\therefore (2\cos n - 1)(\cos n + 1) = 0$, $\therefore \cos n = 1/2$ or $\cos n =$

$\therefore n = \pi/3, 5\pi/3$ or $n = \pi$

$\therefore n = \pi/3$ $(0 < n < \pi)$

$\therefore a\cos\pi/3 = 1$, $\therefore a(1/2) = 1$.

$\therefore a = 2$

(ii) Amplitude $= a = 2$ metres.

Period $= \frac{2\pi}{n} = \frac{2\pi}{\pi/3} = 6$ seconds.

QUESTION 7

(i) $(1+x)^n$

$= 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + x^n$

$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$

putting $x = 1$,

${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$

(ii) Total number of groups of 1 or more digits

${}^{10}C_1$ (groups of 1 digit) + ${}^{10}C_2$ (groups of 2 digits)

$\dots + {}^{10}C_{10}$ (groups of 10 digits)

$= {}^{10}C_1 + {}^{10}C_2 + \dots + {}^{10}C_{10}$

$= 2^{10} - {}^{10}C_0$

$= 1024 - 1$

$= 1023$.

(i) Horizontally $a_x = 0$

$v_x = \int 0\, dt$

$v_x = c_1$

When $t = 0$, $v_x = V\cos\alpha$ $\therefore c_1 = V\cos\alpha$

$v_x = V\cos\alpha$

$\therefore x = \int V\cos\alpha\, dt$

$\therefore x = (V\cos\alpha)t + c_2$

When $t = 0, x = 0$ $\therefore c_2 = 0$

$x = Vt\cos\alpha$

Vertically $a_y = -g$

$v_y = \int -g\, dt$

$v_y = -gt + c_3$

When $t = 0$, $v_y = V\sin\alpha$ $\therefore c_3 = V\sin\alpha$

$\therefore v_y = V\sin\alpha - gt$

$\therefore y = \int (V\sin\alpha - gt)\, dt$

$y = (V\sin\alpha)t - \frac{1}{2}gt^2 + c_4$

When $t = 0, y = 0$ $\therefore c_4 = 0$

$\therefore y = Vt\sin\alpha - \frac{1}{2}gt^2$.

(ii) If the shell hits the aircraft at time T then horizontal distance AC (for shell)

= horizontal distance AB

+ horizontal distance BC (for aircraft)

$\therefore VT\cos\alpha = \frac{h}{\tan\alpha} + UT$ (1)

(iii) If the shell hits the aircraft then also vertical distance (for shell)

= height h (for aircraft)

$\therefore VT\sin\alpha - \frac{1}{2}gT^2 = h$ (2)

From (1), $VT\cos\alpha - UT = \frac{h}{\tan\alpha}$

$\therefore T(V\cos\alpha - U) = \frac{h}{\tan\alpha}$, $\therefore T = \frac{h}{(V\cos\alpha - U)\tan\alpha}$

In (2),

$\frac{V\sin\alpha \cdot h}{(V\cos\alpha - U)\tan\alpha} - \frac{gh^2}{2(V\cos\alpha - U)^2\tan^2\alpha} = h$

$\therefore V\cos\alpha \cdot 2(V\cos\alpha - U)\tan^2\alpha - gh$

$= 2(V\cos\alpha - U)^2\tan^2\alpha$

$\therefore 2V\cos\alpha(V\cos\alpha - U)\tan^2\alpha$

$- 2(V\cos\alpha - U)^2\tan^2\alpha = gh$

$\therefore 2(V\cos\alpha - U)\tan^2\alpha[V\cos\alpha - (V\cos\alpha - U)]$

$\therefore 2U(V\cos\alpha - U)\tan^2\alpha = gh$.

1994 Catholic 3/4 Unit Trial H.S.C

QUESTION 1

(a) If the positive numbers a, b, c are three consecutive terms in a geometric sequence show that $\log_e a, \log_e b, \log_e c$ are three consecutive terms in an arithmetic sequence.

(b) (i) Write down the expansion of $\cos(\alpha + \beta)$.

(ii) Write down the exact values of $\cos 30^\circ$ and $\cos 45^\circ$.

(iii) Hence find the exact value of $\cos 75^\circ$.

(c) The equation $x^3 - 2x^2 + 4x - 5 = 0$ has roots α, β, γ.

(i) Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

(ii) Hence find the value of $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$.

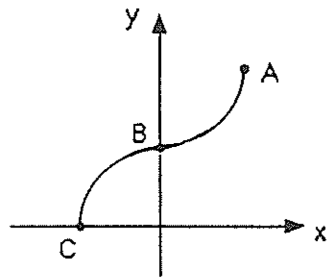
QUESTION 2

(a) (i) Find $\frac{d}{dx} e^{3x^2}$ $6x e^{3x^2}$

(ii) Hence find $\int x e^{3x^2} \, dx$ $\frac{e^{3x^2}}{6}$

(b) Use the substitution $u = \log_e x$ to evaluate $\int_1^e \frac{(\log_e x)}{x} \, dx$ $\int_1^e$

(c) The diagram below shows the graph of $y = \pi + 2\sin^{-1} 3x$



(i) Write down the coordinates of the endpoints A and C.

(ii) Write down the coordinates of the point B.

(iii) Find the equation of the tangent to the curve $y = \pi + 2\sin^{-1} 3x$ at the point B.

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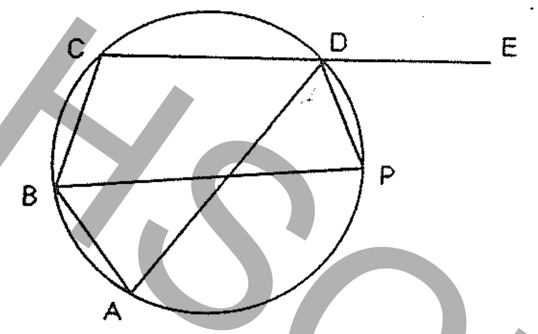
QUESTION 3

(a)



In the diagram above ABCD is a cyclic quadrilateral. CD is produced to E. P is a point on the circle through A, B, C, D such that $\angle ABP = \angle PBC$.

(i) Copy the diagram showing the above information.

(ii) Explain why $\angle ABP = \angle ADP$.

(iii) Show that PD bisects $\angle ADE$.

(iv) If, in addition, $\angle BAP = 90^\circ$ and $\angle APD = 90^\circ$, explain where the centre of the circle is located.

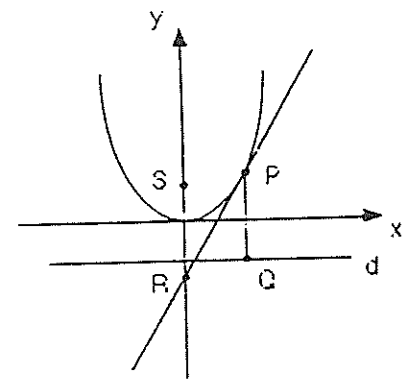
(b) For the function $y = x + e^{-x}$

(i) find the coordinates and the nature of any stationary points on the graph of $y = f(x)$ and show that the graph is concave upwards for all values of x.

(ii) sketch the graph of $y = f(x)$ showing clearly the coordinates of any turning points and the equations of any asymptotes.

QUESTION 4

(a)



$P(2at, at^2)$ is a point on the parabola $x^2 = 4ay$. S is the focus of the parabola. PQ is the perpendicular from P to the directrix d of the parabola. The tangent at P to the parabola cuts the axis of the parabola at the point R.

(i) Show that the tangent at P to the parabola has equation $tx - y - at^2 = 0$.

(ii) Show that PR and QS bisect each other.

(iii) Show that PR and QS are perpendicular to each other.

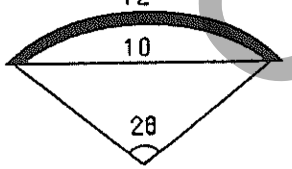
(iv) State with reason what type of quadrilateral PQRS is.

(b) In the expansion of $(1-2x)(1+ax)^{10}$ the coefficient of x^5 is 0. Find the value of a.

QUESTION 5

(a) A body is moving in a straight line. At time t seconds its displacement is x metres from a fixed point O on the line and its velocity is v ms^{-1}. If $v = \frac{1}{x}$ find its acceleration when $x = 0.5$.

(b) A pipe which is 12 metres long is bent into a circular arc which subtends an angle of 2θ radians at the centre of the circle. The chord of the circle joining the ends of the arc is 10 metres long.



(i) Show that $6 \sin \theta - 5\theta = 0$.

(ii) Show that $\theta_0 = 1$ radian is a good first approximation to the value of θ.

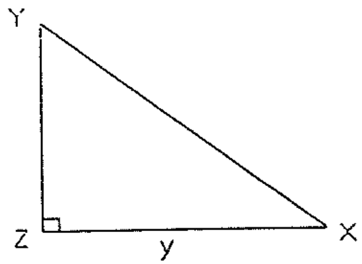
(iii) Use one application of Newton's method to find a better approximation θ_1 to the value of θ.
Use this value of θ_1 to find an approximation to the length of the radius of the arc, rounding off this approximation correct to two decimal places.

QUESTION 6

(a) (i) Write down the expression for $\tan 2a$ in terms of $\tan a$.

(ii) If $f(a) = a \cot a$ show that $f(2a) = (1 - \tan^2 a) f(a)$.

(b)



In ΔXYZ, $ZX = y$ and $\angle YZX = 90^\circ$.

(i) Show that the area A and perimeter P of the triangle are given by $A = \frac{1}{2} y^2 \tan X$ and $P = y(1 + \tan X + \sec X)$ respectively.

(ii)
(α) If $X = \frac{\pi}{4}$ radians and y is increasing at a constant rate of 0.1 cm s^{-1} find the rate at which the area of the triangle is increasing at the instant when $y = 20$ cm.

(β) If $y = 10$ cm and X is increasing at a constant rate of 0.2 radians s^{-1} find the rate at which the perimeter of the triangle is increasing when $X = \frac{\pi}{6}$ radians.

QUESTION 7

(a) An employer wishes to choose two people for a job. There are eight applicants, three of whom are women and five of whom are men.

(i) If each applicant is interviewed separately and all of the women are interviewed before any of the men, find how many ways there are of carrying out the interviews.

(ii) If the employer chooses two of the applicants at random, find the probability that at least one of those chosen is a woman.

(b) A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds its displacement x metres from a fixed point O on the line is given by $x = 2\cos^2 t$.

(i) Show that its velocity v ms^{-1} and its acceleration $\ddot{x}$ ms^{-2} are given by $v^2 = 4(2x - x^2)$ and $\ddot{x} = -4(x - 1)$ respectively.

(ii) Find the centre, amplitude and period of the motion.

C.S.S.A. MATHEMATICS 3 UNIT PAPER SOLUTIONS

1994

QUESTION 1

(a) a, b, c are three consecutive terms in a geometric sequence

$\therefore \frac{b}{a} = \frac{c}{b}$ (= common ratio)

$\therefore \log_e\left(\frac{b}{a}\right) = \log_e\left(\frac{c}{b}\right)$

$\therefore \log_e b - \log_e a = \log_e c - \log_e b$
(= common difference)

$\therefore \log_e a, \log_e b, \log_e c$ are three consecutive terms in an arithmetic sequence.

(b) (i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(ii) $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\cos 45^\circ = \frac{1}{\sqrt{2}}$

(iii) $\cos 75^\circ = \cos(30^\circ + 45^\circ)$

$= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

(c) $x^3 - 2x^2 + 4x - 5 = 0$ has roots α, β, γ

(i) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{+4}{1} = 4$

$\alpha\beta\gamma = \frac{-(-5)}{1} = 5$

(ii) $\alpha^{-1} + \beta^{-1} + \gamma^{-1} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{4}{5}$

QUESTION 2

(a) (i) $\frac{d}{dx} e^{3x^2} = e^{3x^2} \cdot \frac{d}{dx} 3x^2$

$= 6x e^{3x^2}$

(ii) $\int x e^{3x^2}\,dx = \frac{1}{6}\int 6x e^{3x^2}\,dx$

$= \frac{1}{6} e^{3x^2} + c$

(b) $u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}$ $\quad \therefore \frac{1}{x}\,dx = du$

When $x = 1$, $u = \log_e 1 = 0$

When $x = e$, $u = \log_e e = 1$

$\therefore \int_1^e \frac{\log_e x}{x}\,dx = \int_1^e \log_e x \,\frac{1}{x}\,dx$

$= \int_0^1 u\,du$

$= \left[\frac{1}{2}u^2\right]_0^1$

$= \frac{1}{2}$.

(c) $y = \pi + 2\sin^{-1} 3x$

(i) $-1 \le 3x \le 1$ $\quad \therefore -\frac{1}{3} \le x \le \frac{1}{3}$

at A $x = \frac{1}{3}$, $y = \pi + 2\sin^{-1}(1) = \pi + 2(\pi/2) = 2\pi$

$\therefore A(\frac{1}{3}, 2\pi)$

at C $x = -\frac{1}{3}$, $y = \pi + 2\sin^{-1}(-1) = \pi + 2(-\pi/2) = 0$

$\therefore C(-\frac{1}{3}, 0)$

(ii) at B $x = 0$, $y = \pi + 2\sin^{-1}(0) = \pi + 2(0) = \pi$

$\therefore B(0, \pi)$

(iii) $\frac{dy}{dx} = 2\cdot\frac{1}{\sqrt{1-(3x)^2}}\cdot 3 = \frac{6}{\sqrt{1-9x^2}}$

$\therefore$ at $B(0,\pi)$

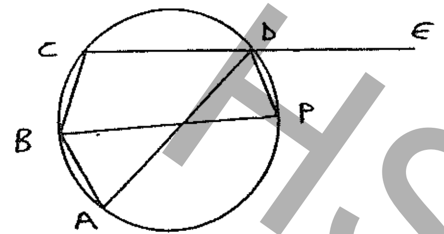
gradient of tangent = value of dy/dx
$= 6$

and equation of tangent is $y - \pi = 6(x-0)$

ie $6x - y + \pi = 0$.

QUESTION 3

(a)(i)



(ii) $\angle ABP = \angle ADP$ because angles in the same segment standing on the arc AP are equal.

(iii) $\angle PDE = \angle PBC$ (exterior angle of cyclic quad. PBCD = interior opposite angle)

$\angle PBC = \angle PBA$ (given)

$\angle PBA = \angle PDA$ (proved above in (ii))

$\therefore \angle PDE = \angle PDA$

$\therefore$ PD bisects $\angle ADE$.

(iv) BP and AD are both diameters because an angle of $90°$ stands in a semicircle.

$\therefore$ the centre of the circle is located at the intersection of the diameters BP and AD.

(b) $y = x + e^{-x}$

(i) $\dfrac{dy}{dx} = 1 - e^{-x}$, $\dfrac{d^2y}{dx^2} = e^{-x}$

$\dfrac{dy}{dx} = 0$ when $x = 0$; when $x = 0$, $y = 1$

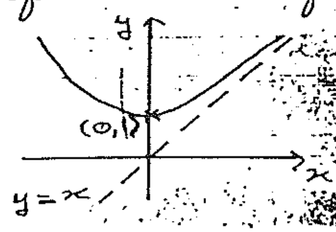
When $x = 0$ $\dfrac{d^2y}{dx^2} > 0$ $\therefore$ $(0,1)$ is a minimum turning point.

$\dfrac{d^2y}{dx^2} > 0$ for all x $\therefore$ graph is concave upwards for all values of x

(ii) As $x \to \infty$

$y = x + e^{-x}$ behaves like $y = x$.

$\therefore$ $y = x$ is an asymptote.



QUESTION 4

(a)(i) $x^2 = 4ay$, $y = \dfrac{x^2}{4a}$ $\therefore \dfrac{dy}{dx} = \dfrac{2x}{4a} = \dfrac{x}{2a}$

At $P(2at, at^2)$ $\dfrac{dy}{dx} = \dfrac{2at}{2a} = t$

$\therefore$ gradient of tangent $= t$

$\therefore$ equation of tangent is

$y - at^2 = t(x - 2at)$ ie $tx - y - at^2 = 0$.

now P is $(2at, at^2)$,

focus S is $(0, a)$,

directrix d is $y = -a$ $\therefore$ Q is $(2at, -a)$,

R is where $tx - y - at^2 = 0$ intersects $x = 0$

$\therefore$ R is $(0, -at^2)$

(ii) PR has midpoint $(at, 0)$

QS has midpoint $(at, 0)$

$\therefore$ PR and QS have the same midpoint.

$\therefore$ PR and QS bisect each other

(iii) gradient PR . gradient QS $= (t).\left(\dfrac{-1}{t}\right) = -1$

$\therefore$ PR and QS are perpendicular to each other

(iv) PQRS is a rhombus because the diagonals PR and QS bisect each other at right angles

(b) $(1-2x)(1+ax)^{10}$

$= (1-2x)(1 + \dots + {}^{10}C_5(ax)^5 + {}^{10}C_6(ax)^6 + \dots + (ax)^{10})$

coefficient of $x^6 = {}^{10}C_6\, a^6 - 2 . {}^{10}C_5\, a^5$

$\therefore {}^{10}C_6\, a^6 - 2 . {}^{10}C_5\, a^5 = 0$

${}^{10}C_6\, a^6 = 2 . {}^{10}C_5\, a^5$

$\therefore 210 a^6 = 2 . 252 a^5$

$\therefore a = \dfrac{504}{210} = \dfrac{12}{5}$

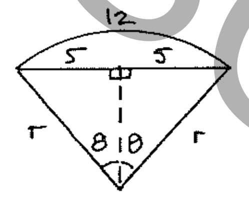
QUESTION 5

(a) acceleration $= v \cdot \dfrac{dv}{dx}$

$= \dfrac{1}{x} \cdot \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{1}{x} \cdot \dfrac{-1}{x^2} = \dfrac{-1}{x^3}$

When $x = 0.5$
acceleration $= \dfrac{-1}{(0.5)^3} = -8 \text{ ms}^{-2}$

(b)



(i) arc length $12 = r \cdot 2\theta \quad \therefore r = \dfrac{6}{\theta}$

Also $\sin\theta = \dfrac{5}{r} \quad \therefore r = \dfrac{5}{\sin\theta}$

$\therefore \dfrac{6}{\theta} = \dfrac{5}{\sin\theta}$ ie $6\sin\theta - 5\theta = 0$

(ii) $f(\theta) = 6\sin\theta - 5\theta$
$f(\theta)$ is continuous and for example
$f(0.95) = 0.1305$ (to 4 d.p.) (+ve)
$f(1.05) = -0.0455$ (to 4 d.p.) (−ve)
$\therefore \theta_0 = 1$ is a good first approximation to θ

(iii) $f'(\theta) = 6\cos\theta - 5$
$\theta_1 = \theta_0 - \dfrac{f(\theta_0)}{f'(\theta_0)} = 1 - \dfrac{6\sin(1) - 5(1)}{6\cos(1) - 5}$
$\therefore \theta_1 = 1.0278$ (to 4 d.p.)
$\therefore$ radius of arc $\doteqdot \dfrac{6}{\theta_1} = 5.84$ cm (to 2 d.p.)

QUESTION 6

(a)(i) $\tan 2a = \dfrac{2\tan a}{1 - \tan^2 a}$

(ii) $f(a) = a\cot a$
$\therefore f(2a) = 2a\cot 2a$
$= 2a \cdot \dfrac{1}{\tan 2a}$
$= 2a \cdot \dfrac{1 - \tan^2 a}{2\tan a}$
$= (1 - \tan^2 a) \cdot a\cot a$
$= (1 - \tan^2 a) f(a)$ as required

(b)(i) $A = \dfrac{1}{2}(ZX)(YZ) = \dfrac{1}{2}(y)(y\tan X) = \dfrac{1}{2}y^2\tan X$

$P = ZX + XY + YZ$
$= y + y\sec X + y\tan X$
$= y(1 + \tan X + \sec X)$

(ii)(α) $X = \dfrac{\pi}{4} \quad \therefore A = \dfrac{1}{2}y^2\tan\dfrac{\pi}{4} = \dfrac{1}{2}y^2$

$\dfrac{dA}{dt} = \dfrac{dA}{dy} \cdot \dfrac{dy}{dt} = y \cdot (0.1) = (0.1)y$

When $y = 20$, $\dfrac{dA}{dt} = (0.1).(20) = 2$

$\therefore A$ is increasing at $2 \text{ cm}^2\text{s}^{-1}$

(β) $y = 10 \quad \therefore P = 10(1 + \tan X + \sec X)$

$\dfrac{dP}{dt} = \dfrac{dP}{dX} \cdot \dfrac{dX}{dt}$
$= 10 \cdot (\sec^2 X + \sec X\tan X) \cdot (0.2)$
$= 2(\sec^2 X + \sec X\tan X)$

When $X = \dfrac{\pi}{6}$, $\dfrac{dP}{dt} = 2\left(\sec^2\dfrac{\pi}{6} + \sec\dfrac{\pi}{6}\tan\dfrac{\pi}{6}\right)$
$= 2\left(\left(\dfrac{2}{\sqrt{3}}\right)^2 + \left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{1}{\sqrt{3}}\right)\right) = 4$

P is increasing at 4 cms^{-1}

QUESTION 7

(a)(i) number of ways of interviewing the 3 women
$= 3! = 6$.
Number of ways of interviewing the 5 men
$= 5! = 120$
$\therefore$ number of ways of carrying out the interviews
$= 6 \times 120 = 720$.

(ii) P(at least 1 woman chosen)
$= 1 - P(0 \text{ women chosen})$
$= 1 - \dfrac{{}^3C_0 \cdot {}^5C_2}{{}^8C_2} = 1 - \dfrac{1.10}{28} = \dfrac{9}{14}$

(d) $x = 2\cos^2 t$
$\therefore x = \cos 2t + 1$

(i) $v = \dot{x} = -2\sin 2t$
$\therefore v^2 = 4\sin^2 2t$
$= 4(1-\cos^2 2t)$
$= 4(1-(x-1)^2)$
$= 4(2x - x^2)$
and $\ddot{x} = -4\cos 2t$
$= -4(x-1)$

(ii) Since $\ddot{x} = -4(x-1)$ the centre of the motion is at $x=1$ ie 1 metre to the right of O.
Since $v^2 = 4(2x-x^2) = 4x(2-x)$
$v=0$ when $x=0$ or $x=2$
$\therefore$ the extremes of the motion are at $x=0$ and $x=2$, and since the centre is at $x=1$
the amplitude of the motion is 1 metre
Since $\ddot{x} = -4(x-1) = -(2)^2(x-1)$ the period of the motion is $\dfrac{2\pi}{2} = \pi$ seconds.

CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1993

MATHEMATICS

3/4 UNIT COMMON PAPER

(i.e. 3 UNIT COURSE — ADDITIONAL PAPER: 4 UNIT COURSE — FIRST PAPER)

Afternoon session — Friday 13th August 1993.

Time Allowed — Two Hours

EXAMINERS

Glenn Abrahams, Patrician Brothers' College, Fairfield
Graham Arnold, John Paul II Senior High, Marayong
Sandra Hayes, All Saints Catholic Senior High, Casula.

DIRECTIONS TO CANDIDATES :

ALL questions may be attempted.

ALL questions are of equal value.

All necessary working should be shown in every question.

Full marks may not be awarded for careless or badly arranged work.

Approved calculators may be used.

Standard integrals are printed on a separate page.

QUESTION 1

(a) Find $\int \frac{1}{\sqrt{9 - x^2}}\,dx$

(b) If $\sin\alpha = \frac{3}{4}$, $0 < \alpha < \frac{\pi}{2}$

and $\sin\beta = \frac{2}{3}$, $\frac{\pi}{2} < \beta < \pi$

find the exact values of

(i) $\tan 2\alpha$

(ii) $\cos(\alpha - \beta)$

(c) (i) Find the centre and the radius of the circle C whose equation is

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

(ii) Find, in terms of the constant k, the length of the perpendicular from the centre of C to the line L whose equation is $3x + 4y = k$.

(ii) Hence find the values of k for which L is a tangent to C.

QUESTION 2

(a) If $f(x) = x^3 + 3x^2 - 10x - 24$ calculate $f(-2)$ and express $f(x)$ as the product of three linear factors.

(b) Two of the roots of the equation $x^3 + px^2 + qx + r = 0$ are equal in magnitude but opposite in sign.

(i) Show that $x = -p$ is the other root.

(ii) Show that $r = pq$.

(c) Solve the inequality $\dfrac{x}{x^2 - 1} > 0$.

Cont...

QUESTION 3

(a) Find the exact value of $\sin(2\tan^{-1} \tfrac{1}{2})$

(b) Use the substitution $u = 2 - x$, to evaluate $\int_{-1}^{2} x\sqrt{2 - x}\, dx$.

(c) $P(2p, p^2)$ and $Q(2q, q^2)$ are two points on the parabola $x^2 = 4y$. The chord PQ subtends a right angle at the origin O.

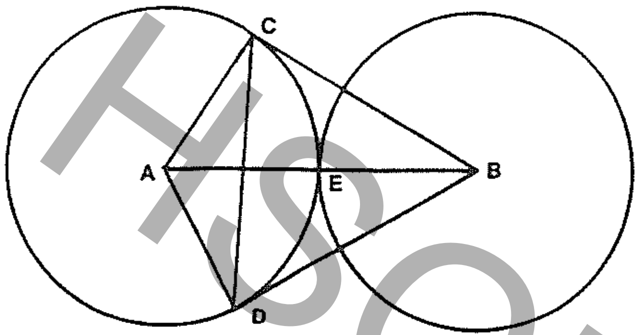
(i) Show that $pq = -4$.

(ii) If M is the mid point of PQ find the locus of M as P and Q move on the parabola.

Cont

QUESTION 4

(a)



Two circles of equal radius and with centres at A and B respectively touch each other externally at E. BC and BD are tangents from B to the circle with centre A.

(i) Copy the diagram.

(ii) Show that BCAD is a cyclic quadrilateral.

(iii) Show that E is the centre of the circle which passes through B, C, A and D.

(iv) Show that $\angle CBA = \angle DBA = 30°$.

(v) Show that triangle BCD is equilateral.

(b) A is the point $(-2, -1)$, B is the point $(1,5)$. Find the co-ordinates of the point Q which divides AB externally in the ratio $5:2$.

Cont.

QUESTION 5

(a) (i) A ball is thrown from a point 0 on the edge of a cliff which is 20 metrres above a beach. The ball is thrown with speed $15\sqrt{2}\ ms^{-1}$ at an angle of 45° above the horizontal.

Taking $g = 10\ ms^{-2}$ show that the ball hits the beach at a point 60 metres along the beach.

(ii) A second ball is thrown horizontally from 0 and hits the beach at the same point as the first ball.

Taking $g = 10\ ms^{-2}$ find the speed of projection of the second ball.

(Standard results about projectile motion can be quoted without proof.)

(b) A sector of a circle with centre O and radius r cm is bounded by the radii OP and OQ, and by the arc PQ. The angle POQ is θ radians.

(i) Given that r and θ vary in such a way that the area of the sector POQ has a constant value of $100\ cm^2$, show that $\theta = \frac{200}{r^2}$

(ii) Given also that the radius is increasing at a constant rate of $0.5\ cm^{-1}$, find the rate at which the angle POQ is decreasing when $r = 10$ cm.

Cont

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QUESTION 6

(a) Find the term independent of x in the expansion of

$$(2x + \frac{1}{x})^{10}$$

(b) The letters of the word CALCULUS are arranged in a row.

(i) How many different arrangements are there?

(ii) If one of these arrangements is selected at random, what is the probability that it begins with 'U' and ends in 'U'?

(c) Use the method of mathematical induction to show that :

$2 \times 1! + 5 \times 2! + 10 \times 3! + \dots + (n^2 + 1)n! = n(n + 1)!$ for all positive integers $n \geq 1$.

Cont..

QUESTION 7

(a) Use the result $\lim_{x \to 0} \frac{\sin x}{x} = 1$

to find $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

(b) A particle moves in a straight line. At time t its displacement from a fixed point O on the line is x, its velocity is v and its acceleration is a.

(i) Show that $a = \frac{d}{dx}(\frac{1}{2} v^2)$.

(ii) If $a = 4x - 4$ and when $t = 0$, $x = 6$ and $|v| = 8$ show that $v^2 = 4x^2 - 8x - 32$.

(iii) Use the expression for v^2 to find the set of possible values of x.

(iv) Describe the motion of the particle in each of the cases

(α) when $t = 0$, $x = 6$ and $v = 8$.

(β) when $t = 0$, $x = 6$ and $v = -8$.

CSSA MATHEMATICS 3 UNIT SOLUTIONS 1993

QUESTION 1

(a) $\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{3^2-x^2}}\,dx$

$= \sin^{-1}\left(\frac{x}{3}\right) + c$.

(b) $\sin\alpha = 3/4$, $0 < \alpha < \pi/2$ $\therefore \cos\alpha > 0$, $\tan\alpha > 0$.

$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-(3/4)^2} = \sqrt{7}/4$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{3/4}{\sqrt{7}/4} = \frac{3\sqrt{7}}{7}$.

$\sin\beta = 2/3$, $\pi/2 < \beta < \pi$ $\therefore \cos\beta < 0$.

$\cos\beta = -\sqrt{1-\sin^2\beta} = -\sqrt{1-(2/3)^2} = -\sqrt{5}/3$.

(i) $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$= \frac{2(3\sqrt{7}/7)}{1-(3\sqrt{7}/7)^2} = -3\sqrt{7}$.

(ii) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \left(\frac{\sqrt{7}}{4}\right)\left(\frac{-\sqrt{5}}{3}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{12}(6-\sqrt{35})$

(c) (i) $x^2+y^2-4x+6y-12=0$

$\therefore (x-2)^2+(y+3)^2 = 5^2$

$\therefore$ centre $(2,-3)$, radius 5.

(ii) distance from $(2,-3)$ to $3x+4y-k=0$

$= \left|\frac{3(2)+4(-3)-k}{\sqrt{3^2+4^2}}\right| = \left|\frac{-k-6}{5}\right| = \frac{|k+6|}{5}$.

(iii) line is a tangent to circle

$\therefore$ distance = radius

$\therefore \frac{|k+6|}{5} = 5$

$\therefore |k+6| = 25$

$\therefore k = -31$ or $k = 19$.

QUESTION 2

(a) $f(x) = x^3 + 3x^2 - 10x - 24$

$\therefore f(-2) = (-2)^3 + 3(-2)^2 - 10(-2) - 24 = 0$.

$\therefore (x+2)$ is a factor of $f(x)$

$\therefore f(x) = (x+2)(x^2 + x - 12)$

$= (x+2)(x+4)(x-3)$.

(b) $x^3 + px^2 + qx + r = 0$ has roots $\alpha, -\alpha, \beta$.

(i) sum of roots

$\alpha + (-\alpha) + \beta = -\frac{p}{1}$ $\therefore \beta = -p$

$\therefore$ the other root is $x = -p$.

(ii) $x = -p$ satisfies $x^3 + px^2 + qx + r = 0$

$\therefore (-p)^3 + p(-p)^2 + q(-p) + r = 0$

$\therefore -p^3 + p^3 - pq + r = 0$

$\therefore r = pq$.

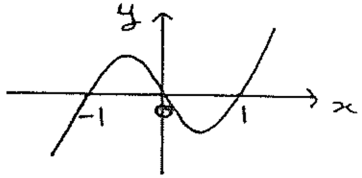
(c) $\frac{x}{x^2-1} > 0$

multiply by $(x^2-1)^2$ which is ≥ 0

$\therefore \frac{x}{x^2-1} \cdot (x^2-1)^2 > 0 \cdot (x^2-1)^2$

$\therefore x(x^2-1) > 0$

$\therefore x(x+1)(x-1) > 0$



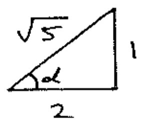
$y = x(x+1)(x-1)$

$\therefore -1 < x < 0$ or $x > 1$.

QUESTION 3

(a) $\sin(2\tan^{-1}\frac{1}{2})$

Let $\alpha = \tan^{-1}\frac{1}{2}$, $\therefore \tan\alpha = \frac{1}{2}$



$\sin\alpha = 1/\sqrt{5} = \sqrt{5}/5$

$\cos\alpha = 2/\sqrt{5} = 2\sqrt{5}/5$

$= \sin 2\alpha$

$= 2\sin\alpha\cos\alpha$

$= 2(\sqrt{5}/5)(2\sqrt{5}/5)$

$= 4/5$.

(b) $\int_{-1}^{2} x\sqrt{2-x}\,dx$

$u = 2 - x$

$\frac{du}{dx} = -1$, $dx = -du$

when $x = -1$, $u = 3$

when $x = 2$, $u = 0$

$\therefore \int_{-1}^{2} x\sqrt{2-x}\,dx$

$= \int_3^0 (2-u)\sqrt{u}\,.(-du)$

$= \int_0^3 (2-u)u^{1/2}\,du$

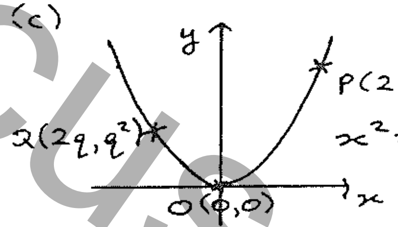
$= \int_0^3 (2u^{1/2} - u^{3/2})\,du$

$= [4/3\, u^{3/2} - 2/5\, u^{5/2}]_0^3$

$= 4/3(3\sqrt{3}) - 2/5(9\sqrt{3})$

$= 2/5\sqrt{3}$.

(c)



(i) $OP \perp OQ$

$\therefore$ gradient OP $\times$ gradient OQ $= -1$

$\therefore \frac{p^2 - 0}{2p - 0} \times \frac{q^2 - 0}{2q - 0} = -1$

$\therefore p/2 \,.\, q/2 = -1$

$\therefore pq = -4$.

(ii) $M(X, Y)$ is mid point PQ

$\therefore X = 1/2(2p + 2q)$

$= p + q$

$Y = 1/2(p^2 + q^2)$

$= 1/2((p+q)^2 - 2pq)$

$= 1/2((p+q)^2 - 2(-4))$

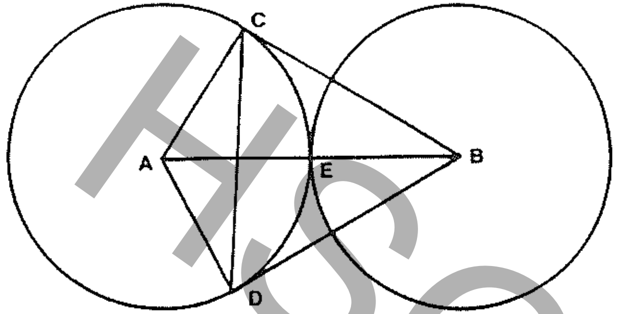
$= 1/2((p+q)^2 + 8)$

$\therefore Y = 1/2(X^2 + 8)$

$\therefore$ the locus of M has equation $y = 1/2(x^2 + 8)$.

QUESTION 4

(a)(i)



(ii) $\angle ACB = 90^\circ$ (tangent ⊥ radius at point of contact)

$\angle ADB = 90^\circ$ (tangent ⊥ radius at point of contact)

$\therefore$ BCAD is cyclic quadrilateral ($\angle ACB$ and $\angle ADB$ are a pair of opposite angles which are supplementary).

(iii) $\angle ACB = \angle ADB = 90^\circ$ (proved above)

$\therefore$ AB is diameter (right angle lies in semicircle)

$\therefore$ mid point AB is centre

But E is mid point AB (A, E, B collinear with AE = BE since equal radii) $\therefore$ E is centre.

(iv) In ΔBCA with $\angle BCA = 90^\circ$

$\sin \angle CBA = {}^{CA}/_{AB} = {}^{1}/_{2}$ $\quad \therefore \angle CBA = 30^\circ$.

In ΔBDA with $\angle BDA = 90^\circ$

$\sin \angle DBA = {}^{DA}/_{AB} = {}^{1}/_{2}$ $\quad \therefore \angle DBA = 30^\circ$.

(v) In ΔBCD

BC = BD (tangents from external point are equal)

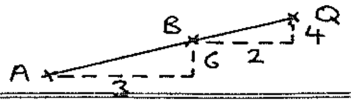
$\therefore \angle BCD = \angle BDC$ (equal angles lie opposite equal sides)

But $\angle CBD = \angle CBA + \angle DBA = 30^\circ + 30^\circ = 60^\circ$

$\therefore \angle BCD = \angle BDC = 60^\circ$ (angle sum $\Delta BCD = 180^\circ$)

$\therefore \Delta BCD$ is equilateral ($\angle CBD = \angle BCD = \angle BDC = 60^\circ$).

(b) $A(-2,-1), B(1,5), Q(X,Y)$.



$QA : QB = 5 : 2$

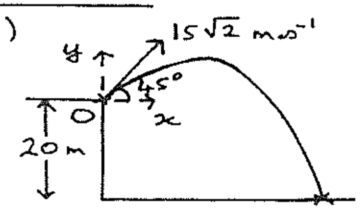
$\therefore BA : QB = 3 : 2$

$\therefore X = 1 + 2 = 3, Y = 5 + 4 = 9$

$\therefore$ Q is the point (3, 9)

QUESTION 5

(a)(i)



$x = (15\sqrt{2} \cos 45^\circ)t$

$\therefore x = 15t$

$y = (15\sqrt{2} \sin 45^\circ)t - {}^{1}/_{2} g t^2$

$\therefore y = 15t - 5t^2$

When the ball hits the beach $y = -20$

$\therefore 15t - 5t^2 = -20$, $\therefore 5t^2 - 15t - 20 = 0$

$\therefore t^2 - 3t - 4 = 0$, $\therefore (t+1)(t-4) = 0$

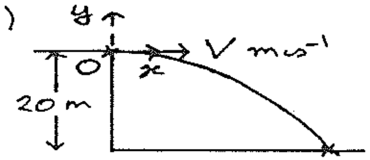
$\therefore$ ~~$t = -1$~~ or $t = 4$

$\therefore$ the ball hits the beach after 4 seconds.

When $t = 4$, $x = 15(4) = 60$

$\therefore$ the ball hits 60 m along the beach.

(ii)



$x = Vt$

$y = -{}^{1}/_{2} g t^2$

$\therefore y = -5t^2$

When the ball hits the beach $y = -20$

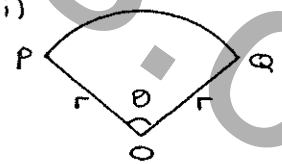
$\therefore -5t^2 = -20$, $\therefore t^2 = 4$, $\therefore$ ~~$t = -2$~~ or $t = 2$

$\therefore$ the ball hits the beach after 2 seconds

When $t = 2$, $x = V(2)$, $\therefore 60 = 2V$, $\therefore V = 30$

$\therefore$ the speed of projection is 60 ms⁻¹.

(b)(i)



Area sector POQ = 100

$\therefore {}^{1}/_{2} r^2 \theta = 100$

$\therefore \theta = \dfrac{200}{r^2}$.

(ii) $\dfrac{d\theta}{dt} = \dfrac{d\theta}{dr} \cdot \dfrac{dr}{dt} = \dfrac{-400}{r^3} \cdot 0.5 = \dfrac{-200}{r^3}$.

When $r = 10$, $\dfrac{d\theta}{dt} = -\dfrac{200}{10^3} = -0.2$

$\therefore \angle POQ$ is decreasing at a rate of 0.2 radians per second.

QUESTION 6

(a) A typical term in the expansion of $(2x + {}^1/_x)^{10}$ is

$^{10}C_r (2x)^{10-r} ({}^1/_x)^r = {}^{10}C_r . 2^{10-r} . x^{10-2r}$.

$\therefore$ the term independent of x has $10-2r=0$, ie $r=5$.

$\therefore$ the term independent of x is

$^{10}C_5 . 2^5 = 252 . 32 = 8064$.

(b) CALCULUS 1 'A', 2 'C', 2 'L', 1 'S', 2 'U'.

(i) The number of different arrangements is

$$\frac{8!}{1!\,2!\,2!\,1!\,2!} = 5040.$$

(ii) The number of these arrangements which begin with 'U' and end in 'U' is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180$$

$\therefore$ P(arrangement begins with 'U' and ends in 'U')

$= {}^{180}/_{5040} = {}^1/_{28}$

(c) $S(n): 2\times1! + 5\times2! + 10\times3! + \dots + (n^2+1)n! = n(n+1)!$

<u>Step 1</u>. When $n=1$, LHS $= 2\times1! = 2$

RHS $= 1\times2! = 2$

$\therefore S(1)$ is true.

<u>Step 2</u>. If $S(k)$ is true for some $k \geq 1$ so that

$2\times1! + 5\times2! + 10\times3! + \dots + (k^2+1)k! = k(k+1)!$, then

$2\times1! + 5\times2! + 10\times3! + \dots + (k^2+1)k! + ((k+1)^2+1)(k+1)!$

$= k(k+1)! + (k^2+2k+1+1)(k+1)!$

$= (k^2+3k+2)(k+1)!$

$= (k+1)(k+2)(k+1)!$

$= (k+1)(k+2)!$

and $S(k+1)$ is also true.

<u>Step 3</u>. $S(1)$ is true, and if $S(k)$ is true for some $k \geq 1$ then $S(k+1)$ is also true.

$\therefore S(1+1)$ is true, ie $S(2)$ is true

$\therefore S(2+1)$ is true, ie $S(3)$ is true ...

$\therefore S(n)$ is true for all $n \geq 1$.

QUESTION 7

(a) $$\lim_{x\to0} \frac{1-\cos 2x}{x^2} = \lim_{x\to0} \frac{2\sin^2 x}{x^2}$$

$$= \lim_{x\to0} 2 . \frac{\sin x}{x} . \frac{\sin x}{x} = 2(1)(1) = 2.$$

(b)(i) $$a = \frac{dv}{dt} = \frac{dv}{dx} . \frac{dx}{dt} = \frac{dv}{dx} . v = v . \frac{dv}{dx}$$

$$\therefore a = \frac{d}{dv}({}^1/_2 v^2) . \frac{dv}{dx} = \frac{d}{dx}({}^1/_2 v^2).$$

(ii) $a = 4x - 4$

$\therefore \frac{d}{dx}({}^1/_2 v^2) = 4x - 4$

$\therefore {}^1/_2 v^2 = \int (4x-4)\,dx = 2x^2 - 4x + c$

When $t=0$, $x=6$, $|v|=8$. $\therefore c = -16$

$\therefore v^2 = 4x^2 - 8x - 32$.

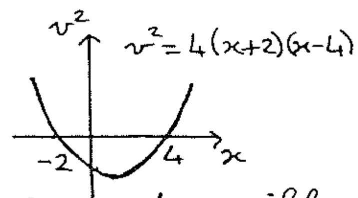
(iii) $v^2 \geq 0$

$\therefore 4x^2 - 8x - 32 \geq 0$

$\therefore 4(x^2 - 2x - 8) \geq 0$

$\therefore 4(x+2)(x-4) \geq 0$

$\therefore x \leq -2$ or $x \geq 4$



But when $t=0$, $x=6$. $\therefore x \leq -2$ is not possible.

$\therefore x \geq 4$

(iv) (α) when $t=0$, $x=6$, $v=8$.

The particle starts 6 units to the right of O. It is moving to the right ($v>0$) and speeding up ($a>0$). It continues to move to the right.

(β) when $t=0$, $x=6$, $v=-8$.

The particle starts 6 units to the right of O. It is moving to the left ($v<0$) and slowing down ($a>0$). It moves to the left until it is 4 units to the right of O where it stops ($v=0$). It then moves to the right ($v>0$) and speeds up ($a>0$), and continues to move to the right.

CATHOLIC 1992 3 UNIT TRIAL

QUESTION 1

(a) Solve the equation

$\cos 2A = \cos A$ where $0° \leq A \leq 360°$.

(b) Evaluate $\displaystyle\int_3^5 x\sqrt{x^2 - 9}\,dx$

using the substitution $u = x^2 - 9$.

(c) (i) Sketch the graph of the function

$y = \sin^{-1}\left(\frac{x}{2}\right)$

(ii) State the domain and the range of the function.

(iii) Find the exact equation of the tangent to the curve

$y = \sin^{-1}\left(\frac{x}{2}\right)$

at the point where $x = 1$.

Cont.

QUESTION 2

(a) Given that $\displaystyle\int_0^1 \frac{1}{x^2 + 3}\,dx = k\pi$

find the value of the constant k.

(b) α, β and γ are the roots of the equation $x^3 + 2x^2 - 3x + 5 = 0$.

(i) State the values of

$\alpha + \beta + \gamma$, $\alpha\beta + \alpha\gamma + \beta\gamma$ and $\alpha\beta\gamma$.

(ii) Find the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$.

(c) A particle is moving in a straight line with Simple Harmonic Motion. If the amplitude of the motion is 4 cm and the period of the motion is 3 seconds, calculate

(i) the maximum velocity of the particle,

(ii) the maximum acceleration of the particle,

(iii) the speed of the particle when it is 2cm from the centre of the motion.

Cont

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QUESTION 3

(a) Solve the inequality $\frac{2x+3}{x-4} > 1$.

(b) (i) Show that the equation

$$5x^4 - 4x^3 - 0 \cdot 9 = 0$$

has a root near $x = 1$.

(ii) Starting with the approximation $x_0 = 1$ attempt to find an improved value for this root using Newton's Method. Explain why this attempt fails.

(c) (i) At any time t the rate of cooling of the temperature T of a body, when the surrounding temperature is P, is given by the equation

$$\frac{dT}{dt} = -k(T - P), \text{ for some constant } k.$$

Show that the solution

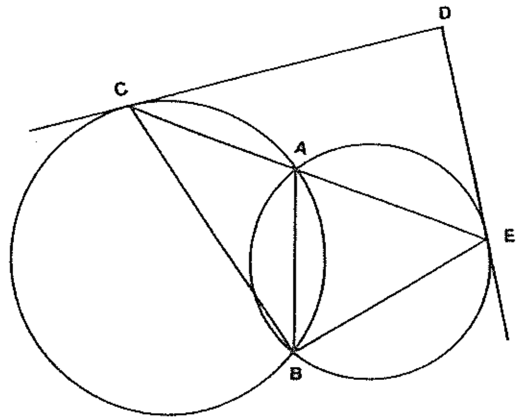
$T = P + Ae^{-kt}$, for some constant A, satisfies this equation.

(ii) A metal bar has a temperature of 1340° C and cools to 1010° C in 12 minutes, when the surrounding temperature is 25° C. Find how much longer it will take the bar to cool to 60° C, giving your answer correct to the nearest minute.

Cont...

QUESTION 4

(a)



Two circles intersect at A and B. CAE is a straight line where C is a point on the first circle and E is a point on the second circle. The tangent at C to the first circle and the tangent at E to the second circle meet at D.

(i) Copy the diagram.

(ii) Prove that BCDE is a cyclic quadrilateral.

(b) (i) Solve the equation

$$\sqrt{3}\cos x - \sin x = 1 \text{ for } 0 \le x \le 2\pi.$$

(ii) What is the general solution of the equation?

Cont..

QUESTION 5

(a) A is the point $(-4,1)$ and B is the point $(2,4)$. Q is the point which divides AB internally in the ratio 2:1 and R is the point which divides AB externally in the ratio 2:1. $P(x,y)$ is a variable point which moves so that $PA = 2PB$.

(i) Find the coordinates of Q and R.

(ii) Show that the locus of P is a circle on QR as diameter.

(b) Twelve people are going to the local swimming pool. Five are to go by car and the rest are to walk to the pool.

(i) How many different groups of five of the people can be found to fill the car?

(ii) In one of these groups it is found that only one person can drive. In how many ways can the seats be filled in this group under this condition?

Cont...

QUESTION 6

(a) (i) Write down the first three terms in the expansion of $(1 + ax)^n$ where a is a constant and n is a positive integer.

(ii) If the first three terms of the expansion are $1 - 12x + 63x^2$

(α) find the values of a and n.

(β) find the next term in the expansion.

(b) Use the Principle of Mathematical Induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n.

Cont.

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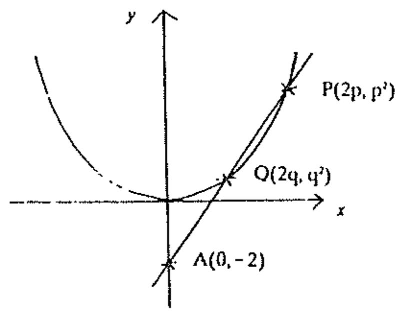
QUESTION 7

$\begin{cases} x = 2t \\ y = t^2 \end{cases}$



(i) Find the equations of the normals to the parabola

$\begin{cases} x = 2t \\ y = t^2 \end{cases}$ at the points $P(2p, p^2)$ and $Q(2q, q^2)$,

where $p \neq q$.

Hence show that these normals intersect at the point $R(X, Y)$ where

$X = -pq(p + q)$ and $Y = (p + q)^2 - pq + 2$.

(ii) If the chord PQ has gradient m and passes through the point $A(0, -2)$ find, in terms of m, the equation of PQ.

Hence show that p and q are the roots of the equation $t^2 - 2mt + 2 = 0$.

(iii) By considering the sum and the product of the roots of this quadratic equation show that the point R lies on the original parabola.

(iv) Find the least value of m^2 for which p and q are real.

Hence find the set of possible values of the y coordinate of R.

C.S.S.A. MATHEMATICS 3 UNIT SOLUTIONS 1992

QUESTION 1

(a) $\cos 2A = \cos A \quad \therefore 2\cos^2 A - 1 = \cos A$

$\therefore 2\cos^2 A - \cos A - 1 = 0 \quad \therefore (2\cos A + 1)(\cos A - 1) = 0$

$\therefore \cos A = -1/2$ or $\cos A = 1$

$A = 120^\circ$ or $240^\circ \qquad A = 0^\circ$ or 360°

$\therefore A = 0^\circ$ or 120° or 240° or $360^\circ \quad (0^\circ \le A \le 360^\circ)$

(b) $\int_3^5 x\sqrt{x^2-9}\,dx \qquad u = x^2 - 9$

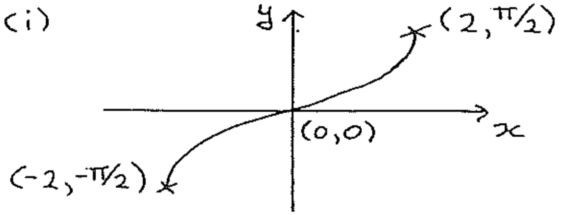
$= \int_0^{16} \sqrt{u}\,.\,1/2\,du \qquad 1/2\,du = x\,dx$

$= \int_0^{16} 1/2\, u^{1/2}\,du \qquad$ when $x = 3,\ u = 0$; when $x = 5,\ u = 16$

$= \left[1/3\, u^{3/2}\right]_0^{16}$

$= 21\,1/3$

(c) (i)



$y = \sin^{-1}\left(\frac{x}{2}\right)$

(ii) domain $\{x \in R : -2 \le x \le 2\}$

range $\{y \in R : -\pi/2 \le y \le \pi/2\}$

(iii) $y = \sin^{-1}\left(\frac{x}{2}\right) \quad \therefore \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$

when $x = 1,\ y = \pi/6$

when $x = 1,\ dy/dx = \sqrt{3}/3$

$\therefore$ gradient of tangent at $(1, \pi/6)$ is $\sqrt{3}/3$

$\therefore$ equation of tangent at $(1, \pi/6)$ is

$y - \pi/6 = \sqrt{3}/3\,(x - 1)$

$\therefore 2\sqrt{3}x - 6y + \pi - 2\sqrt{3} = 0$

QUESTION 2

(a) $\int_0^1 \frac{1}{x^2+3}\,dx = \int_0^1 \frac{1}{x^2+(\sqrt{3})^2}\,dx$

$= \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1 = \frac{1}{\sqrt{3}}(\pi/6 - 0)$

$= \frac{\sqrt{3}}{18}\pi \qquad \therefore k = \frac{\sqrt{3}}{18}$.

(b) $x^3 + 2x^2 - 3x + 5 = 0$ has roots α, β and γ.

(i) $\alpha + \beta + \gamma = -(2)/(1) = -2$

$\alpha\beta + \alpha\gamma + \beta\gamma = (-3)/(1) = -3$

$\alpha\beta\gamma = -(5)/(1) = -5$

(ii) $(\alpha - 1)(\beta - 1)(\gamma - 1)$

$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1$

$= (-5) - (-3) + (-2) - 1$

$= -5$

(c) S H M.

$a = 4$

$T = \frac{2\pi}{n} \quad \therefore 3 = \frac{2\pi}{n} \quad \therefore n = \frac{2\pi}{3}$

(i) $v^2 = n^2(a^2 - x^2)$

at $x = 0$ maximum velocity is

$na = \frac{2\pi}{3} \times 4 = \frac{8\pi}{3}$ cm s^{-1}.

(ii) $\ddot{x} = -n^2 x$

at $x = -a$ maximum acceleration is

$n^2 a = \left(\frac{2\pi}{3}\right)^2 \times 4 = \frac{16\pi^2}{9}$ cm s^{-2}

(iii) $v^2 = n^2(a^2 - x^2)$

when $|x| = 2$

$v^2 = \left(\frac{2\pi}{3}\right)^2 (4^2 - 2^2) = \frac{48\pi^2}{9} \quad \therefore |v| = \frac{4\sqrt{3}\pi}{3}$

$\therefore$ speed is $\frac{4\sqrt{3}}{3}\pi$ cm s^{-1}

QUESTION 5

(a) $A(-4,1)$, $B(2,4)$, $P(x,y)$

(i) Q divides AB internally in the ratio 2 : 1

$\therefore Q\left(-4+\tfrac{2}{3}[2-(-4)],\ 1+\tfrac{2}{3}[4-1]\right)$ ie $Q(0,3)$.

R divides AB externally in the ratio 2 : 1

$\therefore R\left(-4+2[2-(-4)],\ 1+2[4-1]\right)$ ie $R(8,7)$.

(ii) $PA = 2PB$

$\therefore \sqrt{(x-(-4))^2+(y-1)^2} = 2\sqrt{(x-2)^2+(y-4)^2}$

$\therefore (x+4)^2+(y-1)^2 = 4\{(x-2)^2+(y-4)^2\}$

$\therefore x^2+8x+16+y^2-2y+1 = 4\{x^2-4x+4+y^2-8y+16\}$

$= 4x^2-16x+16+4y^2-32y+64$

$\therefore 3x^2-24x+3y^2-30y+63 = 0$

$\therefore x^2-8x+y^2-10y = -21$

$\therefore (x-4)^2+(y-5)^2 = 20$

a circle with centre $C(4,5)$ and radius $r = 2\sqrt{5}$

now midpoint $QR = \left(\tfrac{1}{2}(0+8), \tfrac{1}{2}(3+7)\right) = C(4,5)$

and $QC = RC = r = 2\sqrt{5}$

$\therefore$ locus of P is a circle on QR as diameter.

(b) (i) groups of five of the people can be found to fill the car in

$${}^{12}C_5 = \frac{12\times11\times10\times9\times8}{5\times4\times3\times2\times1} = 792 \text{ different ways}$$

In a group with only one driver the seats can be filled in

$1\times4! = 24$ ways.

QUESTION 6

(a) (i) $(1+ax)^n$

$= 1+{}^nC_1(ax)+{}^nC_2(ax)^2+\ldots$

$= 1+nax+\tfrac{1}{2}n(n-1)a^2x^2+\ldots$

(ii) $1+nax+\tfrac{1}{2}n(n-1)a^2x^2 \equiv 1-12x+63x^2$

$\therefore na = -12 \quad (1)$

$\tfrac{1}{2}n(n-1)a^2 = 63 \quad (2)$

$\therefore n^2a^2-na^2 = 126$

$\therefore 144+12a = 126$

$\therefore a = -\tfrac{3}{2}$ and $n = 8$

The next term in the expansion of $(1-\tfrac{3}{2}x)^8$ is

$${}^8C_3(-\tfrac{3}{2}x)^3 = \frac{8\times7\times6}{3\times2\times1}\left(\frac{-3}{2}\right)^3x^3 = -189x^3$$

(b) $f(n) = 9^{n+2}-4^n$

To show that $f(n)$ is divisible by 5 for $n\geq1$.

When $n=1$ $f(1) = 725 = 5(145)$

which is divisible by 5

$\therefore$ statement is true when $n=1$.

Assume $f(k) = 9^{k+2}-4^k$ is divisible by 5

ie $f(k) = 5M$ for some integer M

$\therefore f(k+1) = 9^{k+3}-4^{k+1}$

$= 9.9^{k+2}-4.4^k$

$= 4.(9^{k+2}-4^k)+5.9^{k+2}$

$= 4.f(k)+5.9^{k+2}$

$= 4.5M+5.9^{k+2}$

$= 5(4M+9^{k+2})$

$= 5N$ for some integer N

which is divisible by 5.

$\therefore$ if statement is true when $n=k$ then statement is also true when $n=k+1$.

$\therefore$ By the Principle of Mathematical Induction statement is true for all $n\geq1$

$\therefore 9^{n+2}-4^n$ is divisible by 5 for all $n\geq1$.

QUESTION 7

(i) $x = 2t$, $\dot{x} = 2$ $\therefore \dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{2t}{2} = t$

$y = t^2$, $\dot{y} = 2t$

$\therefore$ at $P(2p, p^2)$, gradient of normal $= -1/p$ and equation of normal is $y - p^2 = -1/p\,(x - 2p)$

ie $x + py - 2p - p^3 = 0$ (1)

Similarly at $Q(2q, q^2)$ equation of normal is

$x + qy - 2q - q^3 = 0$ (2)

$q(1) - p(2)$ gives $(q-p)x - pq(p-q)(p+q) = 0$

$\therefore x = -pq(p+q)$

$(1)-(2)$ gives $(p-q)y - 2(p-q) - (p-q)(p^2+pq+q^2) = 0$

$\therefore y = (p+q)^2 - pq + 2$

$\therefore$ the normals intersect at the point $R(X, Y)$ where

$X = -pq(p+q)$, $Y = (p+q)^2 - pq + 2$.

(ii) PQ has equation $y - (-2) = m(x - 0)$

ie $mx - y - 2 = 0$

When PQ cuts the parabola $x = 2t$, $y = t^2$

$m(2t) - (t^2) - 2 = 0$ ie $t^2 - 2mt + 2 = 0$.

$\therefore$ p and q are the roots of the equation

$t^2 - 2mt + 2 = 0$.

(iii) sum of roots $p + q = -(-2m)/(1) = 2m$

product of roots $pq = (2)/(1) = 2$

$\therefore$ the point R has coordinates

$X = -(2)(2m) = -4m$, $Y = (2m)^2 - (2) + 2 = 4m^2$

which is of the form

$X = 2t$, $Y = t^2$ with $t = -2m$

$\therefore$ R lies on the original parabola .

(iv) For $t^2 - 2mt + 2 = 0$ to have real roots p and q

$\Delta = (-2m)^2 - 4(1)(2) = 4(m^2 - 2) \geq 0$, $\therefore m^2 \geq 2$

$\therefore$ the least value of m^2 is 2 .

$\therefore$ the y coordinate of $R = 4m^2 \geq 8$

QUESTION 3

(a) $\frac{2x+3}{x-4} > 1$ (multiply both sides by $(x-4)^2$)

$\therefore (2x+3)(x-4) > (x-4)^2$

$\therefore (2x+3)(x-4) - (x-4)^2 > 0$

$\therefore (x-4)\{(2x+3)-(x-4)\} > 0$

$\therefore (x-4)(x+7) > 0$

$\therefore x < -7$ or $x > 4$

(b)(i) $f(x) = 5x^4 - 4x^5 - 0.9$

$f(1) = 0.1\ (>0)$, $f(1.1) = -0.02154\ (<0)$

$f(x)$ is a continuous function

$\therefore f(x) = 0$ ie $5x^4 - 4x^5 - 0.9 = 0$

has a root near $x = 1$.

(ii) $f(x) = 5x^4 - 4x^5 - 0.9$, $f'(x) = 20x^3 - 20x^4$

$\therefore f(1) = 0.1$, $f'(1) = 0$

using Newton's method with $x_0 = 1$ gives

$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{0.1}{0}$

which does not work since $x_0 = 1$ lies at a stationary point of $f(x)$

(c)(i) $\frac{dT}{dt} = -k(T-P)$, $T = P + Ae^{-kt}$

$LHS = \frac{dT}{dt} = \frac{d}{dt}(P + Ae^{-kt})$

$= -k.Ae^{-kt} = -k(T-P) = RHS$

(ii) $T = P + Ae^{-kt}$, $P = 25$ $\therefore T = 25 + Ae^{-kt}$

when $t = 0$, $T = 1340$ $\therefore A = 1315$ $\therefore T = 25 + 1315e^{-kt}$

when $t = 12$, $T = 1010$

$\therefore 1010 = 25 + 1315e^{-12k}$ $\therefore 985 = 1315e^{-12k}$

$\therefore e^{12k} = 1315/985$ $\therefore k = 1/12 \ln 1315/985$

when $T = 60$

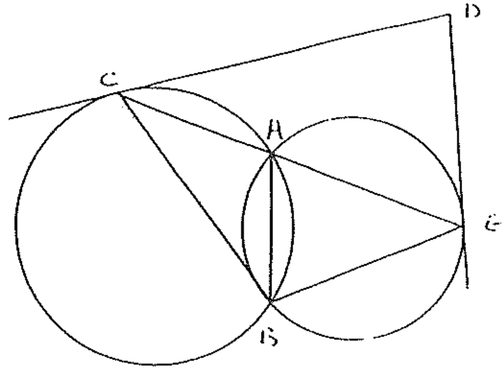
$60 = 25 + 1315e^{-kt}$ $\therefore 35 = 1315e^{-kt}$

$\therefore e^{kt} = 1315/35$ $\therefore t = 1/k \ln 1315/35 = 151$

$\therefore$ it cools to 60°C after a further 139 minutes.

QUESTION 4

(a)(i)



(ii) $\angle DCA = \angle CBA = x°$ (alternate segment theorem)

$\angle DEA = \angle EBA = y°$ (alternate segment theorem)

$\angle CDE = 180° - (\angle DCE + \angle DEC)$ (angle sum of $\Delta CDE = 180°$)

$= 180° - (x° + y°)$

$\angle EBC = \angle CBA + \angle EBA$

$= (x° + y°)$

$\therefore \angle CDE + \angle EBC = 180° - (x°+y°) + (x°+y°) = 180°$

$\therefore BCDE$ is a cyclic quadrilateral (the pair of opposite angles $\angle CDE$ and $\angle EBC$ are supplementary)

(b)(i) $\sqrt{3}\cos x - \sin x = 1$

$\therefore 2\{\cos x.(\sqrt{3}/2) - \sin x(1/2)\} = 1$

$2(\cos x \cos \pi/6 - \sin x \sin \pi/6) = 1$

$2\cos(x + \pi/6) = 1$

$\therefore \cos(x + \pi/6) = 1/2$

$\therefore x + \pi/6 = \pi/3$ or $5\pi/3$

$\therefore x = \pi/6$ or $3\pi/2$ $(0 \le x \le 2\pi)$

(ii) the general solution is

$x = \pi/6 + 2n\pi$ $(n = 0, \pm1, \pm2, ..)$ or

$x = 3\pi/2 + 2n\pi$ $(n = 0, \pm1, \pm2, ..)$

$\therefore x = (12n+1)\pi/6$ or $(4n+3)\pi/2$ $(n = 0, \pm1, \pm2, ..)$

YEAR TWELVE FINAL TESTS 1991

MATHEMATICS
3/4 UNIT COMMON PAPER
(i.e. 3 UNIT COURSE – ADDITIONAL PAPER; 4 UNIT COURSE – FIRST PAPER)

Afternoon session

Two Hours

Friday 16th August 1991

EXAMINERS

J. Hennessy, Marcellin College, Randwick.
M. Campbell, St. Patrick's College, Sutherland
G. Arnold, John Paul II Senior High, Marayong
L. Brooks, Aurora College, Moss Vale.

DIRECTIONS TO CANDIDATES :

ALL questions may be attempted.

ALL questions are of equal value.

In every question, all necessary working should be shown.

Marks may not be awarded for careless or badly arranged work.

Approved slide rules or calculators may be used.

Mathematical templates, such as "Math-o-Matt" may be used provided that they contain only numbers printed on them and no words or letters of any alphabet or formulae or equations of any kind whatsoever.

Answers to the seven questions in this paper are to be returned in separate writing booklets clearly marked : Question 1, Question 2, etc. on the cover.

If required, additional booklets may be obtained from the examination supervisor upon request.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination. However the committees responsible for the preparation of these 'Trial Examinations' do hope that they will provide a positive contribution to your preparation for the final examinations.

QUESTION 1

(a) (i) On the same set of axes, sketch the graphs of
$$y = 2\,|\,x\,| \text{ and } y = |\,x + 3\,|$$

(ii) On your diagram, shade in the region where $y \le 2\,|\,x\,|$ and $y \ge |\,x + 3\,|$ hold simultaneously.

(b) If $2^x = 5^y = 10^z$, show that $\dfrac{1}{z} = \dfrac{1}{x} + \dfrac{1}{y}$

(c) In a tennis club, there are five (5) married couples available to play a "mixed doubles" match, that is, a match in which a combination of one man and one woman play against a combination of another man and another woman. In how many ways can a group of four (4) persons be chosen for this match if :

(i) a man and his wife may play in the match but not as partners.

(ii) a man and his wife may not play in the match either as partners or opponents.

(d) In an Arithmetic Progression, whose first term and common difference are both non-zero, U_n denotes the n^{th} term and Sn denotes the sum of n terms. If U_4, U_6, U_{10} form a Geometric Progression :

(i) show that $S_{10} = 0$

(ii) show that $S_6 + S_{12} = 0$

(iii) deduce that $U_7 + U_8 + U_9 + U_{10} = U_{11} + U_{12}$

Cont

QUESTION 2

(a) Show that

$$\int_{1}^{\sqrt{2}} \frac{dx}{\sqrt{4-2x^2}} = 6 \times \int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{4+2x^2}$$

(b) The lengths of the sides of a scalene triangle are in Arithmetic Progression. If the largest angle in such a triangle measures 120°, show that the smallest angle measures approximately 22°.

(c) $P(x)$ denotes the quadratic polynomial $k x^2 + (k-1)x - (2k-1)$, where k is a real, rational number.

(i) Show that the equation $P(x) = 0$ always has real, rational roots for all values of k.

(ii) Find the value of k for which the roots of $P(x) = 0$ are equal.

(iii) Find the value (or values) of k, for which one of the roots of $P(x) = 0$ will be double the other root.

Cont..

QUESTION 3

(a) Solve for θ in the range $0° \le \theta \le 360°$:

$$6\cot^2\theta - 4\cos^2\theta = 1$$

(b) Evaluate the definite integral

$$\int_{0}^{\frac{1}{\sqrt{2}}} \frac{2x^3\,dx}{\sqrt{1-x^4}}$$

by means of one of the substitutions

$$u = x^4 \quad \text{or} \quad x^2 = \sin\theta$$

(c) When $(3+2x)^n$ is expanded in increasing powers of x, it is found that the coefficients of x^5 and x^6 have the same value. Find the value of n and show that the two coefficients mentioned are greater than all other coefficients in the expansion.

Co..

QUESTION 4

(a) O is a fixed point on a given straight line. A particle moves along this line and its displacement, x cms, from O at a given time, t secs, after its start of motion is given by :

$$x = 2 + \cos^2 t$$

(i) Show that the acceleration is given by :

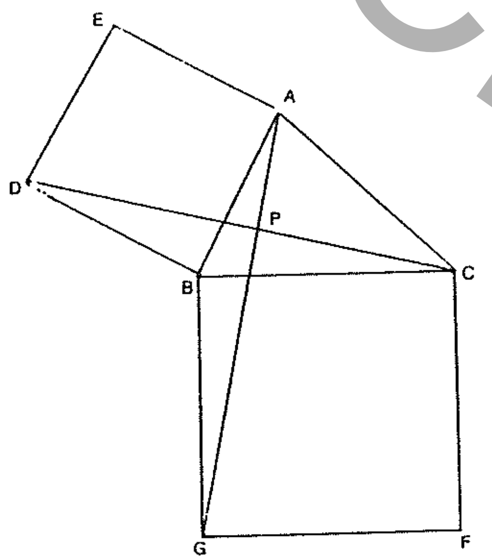
$$\ddot{x} = 10 - 4x$$

(ii) State the centre of motion.

(iii) State the first two occasions when the particle is at rest and the displacements on these occasions.

(iv) State the amplitude and period of motion.

(b)



In the above diagram, ABC is an acute-angled triangle and ABDE and BCFG are squares constructed on AB and BC, respectively, as sides and lying wholly outside the triangle. AG meets CD at the point P.

Prove that :

(i) triangles ABG and DBC are congruent.

(ii) the points B, P, C, G are concyclic.

(iii) AG and DC are perpendicular to each other

(iv) BP bisects angle DPG

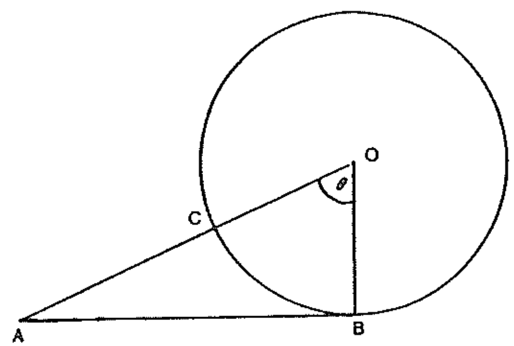
QUESTION 5

(a) A stone is thrown from a point O which is at the top of a cliff 20 metres above a horizontal beach. The stone is thrown at an angle of elevation $\theta°$ above the horizontal and with a speed of 35 ms^{-1}. The stone hits the beach at a point which is distant 140 metres horizontally from the point O.

(i) Taking g, the gravitational constant, as 10 ms^{-2}, show that $\tan\theta = \frac{3}{4}$ or $\tan\theta = 1$.

(ii) Hence find the two possible times for which the stone is in the air, giving your answers in exact form.

(b)



In the above diagram, O is the centre of a circle and AB is a tangent to the circle, meeting it at a point B. The line interval OA cuts the circumference of the circle at a point C.

(i) If the arc of the circle CB divides the triangle AOB into two portions of equal area and if the angle AOB is denoted by θ, show that $\tan\theta = 2\theta$.

(ii) If $\theta = 1.15$ radians is an approximate solution to the equation in (i) above, use one application of Newton's method to find a better approximation, correct to three (3) decimal places.

QUESTION 6

(a) Using the principles of Mathematical Induction, show that $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers n greater than or equal to 1.

(b) P is a point in the first quadrant of the co-ordinate plane, lying on the parabola $x^2 = 4y$. The normal to the parabola at P meets the parabola again at a point Q in the second quadrant. The tangents to the parabola at P and Q meet at a point T. If S is the focus of the parabola and if QS = 2 PS, show that :

(i) QP subtends a right-angle at S.

(ii) PQ = PT.

Cont

QUESTION 7

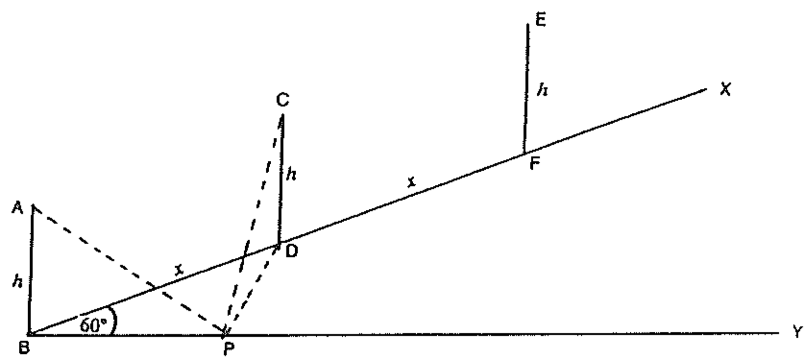
(a) Two of the roots of the equation $x^3 + ax^2 + b = 0$ are reciprocals of each other (a, b both real).

(i) Show that the third root is equal to $-b$.

(ii) Show that $a = b - \frac{1}{b}$

(iii) Show that the two roots, which are reciprocals, will be real if $-\frac{1}{2} \le b \le \frac{1}{2}$.

(b)



In the above diagram, BX and BY represent two roads intersecting at an angle of 60°. On the road BX are situated three telegraph poles AB, CD and EF, all of equal height, the same distance, x metres, apart (i.e. BD = DF = x).

P is a point on the road BY and the angles of elevation of A and C from P are 45° and 30° respectively.

(i) Show that BP = h and DP = $h\sqrt{3}$.

(ii) By the use of the Sine Rule in triangle BDP, show that angle BDP = 30° and hence that triangle BDP is right-angled at P.

(iii) Prove that $x = 2h$.

(iv) By the use of the Cosine Rule in triangle PDF, show that PF = $h\sqrt{13}$ and hence show also that the angle of elevation of E from P is approximately $15\frac{1}{2}$°.

END OF PAPER

QUESTION 7

(a) Two of the roots of the equation $x^3 + ax^2 + b = 0$ are reciprocals of each other (a, b both real).

(i) Show that the third root is equal to $-b$.

(ii) Show that $a = b - \frac{1}{b}$

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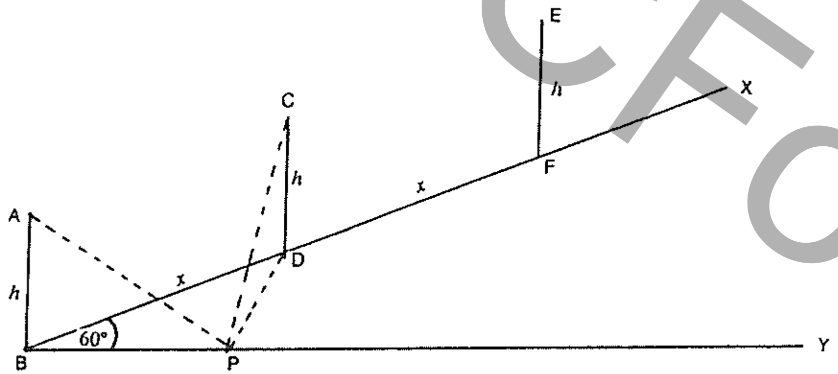
(b)



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(iv) By the use of the Cosine Rule in triangle PDF, show that $PF = h\sqrt{13}$ and hence show also that the angle of elevation of E from P is approximately 15½°.

END OF PAPER

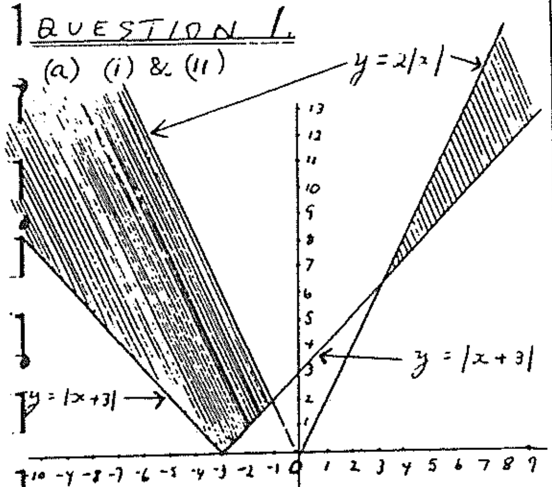
YEAR 12 FINAL EXAMINATIONS – AUGUST 1991

3 UNIT/4 UNIT COMMON MATHS (i.e. 3 UNIT (ADDITIONAL PAPER), 4 UNIT (FIRST PAPER))

SOLUTION

QUESTION 1.

(a) (i) & (ii)



$y \le 2|x|$ and $y \ge |x+3|$ hold simultaneously in the shaded regions

(b) $2^x = 10^z$

Taking logs to base 10

$x \log_{10} 2 = z$

$\therefore \frac{1}{x} = \frac{\log_{10} 2}{z}$

$5^y = 10^z$

Taking logs to base 10

$y \log_{10} 5 = z$

$\therefore \frac{1}{y} = \frac{\log_{10} 5}{z}$

$\therefore \frac{1}{x} + \frac{1}{y} = \frac{\log_{10} 2 + \log_{10} 5}{z}$

$= \frac{\log_{10} 10}{z}$

$= \frac{1}{z}$

(c) This problem is probably best done by a direct counting technique

Let M_1, M_2, M_3, M_4, M_5 denote the five men and let F_1, F_2, F_3, F_4, F_5 respectively denote their wives

For the match take a combination of 2 males (order is not important – it does not matter on which side of the net they stand!) There are ${}^5C_2$ (or 10) such combinations

Let M_1, M_2 be a typical combination

(i) If a man and his wife can play in a match as opponents but not as partners, then only arrangements for a given pair of males M_1, M_2 are

$M_1 \xrightarrow{\text{with}} F_2\ F_2\ F_2\ F_2\ \ F_3\ F_3\ F_3\ F_4\ F_4\ F_4\ F_5\ F_5$

$M_2 \xrightarrow{\text{with}} F_1\ F_3\ F_4\ F_5\ \ F_1\ F_4\ F_5\ F_1\ F_3\ F_5\ F_1\ F_3$

a total of 13 arrangements

And since there are 10 combinations of two males there are therefore 130 arrangements in total

1

(ii) if a man and his wife cannot play in a match either as partners or opponents then the only arrangements for a given pair of males M_1, M_2 are :-

$M_1 - F_3\ F_3 \quad F_4\ F_4 \quad F_5\ F_5$

$M_2 - F_4\ F_5 \quad F_3\ F_5 \quad F_3\ F_4$

a total of 6 arrangements and since there are 10 combinations of two males, there are therefore 60 arrangements in total

(d) Let first term $= a$

common difference $= d$

U_6, U_4, U_{10} form a G.P.

$\therefore (a+3d)^2 = (a+5d)(a+9d)$

$\therefore a^2 + 6ad + 9d^2 = a^2 + 14ad + 45d^2$

$\therefore 8ad + 36d^2 = 0$

$\therefore 4d(2a + 9d) = 0$

And since $d \neq 0$

$2a + 9d = 0$

(i) now $S_{10} = \frac{10}{2}(2a + 9d)$

$= 5 \times 0$

$= 0$

(ii) $S_6 + S_{12} = 3(2a+5d) + 6(2a+11d)$

$= 18a + 81d$

$= 9(2a + 9d)$

$= 9 \times 0$

$= 0$

(iii) Now $S_6 = S_{10} - u_{10} - u_9 - u_8 - u_7$

$= -u_{10} - u_9 - u_8 - u_7$

and $S_{12} = S_{10} + u_{11} + u_{12}$

$= u_{11} + u_{12}$

$\therefore -u_{10} - u_9 - u_8 - u_7 + u_{11} + u_{12} =$

$\therefore u_7 + u_8 + u_9 + u_{10} = u_{11} + u_{12}$

QUESTION 2

(a) $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}}\int_1^{\sqrt{2}} \frac{dx}{\sqrt{2-x^2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_1^{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[\sin^{-1}1 - \sin^{-1}\frac{1}{\sqrt{2}}\right]$

$= \frac{1}{\sqrt{2}}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{4\sqrt{2}}$

$\displaystyle\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{4+2x^2} = \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{2+x^2}$

$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\right]_{\sqrt{2}}^{\sqrt{6}}$

$= \frac{1}{2\sqrt{2}} \times \left[\tan^{-1}\sqrt{3} - \tan^{-1}1\right]$

$= \frac{1}{2\sqrt{2}}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{24\sqrt{2}}$

$\therefore \displaystyle\int_1^{\sqrt{2}} \frac{dx}{\sqrt{4-2x^2}} = 6 \times \int_{\sqrt{2}}^{\sqrt{6}} \frac{dx}{4+2x^2}$

(b) Let a, b, c be the sides in descending order of magnitude and hence angles A, B, C are also in descending order

Now since a, b, c are in A.P.

$\therefore 2b = a + c \quad \ldots\ldots (i)$

and $\hat{B} = 120°$ (be... a)

By Cosine Rule

$a^2 = b^2 + c^2 - 2bc\cos 120°$

$= b^2 + c^2 + bc \quad \ldots\ldots (ii)$

Substituting $b = \frac{a+c}{2}$ from (i)

$a^2 = \frac{a^2+2ac+c^2}{4} + c^2 + \frac{c(a+c)}{2}$

mult by 4

$4a^2 = a^2 + 2ac + c^2 + 4c^2 + 2ac + 2c^2$

$\therefore 7c^2 + 4ac - 3a^2 = 0$

$(c + a)(7c - 3a) = 0$

$\therefore c = -a$ or $7c = 3a$

i.e. $\frac{c}{a} = -1$ or $\frac{3}{7}$

Discarding negative value

$\frac{c}{a} = \frac{3}{7}$

But from Sine Rule

$\frac{c}{a} = \frac{\sin C}{\sin A} = \frac{\sin C}{\frac{\sqrt{3}}{2}}$

$\therefore \frac{2\sin C}{\sqrt{3}} = \frac{3}{7}$

$\therefore \sin C = \frac{3\sqrt{3}}{14}$

$\therefore C = 21.787°$

$\doteqdot 22°$

(obtuse value for C being discarded)

(c)(i) $P(x) = kx^2 + (k-1)x - (2k-1)$

$\Delta = (k-1)^2 + 4k(2k-1)$

$= 9k^2 - 6k + 1$

$= (3k-1)^2$

Since Δ is a perfect square, roots are always real

(ii) For equal roots, $\Delta = 0$

$\therefore (3k-1)^2 = 0$

$\therefore k = \frac{1}{3}$

(iii) Let roots of $P(x) = 0$ be α and 2α

Sum of roots $= 3\alpha = -\frac{k-1}{k}$

$\alpha = -\frac{k-1}{3k}$

Product of roots $= 2\alpha^2 = -\frac{2k-1}{k}$

$\therefore 2\left(-\frac{k-1}{3k}\right)^2 = -\frac{2k-1}{k}$

$\therefore \frac{2(k-1)^2}{9k^2} = -\frac{2k-1}{k}$

$2k^3 - 4k^2 + 2k = -18k^3 + 9k^2$

$\therefore 20k^3 - 13k^2 + 2k = 0$

$\therefore k(4k-1)(5k-2) = 0$

$\therefore k = 0, \frac{1}{4}, \frac{2}{5}$

Discarding $k = 0$, we have

$k = \frac{1}{4}$ or $\frac{2}{5}$

Checking

$k = \frac{1}{4}$ gives $\frac{1}{4}x^2 - \frac{3}{4}x + \frac{1}{2}$ whose roots are 1 and 2

$k = \frac{2}{5}$ gives $\frac{2}{5}x^2 - \frac{3}{5}x + \frac{1}{5}$ whose roots are $\frac{1}{2}$ and 1

QUESTION 3

(a) $6\cot^2\theta - 4\cos^2\theta = 1$

Mult. by $\sin^2\theta$ $(\sin\theta \neq 0)$

$6\cos^2\theta - 4\cos^2\theta\sin^2\theta = \sin^2\theta$

$\therefore 6\cos^2\theta - 4\cos^2\theta(1-\cos^2\theta) = 1-\cos^2\theta$

$\therefore 4\cos^4\theta + 3\cos^2\theta - 1 = 0$

$\therefore (\cos^2\theta + 1)(4\cos^2\theta - 1) = 0$

$\cos^2\theta = -1$ or $\frac{1}{4}$

Discarding the negative value

$\cos^2\theta = \frac{1}{4}$

$\therefore \cos\theta = \pm\frac{1}{2}$

$\therefore \theta = 60°, 120°, 240°, 300°$

(b) Let $I = \int_0^{\frac{1}{\sqrt2}} \frac{2x^3\,dx}{\sqrt{1-x^4}}$

Let $u = x^4$

$\therefore du = 4x^3\,dx$

When $x = 0$, $u = 0$

" $x = \frac{1}{\sqrt2}$, $u = \frac{1}{4}$

$\therefore I = \int_0^{\frac14} \frac{\frac12 \cdot du}{\sqrt{1-u}}$

$= -\left[\sqrt{1-u}\right]_0^{\frac14}$

$= -\left[\frac{\sqrt3}{2} - 1\right]$

$= 1 - \frac{\sqrt3}{2}$

ALTERNATIVELY:

Let $x^2 = \sin\theta$

$\therefore 2x \cdot \frac{dx}{d\theta} = \cos\theta$

$\therefore dx = \frac{\cos\theta \cdot d\theta}{2x}$

When $x = 0$, $\theta = 0$

" $x = \frac{1}{\sqrt2}$, $\theta = \frac{\pi}{6}$

$\therefore I = \int_0^{\frac{\pi}{6}} \frac{2 \cdot x^3 \cdot \frac{\cos\theta\,d\theta}{2x}}{\cos\theta}$

$= \int_0^{\frac{\pi}{6}} x^2 \cdot d\theta$

$= \int_0^{\frac{\pi}{6}} \sin\theta\,d\theta$

$= -\left[\cos\theta\right]_0^{\frac{\pi}{6}}$

$= -\left[\frac{\sqrt3}{2} - 1\right]$

$= 1 - \frac{\sqrt3}{2}$

(c) In expansion of $(3+2x)^n$

$u_{r+1} = {}^nC_r\, 3^{n-r} \cdot 2^r x^r$

Now $\frac{\text{coeff } x^6}{\text{coeff } x^5} = 1$

$\therefore \frac{{}^nC_6\, 3^{n-6} \cdot 2^6}{{}^nC_5 \cdot 3^{n-5} \cdot 2^5} = 1$

$\therefore \frac{n!}{6!(n-6)!} \cdot \frac{5!(n-5)!}{n!} \cdot \frac{2}{3} = 1$

$\therefore \frac{n-5}{6} \cdot \frac{2}{3} = 1$

$\therefore 2n - 10 = 18$

$\therefore n = 14$

In expansion of $(3+2x)^{14}$

$\frac{\text{coeff } u_{r+1}}{\text{coeff } u_r} = \frac{{}^{14}C_r\, 3^{14-r}\, 2^r}{{}^{14}C_{r-1}\, 3^{15-r}\, 2^{r-1}}$

$= \frac{14!}{r!\,(14-r)!} \cdot \frac{(r-1)!\,(15-r)!}{14!} \cdot \frac{2}{3}$

$= \frac{15-r}{r} \cdot \frac{2}{3}$

$= \frac{30-2r}{3r}$

$\therefore \frac{\text{coeff } u_{r+1}}{\text{coeff } u_r} > 1$ when

$\frac{30-2r}{3r} > 1$

i.e. when $30 - 2r > 3r$

" $30 > 5r$

" $6 > r$

" $r < 6$

When $r = 6$, coeff u_7 = coeff u_6 (as already proved)

" $r = 5$, $\frac{\text{coeff } u_6}{\text{coeff } u_5} > 1$

coeff u_6 > coeff u_5

Similarly coeff u_5 > coeff u_4 etc etc

When $r = 7$, $\frac{\text{coeff } u_8}{\text{coeff } u_7} < 1$

i.e. coeff u_8 < coeff u_7

Similarly coeff u_9 < coeff u_8 etc etc

The two coeffs of x^5 and x^6 (i.e. terms 6 and 7) are greater than all other.

QUESTION 4

(a)

(i) $x = 2 + \cos^2 t$

$\dot{x} = -2\cos t \sin t$

$= -\sin 2t$

$\ddot{x} = -2\cos 2t$

$= -2(2\cos^2 t - 1)$

$= -4\cos^2 t + 2$

$= -4(x-2) + 2$

$= 10 - 4x$

(ii) Since $\ddot{x} = 10 - 4x$

$= -4\left(x - \frac{5}{2}\right)$

then motion is S H M and centre of motion lies at $x = \frac{5}{2}$

(iii) At rest $\dot{x} = 0$

$\therefore \sin 2t = 0$

$2t = 0, \pi$

$t = 0, \frac{\pi}{2}$

When $t = 0$, $x = 2 + 1 = 3$

" $t = \frac{\pi}{2}$, $x = 2 + 0 = 2$

(iv) From (iii) path of particle is from 2 to 3 i.e. 1 cm

Amplitude = $\frac{1}{2}$ cm

Since eq'n is of form $\ddot{x} = -n^2(x - b)$

where $n = 2$

then period of motion

$= \frac{2\pi}{2} = \pi$ secs

(b)

(i) $D\hat{B}C = D\hat{B}A + A\hat{B}C$

$= 90° + A\hat{B}C$

$A\hat{B}G = G\hat{B}C + A\hat{B}C$

$= 90° + A\hat{B}C$

Hence $D\hat{B}C = A\hat{B}G$

Now in Δ's ABG, DBC

$AB = DB$ (sides of square)

$BG = BC$ (sides of square)

$A\hat{B}G = D\hat{B}C$ (proved)

$\therefore \Delta ABG \equiv \Delta DBC$ (S A S)

$\therefore\ B\hat{G}A = B\hat{C}D$ (∠'s opp correspt sides)

Hence BP subtends equal angles at G and C

Points B, P, C, G are concyclic

Hence also GC must subtend equal angles at B and P

But $G\hat{B}C = 90^\circ$ (∠ between sides of square)

$G\hat{P}C = 90^\circ$

ie AG and DC are perpendicular to each other

Since B, P, C, G are concyclic (proved)

$\therefore$ BG subtends equal angles at P and C

But $B\hat{C}G = 45^\circ$, since diagonal GC of square BCFG bisects vertical angle $B\hat{C}F\ (=90^\circ)$

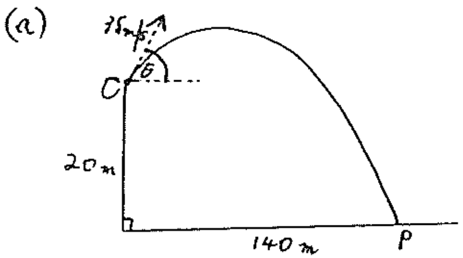
$\therefore\ B\hat{P}G = 45^\circ$

and since $D\hat{P}G = 90^\circ$ (proved)

$\therefore$ BP bisects $D\hat{P}G$

QUESTION 5

(a)



Forces acting on stone are nil horizontally and gravity vertically downwards

$m\ddot{x} = 0$ and $m\ddot{y} = -mg = -10m$

$\therefore \ddot{x} = 0 \qquad \ddot{y} = -10$

$\dot{x} = \int 0\,dt = C_1 \qquad \dot{y} = \int -10\,dt = -10t + C_2$

when $t = 0$, $\dot{x} = 35\cos\theta$ and $\dot{y} = 35\sin\theta$

Hence $C_1 = 35\cos\theta$, $C_2 = 35\sin\theta$

$\therefore \dot{x} = 35\cos\theta \qquad \dot{y} = -10t + 35\sin\theta$

$\therefore x = \int 35\cos\theta\,dt = 35t\cos\theta + C_3 \qquad y = \int(-10t + 35\sin\theta)\,dt = -5t^2 + 35t\sin\theta + C_4$

when $t = 0$, $x = 0$, $y = 0$ (at C)

$\therefore C_3 = 0$ and $C_4 = 0$

$\therefore x = 35t\cos\theta \qquad y = -5t^2 + 35t\sin\theta$

At the point P on beach,

$x = 140 \qquad y = -20$

$\therefore 140 = 35t\cos\theta \qquad -20 = -5t^2 + 35t\sin\theta$

$\therefore 4 = t\cos\theta$ --(i) $\qquad -4 = -t^2 + 7t\sin\theta$ -----(ii)

Subst $t = 4\sec\theta$ from (i) into (ii)

$-4 = -16\sec^2\theta + 28\tan\theta$

$\therefore -1 = -4\sec^2\theta + 7\tan\theta$

$= -4 - 4\tan^2\theta + 7\tan\theta$

$\therefore 4\tan^2\theta - 7\tan\theta + 3 = 0$

$\therefore (\tan\theta - 1)(4\tan\theta - 3) = 0$

$\therefore \tan\theta = 1$ or $\frac{3}{4}$

If $\tan\theta = 1$, $\sec\theta = \sqrt{2}$

and so time $(t) = 4\sec\theta = 4\sqrt{2}$ secs

If $\tan\theta = \frac{3}{4}$, $\sec\theta = \frac{5}{4}$

and so time $(t) = 4\sec\theta = 5$ secs.

(b)(i) Let radius of circle = r

Then in ΔACB, since $A\hat{B}O = 90^\circ$ (angle between tangent and radius at point of contact)

$r\tan\theta = \frac{AB}{CB} = \frac{AB}{r}$

$\therefore AB = r\tan\theta$

$\therefore$ Area $\Delta AOB = \frac{1}{2}AB \cdot OB$

$= \frac{1}{2} \cdot r\tan\theta \cdot r$

$= \frac{1}{2}r^2\tan\theta$

Area sector $OCB = \frac{1}{2}r^2\theta$

Area of $\Delta ACB = 2\times$ sector OCB

$\therefore \frac{1}{2}r^2\tan\theta = 2\times\frac{1}{2}r^2\theta$

$\therefore \tan\theta = 2\theta$

(ii) Let $f(\theta) = \tan\theta - 2\theta = 0$

$\therefore f'(\theta) = \sec^2\theta - 2$

If $\theta_1 = 1.15$ is first approx then second approx θ_2 is

$\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)}$

$= 1.15 - \frac{\tan 1.15 - 2.3}{\sec^2 1.15 - 2}$

$= 1.15 - \frac{-0.065503}{3.992977}$

$= 1.15 + 0.0164$

$= 1.166$ (to 3 dec places)

QUESTION 6

(a) Let $P_n = 3^{3n} + 2^{n+2}$

(i) Let $n = 1$, then $P_1 = 27 + 8 = 35$ which is divisible by 5

Hence proposition is true for $n = 1$

(ii) Assume prop'n is true for $n = k$

ie assume $P_k = 3^{3k} + 2^{k+2}$ is divisible by 5

If so, let $3^{3k} + 2^{k+2} = 5M$ where M is some integer

$\therefore 3^{3k} = 5M - 2^{k+2}$

Now $P_{k+1} = 3^{3k+3} + 2^{k+3}$

$= 27\cdot 3^{3k} + 2^{k+3}$

$= 27(5M - 2^{k+2}) + 2^{k+3}$

$= 135M - 27.2^{k+2} + 2^{k+3}$

$= 135M - 2^{k+2}(27 - 2)$

$= 135M - 25.2^{k+2}$

$= 5(27M - 5.2^{k+2})$

and so P_{k+1} is divisible by 5

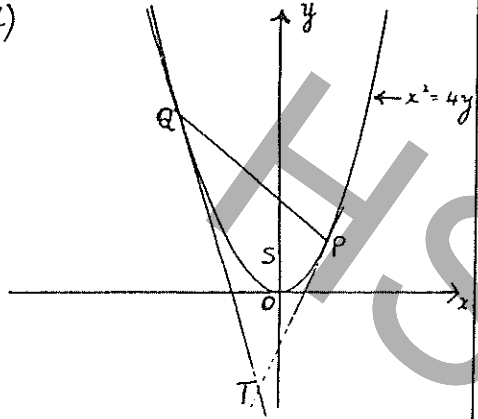
(iii) But prop'n is true for $n = 1$

$\therefore$ It is true for $n = 2$

and so it is true for $n = 3$

and so on for all n

(b)



Let P be pt. $(2p, p^2)$
and Q be pt. $(2q, q^2)$
S, the focus, is the pt. $(0,1)$

$y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{x}{2}$

$= p$ at pt P

$\therefore$ Eqn tangent at P is

$y - p^2 = p(x-2p)$

i.e. $y = px - p^2$

Eqn normal at P is

$y - p^2 = -\frac{1}{p}(x-2p)$

i.e. $py - p^3 = -x + 2p$

i.e. $x + py = p^3 + 2p$

Since Q lies on normal at P

$2q + pq^2 = p^3 + 2p$

$\therefore p^3 - pq^2 = -2p + 2q$

$\therefore p(p^2 - q^2) = -2(p-q)$

$\therefore p(p+q) = -2$

$p^2 + pq = -2 \quad \text{------} (1)$

Now $QS = 2PS$

or $QS^2 = 4PS^2$

(8)

$\therefore 4q^2 + (q^2-1)^2 = 4[4p^2 + (p^2-1)^2]$

$\therefore (q^2+1)^2 = 4(p^2+1)^2$

$\therefore q^2 + 1 = 2(p^2+1)$

$q^2 = 2p^2 + 1 \text{ ....} (11)$

From (1), $q = -\frac{p^2+2}{p}$

$\therefore \frac{(p^2+2)^2}{p^2} = 2p^2 + 1$

$\therefore p^4 + 4p^2 + 4 = 2p^4 + p^2$

$p^4 - 3p^2 - 4 = 0$

$\therefore (p^2+1)(p^2-4) = 0$

$p^2 = -1, 4$

Discarding negative value

$p^2 = 4$

$p = 2$ (since P lies in 1st quadrant)

and so $q = -3$

P is pt $(4,4)$, Q is pt $(-6,9)$

Grad PS × Grad QS

$= \frac{3}{4} \times -\frac{4}{3} = -1$

$\therefore$ PS and QS are perpendicular

i.e. QP subtends right angle at S

Tang at P is $y = 2x - 4$

Similarly tang at Q is $y = -3x - 9$

Solving simultaneously $x = -1, y = -6$

Hence T is pt. $(-1,-6)$

$PQ = \sqrt{100+25} = 5\sqrt{5}$

$PT = \sqrt{25+100} = 5\sqrt{5}$

$\therefore PQ = PT$

QUESTION 7

(a) $x^3 + ax^2 + b = 0$

(1) Let roots be $\alpha, \frac{1}{\alpha}, \beta$

Product of Roots $= \beta = -b$

$\therefore$ Third Root $(\beta) = -b$

(ii) Sum of Roots (one at a time)

$= \alpha + \frac{1}{\alpha} + \beta = -a$

Hence $\alpha + \frac{1}{\alpha} = -a - \beta = -a + b$

Sum of Roots (two at a time)

$= 1 + \alpha\beta + \frac{\beta}{\alpha} = 0$

$1 + \beta(\alpha + \frac{1}{\alpha}) = 0$

$1 - b(-a+b) = 0$

$ab = b^2 - 1$

$a = b - \frac{1}{b}$

(iii) Since $-b$ is a root of $x^3 + ax^2 + b = 0$ (proved), then $(x+b)$ is a factor

Let $x^3 + ax^2 + b = (x+b)(x^2 + Px + Q)$

Equating consts. $b = bQ$

$Q = 1$

Equating coeffs of x; $0 = Q + Pb$

$\therefore P = -\frac{Q}{b} = -\frac{1}{b}$

Hence $\alpha, \frac{1}{\alpha}$ are roots of the quadratic equation

$x^2 - \frac{1}{b}x + 1 = 0$

Here, $\Delta = \frac{1}{b^2} - 4$

now roots real if $\Delta \geq 0$

(9)

$\frac{1}{b^2} - 4 \geq 0$

$\frac{1}{b^2} \geq 4$

$b^2 \leq \frac{1}{4}$

i.e. $-\frac{1}{2} \leq b \leq \frac{1}{2}$

(b)

(1) In $\triangle ABP$; $\tan 45° = \frac{AB}{BP} = \frac{h}{BP}$

$\therefore BP = h\cot 45° = h$

In $\triangle CDP$, $\tan 30° = \frac{CD}{DP} = \frac{h}{DP}$

$\therefore DP = h\cot 30° = h\sqrt{3}$

(ii) In $\triangle BDP$, by Sine Rule

$\frac{DP}{\sin 60°} = \frac{BP}{\sin B\hat{D}P}$

$\therefore \sin B\hat{D}P = \frac{BP \cdot \sin 60°}{DP}$

$= \frac{h \cdot \frac{\sqrt{3}}{2}}{h\sqrt{3}}$

$= \frac{1}{2}$

$B\hat{D}P = 30°$

and since $D\hat{B}P = 60°$ (data)

$\therefore B\hat{P}D = 180° - (30° + 60°) = 90°$

(iii) In $\triangle BPD$ by Pythagoras' Theor.

$x^2 = h^2 + (h\sqrt{3})^2 = 4h^2$

$\therefore x = 2h$

(iv) In $\triangle PDF$ by Cosine Rule

$PF^2 = (h\sqrt{3})^2 + (2h)^2 - 2 \cdot h\sqrt{3} \cdot 2h \cos 150°$

$= 3h^2 + 4h^2 - 4\sqrt{3}h^2(-\frac{\sqrt{3}}{2})$

$= 7h^2 + 6h^2$

$= 13h^2$

$\therefore PF = h\sqrt{13}$

$\tan E\hat{P}F = \frac{EF}{PF} = \frac{h}{h\sqrt{13}} = \frac{1}{\sqrt{13}}$

$\therefore E\hat{P}F = 15.5014° \doteqdot 15\frac{1}{2}°$

CATHOLIC 1990 3UNIT TRIAL

QUESTION 1.

(a) Evaluate $\int_0^3 \frac{dx}{x^2 + 9}$, leaving the answer in <u>exact</u> form. Marks (2)

(b) (i) Show that :

$$\cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right)$$

(ii) Hence solve the equation

$$\cos\theta - \sqrt{3}\sin\theta = 1$$

for θ in the interval $\theta \le \theta \le 2\pi$ Marks (3)

(c) Solve the following inequality for x :

$$x - \frac{1}{x} < 0$$

and graph the solution on a number line. Marks (3)

(d) Find the first derivative of the function

$$\log_e\left(\frac{1}{\sqrt{\sin x}}\right)$$

Marks (2)

(e) Find the value of the expression

$$\cos^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$$

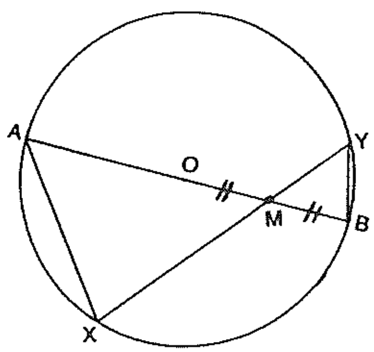
in terms of π. Marks (2)

Cont

QUESTION 2.

(a) Find any values of k, which will make the expression :
$$(k+1)x^2 - 2(k-1)x + (2k-5)$$
a perfect square. Marks (3)

(b) In the diagram below, AB is a diameter of a circle, whose centre is the point O. The chord XY passes through M, the mid-point of OB. AX and BY are joined.



(i) Copy the above diagram carefully into your writing booklet.

(ii) Prove the two triangles formed (triangles AXM and MYB) are similar.

(iii) If XM = 8cms and YM = 6cms, find the length of the radius of the circle. Marks (4)

(c) The equation $(x - 3y + 5) + k(x + 2y) = 0$ represents a family of straight lines passing through a fixed point P.

(i) For what value of k is one of the lines in the family parallel to the straight line $x + y = 2$?

(ii) For what value of k does one of the lines in the family pass through the centre of the circle $x^2 + y^2 - 10y + 21 = 0$?

(iii) Find the co-ordinates of P. Marks (5)

Cont

QUESTION 3.

(a) If α , β , γ are the roots of the equation:

$$x^3 - x^2 + 4x - 1 = 0$$

Find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$. Marks (2)

(b) The positive square root of 50 is approximately 7.
Use one application of Newton's Method, to find a better approximation, correct to 2 decimal places . Marks (2)

(c) In how many ways can the letters of the word **GEOMETRY** be arranged in a straight line if the vowels must occupy the 2nd , 4th , and 6th places .
(**NOTE : The vowels in the English alphabet are the letters A , E , I , O , U**). Marks (2)

(d) (i) Show that the function $f(x) = \frac{x - 4}{x - 2}$ $(x \neq 2)$
is an increasing function for all values of x in its domain.

(ii) Sketch the graph of the function, showing clearly the co-ordinates of any points of intersection with the x - axis and the y - axis, and also the equations of any asymptotes .

(iii) Find the inverse function , $f^{-1}(x)$, and state its range . Marks (6)

Cont ..

QUESTION 4.

(a) A is the point $(-2, -1)$. B is the point $(1, 5)$.
Find the co-ordinates of the point Q , which divides AB externally in the ratio 5 : 3 . Marks (2)

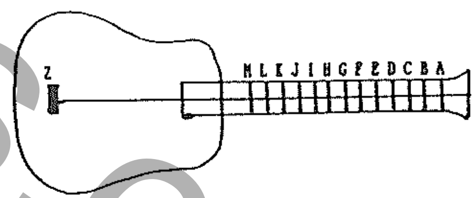
(b) If $2 \sin\left(\theta - \frac{\pi}{3}\right) = \cos\left(\theta + \frac{\pi}{3}\right)$,

express $\tan\theta$ in the form $a + b\sqrt{3}$, where a and b are integers. Marks (2)

(c) Find the value of the term independent of x in the expansion of

$$\left(2x - \frac{1}{x^2}\right)^{12}$$ Marks (3)

(d)



On the keyboard of a guitar, the mark M is exactly half-way between A and Z . The 13 marks lettered A to M are such that their distances from Z are in Geometric Progression. The length AZ is 52cms .

Find, correct to 1 decimal place :

(i) the distance AB

(ii) the distance FG Marks (5)

Cont

QUESTION 5.

(a) If $\tan A$ and $\tan B$ are the roots of the equation $3x^2 - 5x - 1 = 0$, find the value of $\tan(A + B)$. Marks (2)

(b) Find the largest possible domain of the function $f(x) = \log_e\left(\frac{x-2}{x}\right)$ Marks (2)

(c) Use the substitution $u = \sqrt{x}$ to evaluate

$$\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$$, giving the

answer in <u>exact</u> form. Marks (2)

(d) A certain particle moves along the x - axis in accordance with the law :

$$t = 2x^2 - 5x + 3$$

where x is measured in cms and t in seconds.
Initially, the particle is 1.5 cms to the right of 0 and moving away from 0.

(i) Prove that the velocity, v cms/sec, is given by

$$v = \frac{1}{4x - 5}$$

(ii) Find an expression for the acceleration, a cms./secs^2, in terms of x.

(iii) Find the velocity and acceleration of the particle when:

(α) $x = 2$ cms.

(β) $t = 6$ secs.

(iv) Describe carefully in words the motion of the particle. Marks (6)

Cont..

QUESTION 6.

(a) The polynomial $P(x) = x^3 + ax^2 + bx + c$ leaves the same remainder, whether divided by $(x + 1)$, $(x + 2)$ or $(x + 3)$.

Find the values of a, b and c if the polynomial has a zero equal to 1 and then show that it has no other real zeros. Marks (6)

(b) The curve of the function $y = \ln(x - 1)$ meets the line $y = 2$ at P and the x - axis at Q. From P, perpendiculars are drawn to the x - axis and the y - axis, meeting them at R and S respectively.

(i) Show that the normal to the curve at Q passes through S.

(ii) Show that the arc QP of the curve divides the rectangle OSPR into two portions of equal area, where O is the origin.

(iii) Show that the area enclosed between the arc QP and the straight line interval QP equals the area of triangle OSQ. Marks (6)

Cont..

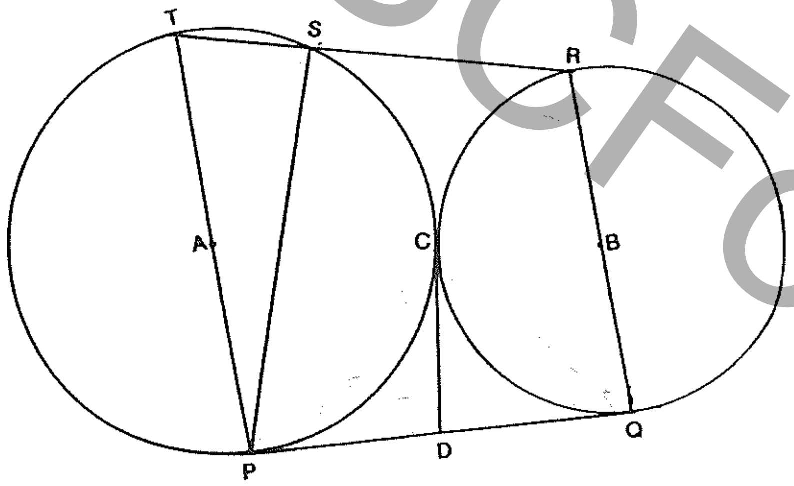
QUESTION 7.

(a) Find: $\lim\limits_{x \to -5} \dfrac{\sqrt{20 - x} - 5}{5 + x}$ Marks (2)

(b) A die is loaded in such a way that in 8 throws of the die, the probability of getting 3 even numbers is four times the probability of getting 2 even numbers.

Find the probability that a single throw of the die results in an even number. Marks (4)

(c)



A circle, centre A, touches a smaller circle, centre B, externally at a point C. PQ is a direct common tangent to the two circles, touching them at points P and Q. The common tangent to both circles passing through C meets PQ at the point D. PA and QB, when produced, meet the circumferences of the two circles at T and R respectively. TR meets the larger circle at S.

(i) Copy the above diagram carefully into your writing booklet.

(ii) Show that the points P, C and R are collinear.

(iii) Show that BD is parallel to the line RCP.

(iv) Show that the points P, Q, R, S are concyclic. Marks (6)

END OF PAPER

CATHOLIC 1990 3 UNIT TRIAL SOLUTIONS

QUESTION 1

(a) $\int_0^3 \frac{dx}{x^2+9} = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$

$= \frac{1}{3}\tan^{-1}1 - 0$

$= \frac{1}{3}\times\frac{\pi}{4}$

$= \frac{\pi}{12}$

(b)(i) $\cos\theta - \sqrt{3}\sin\theta$

$= 2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)$

$= 2\left(\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right)$

$= 2\cos\left(\theta + \frac{\pi}{3}\right)$

(ii) $\cos\theta - \sqrt{3}\sin\theta = 1$

$\therefore 2\cos\left(\theta + \frac{\pi}{3}\right) = 1$

$\therefore \cos\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}$

$\therefore \theta + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\therefore \theta = 0, \frac{4\pi}{3}, 2\pi$

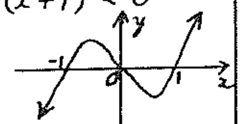
(c) $x - \frac{1}{x} < 0$

Multiply both sides by x^2

$\therefore x^3 - x < 0$

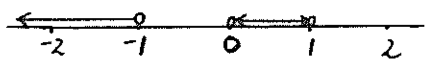
$\therefore x(x-1)(x+1) < 0$

Graph of $y = x(x-1)(x+1)$ $\Longrightarrow$



From graph, function is negative when $x < -1$ or $0 < x < 1$

Number Line Graph of Solution :-



(d) $y = \log_e\left(\frac{1}{\sqrt{\sin x}}\right)$

$= \log_e\left((\sin x)^{-\frac{1}{2}}\right)$

$= -\frac{1}{2}\log_e(\sin x)$

$\therefore \frac{dy}{dx} = -\frac{1}{2}\cdot\frac{1}{\sin x}\cdot\cos x$

$= -\frac{1}{2}\cot x$

(e) $\cos^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

$= \left(\pi - \cos^{-1}\frac{1}{2}\right) - \left(-\sin^{-1}\frac{1}{2}\right)$

$= \pi - \cos^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{2}$

$= \pi - \frac{\pi}{3} + \frac{\pi}{6}$

$= \frac{5\pi}{6}$

QUESTION 2

(a) $(k+1)x^2 - 2(k-1)x + (2k-5)$

Expression is perfect square if :-

(i) discriminant (Δ) is zero

(ii) coeff of x^2 and constant term are both perfect squares

$\Delta = 4(k-1)^2 - 4(k+1)(2k-5)$

$= 4\left[k^2 - 2k + 1 - 2k^2 + 3k + 5\right]$

$= 4(6 + k - k^2)$

$= 4(3-k)(2+k)$

$\therefore \Delta = 0$ if $k = 3$ or -2

If $k = 3$, $k+1 = 4$ & $2k-5 = 1$

$k = -2$, $k+1 = -1$ & $2k-5 = -9$

Hence only valid value of k is 3

(b)(i) In Δ's AXM, MYB

$X\hat{A}M = M\hat{Y}B$ (on same arc XB)

$A\hat{X}M = M\hat{B}Y$ (on same arc AY)

$A\hat{M}X = B\hat{M}Y$ (vertically opposite)

$\therefore$ Triangles are similar, because they are equiangular

(ii) Let radius of circle $= r$

Since Δ's AXM, MYB are similar (proved)

$\therefore \frac{AM}{XM} = \frac{MY}{MB}$

$\therefore \frac{\frac{3r}{2}}{8} = \frac{6}{\frac{r}{2}}$

$\therefore \frac{3r}{16} = \frac{12}{r}$

$\therefore 3r^2 = 192$

$\therefore r^2 = 64$

$\therefore r = 8$

(c)(i) $(x - 3y + 5) + k(x + 2y) = 0$

$(k+1)x + (2k-3)y + 5 = 0$

Gradient $= -\frac{k+1}{2k-3}$

If the line is parallel to line $x + y = 2$, then it has a gradient of -1.

$-\frac{k+1}{2k-3} = -1$

$\therefore k + 1 = 2k - 3$

$\therefore k = 4$

(ii) The eqn of circle $x^2 + y^2 - 10x + 21 = 0$ can be written as

$(x-5)^2 + y^2 = 4$

Hence the centre is pt. $(5, 0)$

If pt $(5,0)$ lies on the line $(x - 3y + 5) + k(x + 2y) = 0$

then $(5 - 0 + 5) + k(5 + 0) = 0$

$k = -2$

(iii) The fixed point, P, through which all lines in the family pass is found by solving simultaneously the two equations

$x - 3y + 5 = 0$

$x + 2y = 0$

By subtraction

$-5y + 5 = 0$

$\therefore y = 1$

and by substitution $x = -2$

Hence P is the point $(-2, 1)$

QUESTION 3

(a) $x^3 - x^2 + 4x - 1 = 0$

Roots are α, β, γ

$\therefore \alpha + \beta + \gamma = 1$

$\alpha\beta + \beta\gamma + \gamma\alpha = 4$

$\alpha\beta\gamma = 1$

Now $(\alpha+1)(\beta+1)(\gamma+1)$

$= (\alpha+1)(\beta\gamma + \beta + \gamma + 1)$

$= \alpha\beta\gamma + \alpha\beta + \gamma\alpha + \alpha + \beta\gamma + \beta + \gamma + 1$

$= \alpha\beta\gamma + (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma) + 1$

$= 1 + 4 + 1 + 1$

$= 7$

ALTERNATIVELY :

Let $y = x + 1$

$x = y - 1$

$\therefore (y-1)^3 - (y-1)^2 + 4(y-1) - 1 = 0$

$\therefore y^3 - 3y^2 + 3y - 1 - y^2 + 2y - 1 + 4y - 4 - 1 = 0$

$y^3 - 4y^2 + 9y - 7 = 0$

The roots of this equation are $(\alpha+1), (\beta+1), (\gamma+1)$

Hence

$(\alpha+1)(\beta+1)(\gamma+1) = 7$

(b) Let $f(x) = x^2 - 50 = 0$, a root of which is exactly $\sqrt{50}$
Taking $x_1 = 7$ as a first approximation, then a second and better approximation, x_2, is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Now $f'(x) = 2x$
So, $f(7) = -1$ and $f'(7) = 14$

$$\therefore x_2 = 7 - \frac{-1}{14} = 7\tfrac{1}{14} = 7.07 \text{ (to 2 dec places)}$$

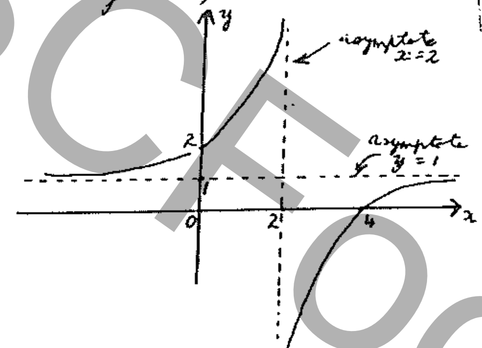
(c) The three vowels E, O, E must be placed in 2^{nd}, 4^{th}, 6^{th} positions. This can be done in $\frac{3!}{2!}$ ways (i.e 3 ways)
The remaining letters (GMTRY) being all different can be arranged in $5!$ or 120 ways
Hence total number of different arrangements $= 3 \times 120 = 360$

(d) $f(x) = \frac{x-4}{x-2} \quad (x \neq 2)$

$$\therefore f'(x) = \frac{(x-2)1 - (x-4)1}{(x-2)^2} = \frac{2}{(x-2)^2}$$

Since $f'(x)$ must always be positive, then $f(x)$ is an increasing function (for all x in its domain i.e all x except $x = 2$)

(ii) Since $f'(x) \neq 0$ there are no stationary points
$x = 2$ is a vertical asymptote
As x gets larger, $y \to 1$ so $y = 1$ is an asymptote
When $x = 0$, $y = 2$
" $y = 0$, $x = 4$



(iii) Let $y = \frac{x-4}{x-2}$
Inverse function given by

$$x = \frac{y-4}{y-2}$$
$$xy - 2x = y - 4$$
$$xy - y = 2x - 4$$
$$y = \frac{2x-4}{x-1}$$

Hence inverse function is

$$y = f^{-1}(x) = \frac{2x-4}{x-1}$$

Range of this function is all real values except 2, which is value excepted from the <u>domain</u> of the original function

QUESTION 4

(a) $A(-2,-1)$ $\quad B(1,5)$
If Q divides AB externally in the ratio $5:3$, then co-ordinates of Q are

$$\left(\frac{5\times1+(-3)\times(-2)}{5-3}, \frac{5\times5+(-3)\times(-1)}{5-3}\right)$$

i.e $\left(\frac{5+6}{2}, \frac{25+3}{2}\right)$

i.e $\left(5\tfrac{1}{2}, 14\right)$

(b) $2\sin(\theta - \frac{\pi}{3}) = \cos(\theta + \frac{\pi}{3})$

$$2\sin\theta\cos\tfrac{\pi}{3} - 2\cos\theta\sin\tfrac{\pi}{3} = \cos\theta\cos\tfrac{\pi}{3} - \sin\theta\sin\tfrac{\pi}{3}$$

$$\therefore \sin\theta - \sqrt{3}\cos\theta = \tfrac{1}{2}\cos\theta - \tfrac{\sqrt{3}}{2}\sin\theta$$

$$\therefore \left(1 + \tfrac{\sqrt{3}}{2}\right)\sin\theta = \left(\tfrac{1}{2} + \sqrt{3}\right)\cos\theta$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}+\sqrt{3}}{1+\frac{\sqrt{3}}{2}} = \frac{1+2\sqrt{3}}{2+\sqrt{3}} = \frac{(1+2\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = -4 + 3\sqrt{3}$$

(c) $\left(2x - \frac{1}{x^2}\right)^{12}$

General Term $u_{r+1} = {}^{12}C_r (2x)^{12-r}\left(-\frac{1}{x^2}\right)^r = (-1)^r\, {}^{12}C_r\, 2^{12-r}\, x^{12-3r}$

For term independent of x,

$$12 - 3r = 0$$
$$\therefore r = 4$$

$\therefore$ Required Term $= (-1)^4\, {}^{12}C_4 \cdot 2^8 = 1,495 \cdot 256 = 126,720$

(d) $AZ = 52$ cms
and since M is mid-point of AZ then $MZ = 26$ cms
Now MZ, LZ, KZ, JZ ..... AZ are in Geometric Progression
Let common ratio $= r$
First term (MZ) $= 26$
and since AZ is 13^{th} term

$$52 = 26r^{12}$$
$$\therefore r^{12} = 2$$
$$\therefore r = 2^{\frac{1}{12}}$$

Now $BZ = 12^{th}$ term $= 26 \cdot (2^{\frac{1}{12}})^{11} = 26 \cdot 2^{\frac{11}{12}}$

$$\therefore AB = AZ - BZ = 52 - 26 \cdot 2^{\frac{11}{12}} = 26\left(2 - 2^{\frac{11}{12}}\right) = 26 \times 0.11225 = 2.9 \text{ cms}$$

Now $FZ = 8^{th}$ term, $GZ = 7^{th}$ term

$$\therefore FG = 26(2^{\frac{1}{12}})^8 - 26(2^{\frac{1}{12}})^7 = 26\left(2^{\frac{8}{12}} - 2^{\frac{7}{12}}\right) = 26(1.58740 - 1.49831) = 26 \times 0.08909 = 2.3 \text{ cms}$$

QUESTION 5

(a) $3x^2 - 5x - 1 = 0$

Roots are $\tan A$ and $\tan B$

$\therefore \tan A + \tan B = \frac{5}{3}$

$\tan A \tan B = -\frac{1}{3}$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{5}{3} - \frac{1}{3}}{1 - (\frac{5}{3})\cdot(-\frac{1}{3})} = \frac{4/3}{14/9} = \frac{4}{3}\cdot\frac{9}{14} = \frac{6}{7}$$

(b) $f(x) = \log_e\left(\frac{x-2}{x}\right)$

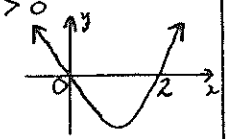
Since domain of $\log_e X$ is $X > 0$, then we have

$\frac{x-2}{x} > 0$

multiply by x^2

$\therefore x(x-2) > 0$

Graph of $x(x-2)$ $\longrightarrow$



From graph, solution is $x < 0$ or $x > 2$

(c) Let $I = \int_1^3 \frac{dx}{(1+x)\sqrt{x}}$

Let $u = \sqrt{x}$

$\therefore \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$

$\therefore dx = 2u \cdot du$

when $x = 1$, $u = 1$

" $x = 3$, $u = \sqrt{3}$

$$\therefore I = \int_1^{\sqrt{3}} \frac{2u \cdot du}{(1+u^2)\cdot u} = 2\int_1^{\sqrt{3}} \frac{du}{1+u^2} = 2\left[\tan^{-1} u\right]_1^{\sqrt{3}} = 2\left[\tan^{-1}\sqrt{3} - \tan^{-1} 1\right] = 2\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{6}$$

(d)(i) $t = 2x^2 - 5x + 3$

$\therefore \frac{dt}{dx} = 4x - 5$

$\therefore v = \frac{dx}{dt} = \frac{1}{\frac{dt}{dx}} = \frac{1}{4x-5}$

(ii) $a = v \cdot \frac{dv}{dx}$

$= \frac{1}{4x-5} \cdot \frac{-1}{(4x-5)^2} \cdot 4$

$= \frac{-4}{(4x-5)^3}$

(iii)(α) When $x = 2$

$v = \frac{1}{8-5} = \frac{1}{3}$ cms/sec

$a = \frac{-4}{3^3} = -\frac{4}{27}$ cms/sec^2

(β) When $t = 6$

$2x^2 - 5x + 3 = 6$

$2x^2 - 5x - 3 = 0$

$\therefore (x-3)(2x+1) = 0$

$\therefore x = 3$ or $-\frac{1}{2}$

But since particle is initially at $1\frac{1}{2}$ cms to right of O and since it is moving away from O and since velocity cannot equal zero (and so no reversing of motion)

$\therefore x = 3$

$\therefore v = \frac{1}{12-5} = \frac{1}{7}$ cm/sec

$a = \frac{-4}{7^3} = -\frac{4}{343}$ cm/sec^2

(iv) Particle starts its motion at $x = 1\frac{1}{2}$ with velocity of 1 cm/sec and acceleration of -4 cm/sec^2. Thereafter velocity is always positive and getting smaller, with acceleration always negative (and getting numerically smaller). Thus particle is slowing down very gradually moving always in a positive direction but being retarded and theoretically not coming to rest.

QUESTION 6

(a) $P(x) = x^3 + ax^2 + bx + c$

$P(-1) = -1 + a - b + c$

$P(-2) = -8 + 4a - 2b + c$

$P(-3) = -27 + 9a - 3b + c$

Since $P(-1) = P(-2)$

$-1 + a - b + c = -8 + 4a - 2b + c$

$\therefore 3a - b = 7$ ....(i)

Since $P(-2) = P(-3)$

$-8 + 4a - 2b + c = -27 + 9a - 3b + c$

$\therefore 5a - b = 19$ ......(ii)

Subtracting (i) from (ii)

$2a = 12$

$\therefore a = 6$

and so $b = 11$

Also since 1 is a zero

$\therefore P(1) = 0$

$\therefore 1 + a + b + c = 0$

$\therefore 1 + 6 + 11 + c = 0$

$\therefore c = -18$

$\therefore P(x) = x^3 + 6x^2 + 11x - 18$

Now $(x-1)$ must be a factor and so by division

$P(x) = (x-1)(x^2 + 7x + 18)$

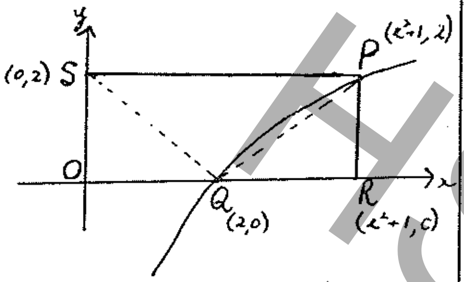
The quadratic expression $x^2 + 7x + 18$ has no real zeros since

$\Delta = 7^2 - 4 \times 18$

$= -23$

Hence $P(x)$ has only one real zero

(b)



$y = \ln(x-1)$

When $y = 0$; $0 = \ln(x-1)$

$\therefore x - 1 = e^0 = 1$

$\therefore x = 2$

Hence Q is pt $(2,0)$

When $y = 2$; $2 = \ln(x-1)$

$\therefore x - 1 = e^2$

$\therefore x = e^2 + 1$

$\therefore$ P is pt $(e^2+1, 2)$

and hence S is pt $(0,2)$

and R is pt. $(e^2+1, 0)$

(i) $\dfrac{dy}{dx} = \dfrac{1}{x-1}$

$= 1$ at pt $Q(2,0)$

$\therefore$ gradient of normal at Q $= -1$

$\therefore$ eqn normal at Q is

$y = -1.(x-2)$

i.e. $x + y = 2$

Subst. $x = 0, y = 2$

$0 + 2 = 2$ which is true

$\therefore S(0,2)$ lies on normal at Q

(ii) $y = \ln(x-1)$

$\therefore x - 1 = e^y$

$\therefore x = e^y + 1$

Area under arc PQ bounded by line intervals OQ, SP, OS

$= \int_0^2 x \cdot dy$

$= \int_0^2 (e^y + 1) \cdot dy$

$= [e^y + y]_0^2$

$= (e^2 + 2) - (e^0 + 0)$

$= e^2 + 1$ (since $e^0 = 1$)

Area rectangle OSPR

$= OR \cdot OS$

$= (e^2+1) \cdot 2$

$= 2(e^2+1)$

Hence arc QP divides rectangle OSPR into two equal portions.

(iii) Area under arc QP bounded by QR, PR

$= e^2 + 1$ (from above)

Area Δ PQR

$= \frac{1}{2} QR \cdot PR$

$= \frac{1}{2} \cdot (e^2 - 1) \cdot 2$

$= e^2 - 1$

$\therefore$ Area enclosed between arc PQ & line interval PQ

$= (e^2+1) - (e^2-1)$

$= 2$ sq. units

Area Δ OSQ $= \frac{1}{2} \cdot OQ \cdot OS$

$= \frac{1}{2} \cdot 2 \cdot 2$

$= 2$ sq. units

QUESTION 7

(a) $\lim\limits_{x \to -5} \dfrac{\sqrt{20-x} - 5}{5 + x}$

$= \lim\limits_{x \to -5} \dfrac{(\sqrt{20-x} - 5).(\sqrt{20-x} + 5)}{(5+x).(\sqrt{20-x} + 5)}$

$= \lim\limits_{x \to -5} \dfrac{(\sqrt{20-x})^2 - 5^2}{(5+x)(\sqrt{20-x} + 5)}$

$= \lim\limits_{x \to -5} \dfrac{20 - x - 25}{(5+x)(\sqrt{20-x} + 5)}$

$= \lim\limits_{x \to -5} \dfrac{-(5+x)}{(5+x)(\sqrt{20+x} + 5)}$

$= \lim\limits_{x \to -5} \dfrac{-1}{\sqrt{20+x} + 5}$

$= \dfrac{-1}{5 + 5}$

$= -0.1$

(b) Let probability of a single throw resulting in an even number $= p$.

$\therefore$ Probability of a single throw resulting in an odd number

$= 1 - p$ ($= q$, say)

In 8 throws of die :-

$P\binom{3 \text{ even}}{\text{numbers}}$ = term in p^3 in expansion of $(q+p)^8$

$= {}^8C_3\, q^5 p^3$

$= 56\, q^5 p^3$

$P\binom{2 \text{ even}}{\text{numbers}}$ = term in p^2 in expansion of $(q+p)^8$

$= {}^8C_2\, q^6 p^2$

$= 28\, q^6 p^2$

and since

$P\binom{3 \text{ even}}{\text{nos}} = 4 \times P\binom{2 \text{ even}}{\text{nos}}$

$\therefore 56\, q^5 p^3 = 4 \times 28 q^6 p^2$

$= 112\, q^6 p^2$

Assuming $p \neq 0$, $q \neq 0$

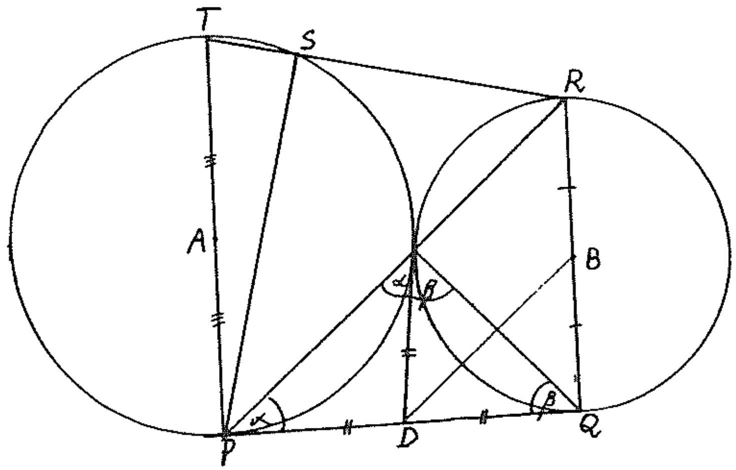
then $p = 2q$

$= 2 - 2p$

$\therefore p = \frac{2}{3}$

$\therefore P\binom{1 \text{ even}}{\text{number}} = \dfrac{2}{3}$

(c)



(i) Join PC, CR, CQ

Since $DP = DC$ (tangents from external pt.)

$\therefore D\hat{P}C = D\hat{C}P$ (isosceles Δ)

$= \alpha$ (say)

Since $DQ = DC$ (tangents from external pt.)

$\therefore D\hat{Q}C = D\hat{C}Q$ (isosceles Δ)

$= \beta$ (say)

But in Δ CPQ

$C\hat{P}Q + C\hat{Q}P + P\hat{C}Q = 180^\circ$ ($\angle$ sum of Δ)

$\therefore \alpha + \beta + \alpha + \beta = 180^\circ$

$\therefore \alpha + \beta = 90^\circ$

And since $P\hat{C}Q = \alpha + \beta$

$\therefore P\hat{C}Q = 90^\circ$

Also $R\hat{C}Q = 90^\circ$ (angle in semi circle)

$\therefore P\hat{C}Q + R\hat{C}R = 180^\circ$

$\therefore$ PCR is a straight angle

$\therefore$ P, C, R are collinear

(ii) Join BD

Since $DP = DC$ (from above)

and $DQ = DC$ (" ")

$\therefore DP = DQ$

$\therefore$ D is mid-pt of PQ

And since B is mid pt of RQ (RQ is diameter, B is centre)

Then in triangle PQR, the line, BD, joining the mid-points of 2 sides must be parallel to the third side PR (Theorem)

(iii) Since PT is a diameter

$\therefore T\hat{S}P = 90^\circ$ ($\angle$ in a semi-circle)

Since BQ is a radius and PQ is a tangent

$\therefore P\hat{Q}B = 90^\circ$ (Theorem)

$\therefore$ In quadrilateral PQRS, exterior angle $T\hat{S}P$ ($= 90^\circ$) equals interior opposite angle $P\hat{Q}R$ ($= 90^\circ$)

$\therefore$ PQRS is cyclic quadrilateral

i.e. points P, Q, R, S are concyclic

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QUESTION 1.

(a) Evaluate $\int_0^{1.5} \frac{dx}{\sqrt{9 - 2x^2}}$, leaving your answer in **EXACT** form.

(b) State the domain and range for the function:

$$y = 2\cos^{-1}(2x)$$

(c) Solve the inequality:

$$\frac{x - 3}{x^2 - x} \geq -2$$

(d) Show by the principles of Mathematical Induction that:—

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

QUESTION 2.

(a) If $f(x) = 2\cos^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sin^{-1}(1 - x^2)$, find $f'(x)$.

Hence, or otherwise, show that:-

$$2\cos^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sin^{-1}(1 - x^2) = \frac{\pi}{2}$$

(b) (i) Write down the expansion for $\sin(A+B)$

(ii) By letting $A = B = x$ in the above expansion, derive an expression for $\sin 2x$ in terms of both $\sin x$ and $\cos x$.

(iii) Find a general solution for the equation:

$$\sin 2x = 2\cos^2 x$$

(c) Use Simpson's Rule, with three function values, to find an approximate value for

$\int_0^1 e^{-x^2}.dx$, to three (3) significant figures.

Cont...

QUESTION 3.

(a) Find the value of. $\int_1^6 x\sqrt{x+3}.dx$, by means of the substitution $u^2 = x + 3$.

(b) A particle moves in a straight line and at time t seconds, its distance x cms. from a fixed origin point, 0, on the line is given by:

$$x = 1 + \frac{1}{2}\cos 2t$$

(i) Sketch a graph of x as a function of t in the domain $0 \leq t \leq 2\pi$.

(ii) Show that the motion of the particle is Simple Harmonic Motion.

(iii) State the centre of motion of the particle.

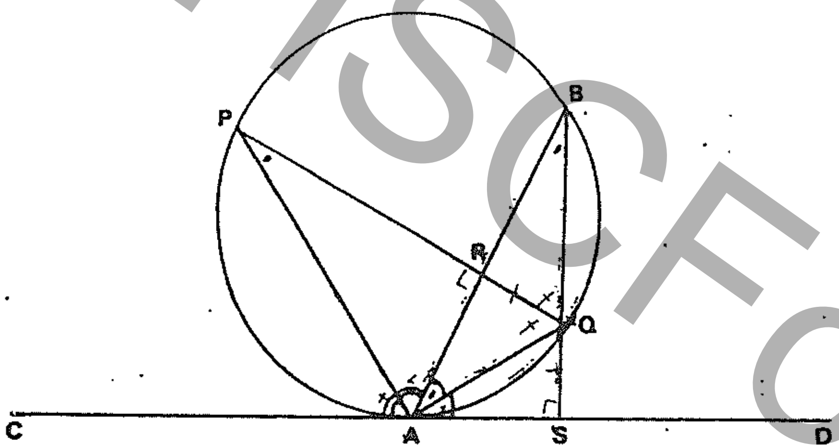
(iv) Find the displacements of the particle when it is at rest and thus determine the length of its path.

(v) State the period of motion for the particle.

QUESTION 4.

(a) Find, expressed as a rational number, the term independent of x in the expansion of $\left(5x^3 + \frac{3}{x^2}\right)^{10}$ and show also that it is the greatest coefficient in the expansion.

(b)



(i) AB is a chord of a circle and CAD is a tangent to the circle at the point A. The bisector of angle BAC meets the circle again at P and the bisector of angle BAD meets the circle again at Q.

Show that:

(α) PQ is a diameter of the circle

(β) PQ is perpendicular to the chord AB.

(ii) PQ meets AB at R and BQ produced meets CD at S.

If BS is perpendicular to CD, prove that:—

(α) $B\hat{A}D = 60°$

(β) $QR = QS$

(γ) $AB = AP$

QUESTION 5.

(a) $P\,(2ap,\ ap^2)$ is a point on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal to the curve of the parabola at the point, P, is:

$$x + py = 2ap + ap^3$$

(ii) Find the co-ordinates of the point Q where the normal at P meets the y-axis.

(iii) Determine the co-ordinates of the point R which divides PQ externally in the ratio 2:1.

(iv) Find the cartesian equation of the locus of R and describe the locus in geometrical terms.

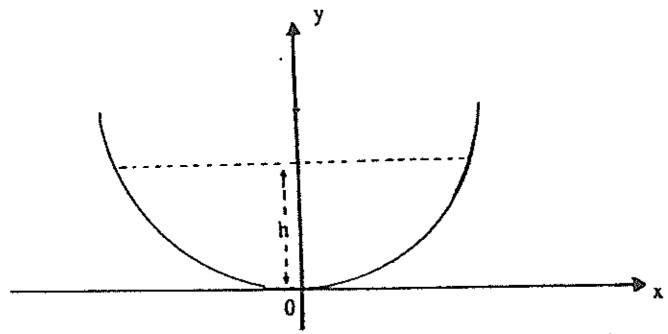
(b) A prize fund is set up with a single investment of \$2000 to provide an annual prize of \$250. The fund accrues interest at the rate of 10% per annum, compounded yearly. The first prize is awarded one year after the investment is initially set up.

(i) If $\$P_n$ denotes the value of the fund at the end of n years (and after the n^{th} prize has been awarded), show that

$$P_n = 2500 - 500\,(1 \cdot 1)^n$$

(ii) Hence find the number of years for which the full amount of the prize can be awarded.

QUESTION 6.



The diagram above shows the lower half of the circle whose equation is $x^2 + y^2 - 20y = 0$.

A hemispherical bowl is obtained by rotating the semi-circle about the y—axis.

(i) Show that when the depth of the water in the bowl is h cms., the volume of water, V, in the bowl is given by:

$$V = \frac{\pi h^2}{3}(30 - h) \text{ cms}^3.$$

and the surface area of the water, S, exposed to the open air is given by:

$$S = \pi h(20 - h) \text{ cms}^2.$$

(ii) If water is poured into the bowl at the rate of 50 cms^3/sec., determine, at the instant when the depth of water is 5cms, :

(α) the rate of increase of the depth of water in cms /sec. (correct to 2 decimal places)

(β) the rate of increase of the open surface area of water in cms^2/sec. (correct to 1 decimal place).

QUESTION 7.

(a) (i) The letters of the word PERSEVERE are arranged in a row. How many **DIFFERENT** arrangements would be possible?

(ii) Out of all the different arrangements found in (i) above, one is chosen at random. Find the probability that this particular arrangement:

(α) will have all the E's together in one group **AND** all the R's together in another group.

(β) will have an 'E' at one end and an 'R' at the other end.

(b) A man 1.8 metres tall, standing due east of a street light, notices that his shadow is 6.5 metres long. He walks on a bearing of 150°T from his first position for a distance of 12 metres and now finds his shadow is 9.1 metres long. Find the height of the street light above ground level (in metres correct to 1 decimal place).

END OF PAPER

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Question 1

a) $\int_0^{1.5} \frac{dx}{\sqrt{9-2x^2}} = \frac{1}{\sqrt{2}}\int_0^{1.5} \frac{dx}{\sqrt{\frac{9}{2}-x^2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{\sqrt{2}x}{3}\right]_0^{1.5}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{\sqrt{2}\cdot\frac{3}{2}}{3} - \sin^{-1}0\right]$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{\sqrt{2}}{2}\right]$

$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{1}{\sqrt{2}}$

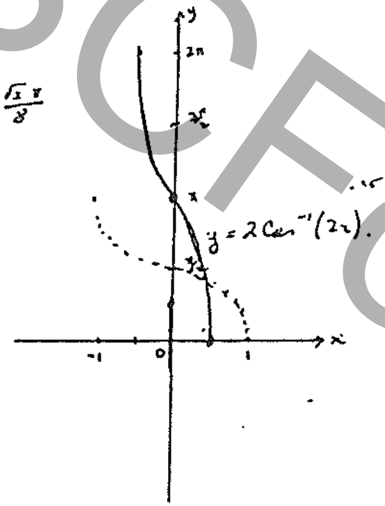
$= \frac{1}{\sqrt{2}} \times \frac{\pi}{4}$

$= \frac{\pi}{4\sqrt{2}}$ or $\frac{\sqrt{2}\pi}{8}$

b). $y = 2\cos^{-1}(2x)$.

$2x = \cos\frac{y}{2}$

$x = \frac{1}{2}\cos\frac{y}{2}$.



D: $\frac{1}{2} \leq x \leq -\frac{1}{2}$

R: $0 \leq y \leq 2\pi$.

c) $\frac{x-3}{x^2-x} \geq -2$.

$\frac{x-3}{x^2-x} + 2 \geq 0$

$\frac{x-3}{x^2-x} + \frac{2x^2-2x}{x^2-x} \geq 0$

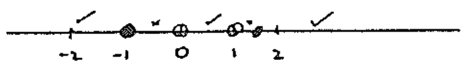
$\frac{2x^2-x-3}{x(x-1)} \geq 0$

$\frac{(2x-3)(x+1)}{x(x-1)} \geq 0$

part soln

1) $2x-3=0$ 2) $x+1=0$ 3) $x \neq 0$ 4) $x \neq 1$

$x = \frac{3}{2}$ $x=-1$



Test. $\boxed{x=-2}$. $\frac{-7\cdot-1}{-2\cdot-3} > 0$ $\frac{7}{6} > 0$ T.

$\boxed{x=-\frac{1}{2}}$ $\frac{-4\cdot\frac{1}{2}}{-\frac{1}{2}\cdot-1\frac{1}{2}} > 0$ F

$\boxed{x=\frac{1}{2}}$ $\frac{-2\cdot1\frac{1}{2}}{\frac{1}{2}\cdot-\frac{1}{2}} > 0$ T.

$\boxed{x=1\frac{1}{4}}$ $\frac{-\frac{1}{2}\cdot\frac{9}{4}}{\frac{5}{4}\cdot\frac{1}{4}} > 0$ F

$\boxed{x=2}$ $\frac{1\cdot3}{2\cdot1} > 0$

$\therefore$ soln is $x \leq -1$, $0 < x < 1$, $x \geq 2$.

d) $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.

For $n=1$

L.H.S $= \frac{1}{3}$ R.H.S $= \frac{1}{2\cdot1+1}$

$= \frac{1}{3}$.

$\therefore$ L.H.S = R.H.S.

Assume true for $n=k$.

i.e $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

R.T.P true for $n=k+1$

i.e $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{(2k+3)}$

L.H.S $= \frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$.

$= \frac{k(2k+3) + 1}{(2k+1)(2k+3)}$.

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$.

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{2k+3}$.

= R.H.S.

$\therefore$ true for $n=k+1$.

$\therefore$ True for $n=1$ also true for $n=k$ and $n=k+1$.

$\therefore$ true $\forall\, n$.

QUESTION 2.

a) $f(x) = 2\cos^{-1}\frac{x}{\sqrt2} - \sin^{-1}(1-x^2)$.

$f'(x) = \frac{-2}{\sqrt{2-x^2}} - \frac{1}{\sqrt{1-(1-x^2)^2}} \cdot -2x$.

$= \frac{-2}{\sqrt{2-x^2}} + \frac{2x}{\sqrt{1-1+2x^2-x^4}}$

$= \frac{-2}{\sqrt{2-x^2}} + \frac{2x}{\sqrt{2x^2-x^4}}$

$= \frac{-2}{\sqrt{2-x^2}} + \frac{2x}{x\sqrt{2-x^2}}$.

$= \frac{-2}{\sqrt{2-x^2}} + \frac{2}{\sqrt{2-x^2}}$

$= 0$.

$\therefore f(x)$ is a horizontal line. $\forall x$.
and cuts y-axis when $x=0$.

$\therefore f(0) = 2\cos^{-1}0 - \sin^{-1}1$

$= 2\times\frac{\pi}{2} - \frac{\pi}{2}$.

$= \pi - \frac{\pi}{2}$

$= \frac{\pi}{2}$.

b)

i) $\sin(A+B) = \sin A\cos B + \sin B\cos A$.

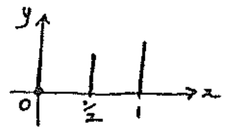
ii) $\sin 2A = \sin A\cos A + \sin A\cos A$. where $A=B$.
$\sin 2A = 2\sin A\cos A$.

i.e $\sin 2x = 2\sin x\cos x$.

iii) $\sin 2x = 2\cos^2 x$.
$2\sin x\cos x = 2\cos^2 x$.
$2\cos^2 x - 2\sin x\cos x = 0$.
$2\cos x(\cos x - \sin x) = 0$.
$\therefore 2\cos x = 0$ or $\cos x = \sin x$.
$\cos x = 0$. $x = 45°, 225°, 405°$..
$\therefore x = 0, 180, 360, \ldots$ $= 45, 5\times 45, 9\times 45°$
$x = 180(n-1)$ $x = 45°(4n-3)$.
where $n=1,2,3,\ldots$ $n = 1,2,3,\ldots$

1, 5, 9, 13, ...

c). $\int_0^1 e^{-x^2}dx$



x	0	$\frac12$	1
y	1	$e^{-\frac14}$ 0.7788	e^{-1} 0.3679

$A = \frac{h}{3}[y_0 + y_3 + 4y_2]$

$= \frac{\frac12}{3}[1 + 0.3679 + 4\times 0.7788]$

$= \frac{1}{6}[4.4831]$

$= 0.747$ (3 sig fig).

QUESTION 3.

a) $\int_1^6 x\sqrt{x+3}\,dx$

Let $u^2 = x+3$.

$\therefore u = \sqrt{x+3} = (x+3)^{\frac{1}{2}}$

~~$du = \frac{1}{2}(x+3)$~~

also $2u\,du = dx$

when $x=1$, $u=\sqrt{4} = 2$.

when $x=6$, $u=\sqrt{9} = 3$.

$\therefore \int_1^6 x\sqrt{x+3}\,dx = \int_2^3 (u^2-3).u.2u\,du$

$= \int_2^3 2u^4 - 6u^2\,du$

$= \left[\frac{2u^5}{5} - \frac{\cancel{6}^2 u^3}{\cancel{3}}\right]_2^3$

$= \left[\frac{2.3^5}{5} - 2.3^3\right] - \left[\frac{2.2^5}{5} - 2.2^3\right]$

$= \frac{486}{5} - 54 - \frac{64}{5} + \cancel{54}\,16.$

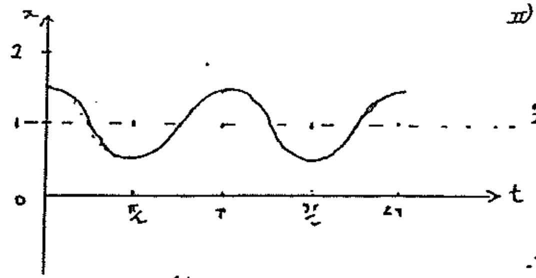
$= \frac{422}{5} - 38$

$= 84\frac{2}{5} - 38$

$= 46\frac{2}{5}$

b) $x = 1 + \frac{1}{2}\cos 2t.$

(I)



(II) $\dot{x}' = -\frac{1}{2}.2\sin 2t.$

$= -\sin 2t.$

$\ddot{x}'' = -2\cos 2t.$

now $\cos 2t = 2x - 2.$

$\therefore \ddot{x} = -2(2x-2).$

$= 4 - 4x$

which is simple harmonic motion

(III) Centre of motion $= 1$

(IV). at rest when $\dot{x}' = 0.$

$\therefore -\sin 2t = 0.$

$\sin 2t = 0$

$2t = 0, \pi, 2\pi, \ldots$

$t = 0, \frac{\pi}{2}, \pi,$

$\therefore$ length of path $= \frac{\pi}{2}$ units.

(V) $T = \frac{2\pi}{n}.$ $n = 2.$

$= \frac{2\pi}{2}.$

$= \pi$ units.

QUESTION 4

a) $(5x^2 + \frac{3}{x^3})^{10}$ typical term is ${}^{10}C_k (5x^2)^k (\frac{3}{x^3})^{10-k}$

$= {}^{10}C_k\, 5^k x^{2k} \cdot \frac{3^{10-k}}{x^{30-3k}}$

$= {}^{10}C_k\, 5^k \cdot 3^{10-k}\; x^{5k-30}$.

independent term has x^0

$\therefore 5k - 30 = 0$.

$5k = 30$

$\therefore k = 6$.

$\therefore$ Term is ${}^{10}C_6\, 5^6\, 3^4$

for greatest term k $\frac{T_k}{T_{k+1}} > 1$.

i.e $\frac{{}^{10}C_k (5x^2)^k (\frac{3}{x^3})^{10-k}}{{}^{10}C_{k+1} (5x^2)^{k+1} (\frac{3}{x^3})^{9-k}} > 1$

$\frac{\frac{10!}{(10-k)!\,k!} \cdot 5^k x^{2k} \cdot \frac{3^{10-k}}{x^{30-3k}}}{\frac{10!}{(10-k-1)!\,(k+1)!}\, 5^{k+1} x^{2k+2} \cdot \frac{3^{9-k}}{x^{30-3k-3}}} > 1$

$\frac{\frac{3}{10-k}\, x^{5k-30}}{\frac{5}{k+1}\, x^{5k-30+3}} > 1$ We are comparing co-efficients $\therefore$ we can ignore this expression.

i.e $\frac{3(k+1)}{5(10-k)} > 1$

$3k + 3 > 50 - 5k$

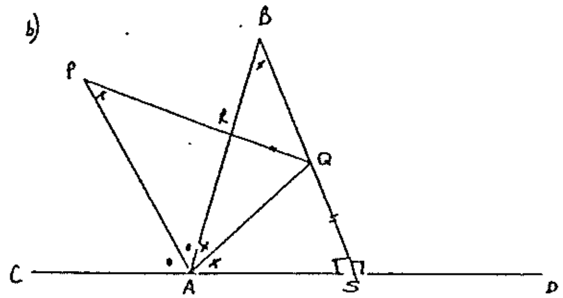
$8k > 47$

$k > 5\frac{7}{8}$

i.e greatest term is when $k = 6$

i.e ${}^{10}C_6\, 5^6\, 3^4$

b)



I). (α) $\angle CAP = \angle PAB$ (PA bisects $\angle CAB$).

$\angle DAQ = \angle QAB$ (AQ bisects $\angle BAD$).

CAD is straight

$\therefore \angle CAP + \angle PAB + \angle DAQ + \angle QAB = 180°$

i.e. $2\angle PAB + 2\angle BAQ = 180°$

$\angle PAB + \angle BAQ = 90°$

$\therefore$ PQ is a diameter (angle in a semi-circle is 90°).

(β) $\angle QAS = \angle APQ$ (Angle in alternate segment).

~~But~~ $\therefore \angle APQ + \angle PAB = 90°$ (from above).

$\therefore \angle PRA = 90$ (angle sum of Δ).

i.e $PQ \perp AB$.

II) (α) $\angle ABS = \angle QAS$ (Angle in Alternate segment).

$\therefore \angle ABS = \angle BAS = \angle BAQ$

$\therefore \angle BAD = 2 \times 30°$

$= 60°$

In ΔARQ and ΔQAS.

(β) $\angle BAQ = \angle QAS$ (AQ bisects $\angle BAD$)

AQ is common

$\angle ARQ = \angle ASQ$ (90°)

$\therefore \Delta ARQ \equiv \Delta AQS$ (AAS)

$\therefore QR = QS$ (Corresponding sides $\equiv \Delta$).

(γ) In ΔBAS & ΔPAR

$AR = AS$ (Corresponding sides $\equiv \Delta$'s (β)).

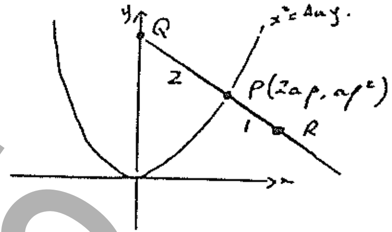
$\angle PRA = \angle BSA = 90°$

$\angle APR = \angle ABS$ (Angles in same segment).

$\therefore \Delta BAS \equiv \Delta PAR$ (AAS) $\therefore AP = AB$ (corresponding sides ...).

GUESTION 5.

(a) $P(2ap, ap^2)$



(I) $x^2 = 4ay$.

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$m_T = \frac{2ap}{2a}$

$= p$

$\therefore m_N = -\frac{1}{p}$. normal $\Rightarrow$ $y - y_1 = m(x - x_1)$.

$y - ap^2 = -\frac{1}{p}(x - 2ap)$.

$py - ap^3 = -x + 2ap$.

$x + py = 2ap + ap^3$

(II) at Q $x = 0$ (y-axis).

$\therefore py = 2ap + ap^3$

$y = 2a + ap^2$ i.e $Q = (0, 2a + ap^2)$

(III) $QR : RP = 3 : -1$.

$R = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

$= \left(\frac{3 \times 2ap + -1 \times 0}{3-1}, \frac{3 \times ap^2 + -1 \times (2a + ap^2)}{3-1}\right)$.

$= \left(\frac{6ap}{2}, \frac{3ap^2 - 2a - ap^2}{2}\right)$.

$= (3ap, ap^2 - a)$

(IV) $x = 3ap \Rightarrow p = \frac{x}{3a}$

$y = ap^2 - a$.

$\therefore y = a \cdot \left(\frac{x}{3a}\right)^2 - a$.

$y = \frac{x^2}{9a} - a$

$9a(y + a) = x^2$.

Parabola with vertex $(0, -a)$

focal length $= \frac{9a}{4}$.

(b) \$2000 annual payt = \$250 10% p.a.

(I) after 1 year $P_1 = 2000(1 \cdot 1) - 250$

after 2 years $P_2 = (2000(1 \cdot 1) - 250)(1 \cdot 1) - 250$

$= 2000(1 \cdot 1)^2 - 250(1 \cdot 1) - 250$.

⋮

after n years $P_n = 2000(1 \cdot 1)^n - 250\underbrace{(1 + 1 \cdot 1 + 1 \cdot 1^2 + \ldots + 1 \cdot 1^{n-1})}$.

GP $a = 1$ $r = 1 \cdot 1$ n terms.

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1(1 \cdot 1^n - 1)}{1 \cdot 1 - 1}$

$= \frac{1 \cdot 1^n - 1}{\cdot 1}$

$\therefore P_n = 2000(1 \cdot 1)^n - 250 \times \frac{((1 \cdot 1)^n - 1)}{\cdot 1}$

$= 2000(1 \cdot 1)^n - 2500(1 \cdot 1)^n + 2500$.

$= 4 \times 500(1 \cdot 1)^n - 5 \times 500(1 \cdot 1)^n + 2500$

$= 2500 - 500(1 \cdot 1)^n[5 - 4]$

$= 2500 - 500(1 \cdot 1)^n$

(II) After n years $P = 0$

$\therefore 2500 - 500(1 \cdot 1)^n = 0$

$500(1 \cdot 1)^n = 2500$

$(1 \cdot 1)^n = 5$

$\log(1 \cdot 1)^n = \log 5$

$n \log 1 \cdot 1 = \log 5$

$n = \frac{\log 5}{\log 1 \cdot 1}$

$= 16 \cdot 886$ years.

$\doteqdot 17$ years.

Question 6.

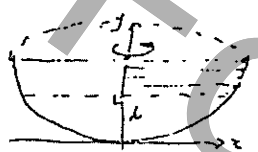
$x^2+y^2-20y=0$. $\quad x^2+(y-10)^2=100$. $\quad (0,10)$. $R=10$.

(I) $V = \pi \int (f(y))^2 \, dy$. $\qquad x^2 = 20y - y^2$; $\quad x = \sqrt{20y-y^2}$.

$\therefore V = \pi \int_0^h \left(\sqrt{20y-y^2}\right)^2 dy$.

$= \pi \int_0^h 20y - y^2 \, dy$.

$= \pi \left[10y^2 - \frac{y^3}{3}\right]_0^h$.



$= \pi\left[\left(10h^2 - \frac{h^3}{3}\right) - 0\right]$.

$= \frac{\pi h^2}{3}(30-h)$. cm^3

~~S.A. = πr²~~ ~~= 2π~~ ~~= 2π × 10²~~ ~~= 200π.~~ $\quad S.A = \pi r^2$.

at height h. $\quad x = \sqrt{20h-h^2}$.

$\therefore S.A = \pi\left(\sqrt{20h-h^2}\right)^2$

$= \pi(20h-h^2)$.

$= \pi h(20-h)$.

II) $\frac{dV}{dt} = 50 \text{ cm}^3/\text{s}$. $\qquad h = 5\text{cm}$.

(α) $\frac{dh}{dt} = ?$ $\qquad \frac{dV}{dt} = \frac{dh}{dt}\cdot\frac{dV}{dh}$.

$\frac{dV}{dh} = \pi(20h-h^2)$.

at $h=5$ $\qquad \therefore 50 = \frac{dh}{dt}\cdot 75\pi$.

$\frac{dV}{dh} = \pi(100-25)$ $\qquad \frac{dh}{dt} = \frac{50}{75\pi}$.

$= 75\pi$ $\qquad = \frac{2}{3\pi}$ cm/sec.

(β) $\frac{dS}{dt} = ?$ $\qquad \frac{dS}{dt} = \frac{dS}{dh}\cdot\frac{dh}{dt}$.

$\frac{dS}{dh} = \pi(20-2h)$. $\qquad \therefore \frac{dS}{dt} = 10\pi \times \frac{2}{3\pi}$

at $h=5$ $\qquad = \frac{20}{3}$ cm²/sec.

$\frac{dS}{dh} = \pi(20-10)$

$= 10\pi$.

Question 7.

a (I) No of arrangements $= \frac{9!}{4!\,2!}$. PERSEVERE

$= 7560$.

(II)(a) $\boxed{4\cdot E's}$ $\boxed{2\cdot R's}$ $\square$ $\square$ $\square$

No of arrangements $= 5!$ (all E's identical, all R's identical)

$\therefore Pr(E \text{ and } R \text{ together}) = \frac{5!}{7560}$

$= \frac{120}{7560}$.

$= \frac{1}{63}$.

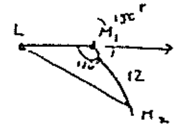
(β) E $\boxed{4}$ $\boxed{7}$ $\boxed{6}$ $\boxed{5}$ $\boxed{4}$ $\boxed{3}$ $\boxed{2}$ $\boxed{1}$ R $\boxed{2}$ (PRSEVEE)

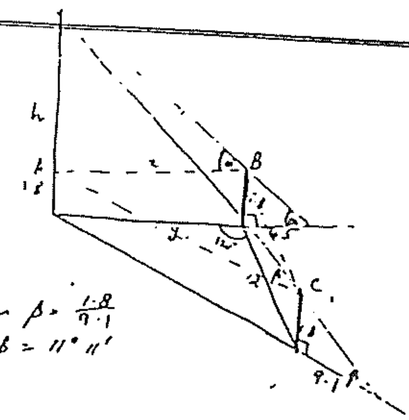
or R $\boxed{2}$ $\boxed{7}$ $\boxed{6}$ $\boxed{5}$ $\boxed{4}$ $\boxed{3}$ $\boxed{2}$ $\boxed{1}$ E $\boxed{3}$?

No of arrangements $= 2 \times (4 \times 2 \times \frac{7!}{3!})$.

$=$

(b)



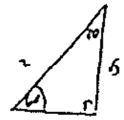


$\tan\alpha = \frac{1.8}{6.5}$ $\tan\beta = \frac{1.8}{9.1}$

$\alpha = 15°29'$ $\beta = 11°11'$

$\tan\alpha = \frac{h}{x}$ $\tan\beta = \frac{h}{y}$.

$\therefore x = \frac{h}{\tan\alpha}$ $y = \frac{h}{\tan\beta}$.



In $\triangle ABC$ $y^2 = x^2 + 12^2 - 2 \times x \times 12 \cos 120°$

$\frac{h^2}{\tan^2\beta} = \frac{h^2}{\tan^2\alpha} + 144 - 24\frac{h}{\tan\alpha} \times -\frac{1}{2}$.

$\frac{h^2}{\tan^2\beta} = \frac{h^2}{\tan^2\alpha} + 144 + \frac{12h}{\tan\alpha}$.

$\therefore 0 = h^2\left(\frac{1}{\tan^2\alpha} - \frac{1}{\tan^2\beta}\right) + \frac{12h}{\tan\alpha} + 144$.

$= h^2\left(\left(\frac{6.5}{1.8}\right)^2 - \left(\frac{9.1}{1.8}\right)^2\right) + \frac{12 \times 6.5}{1.8}h + 144$

$= h^2(6.5^2 - 9.1^2) + 12 \times 6.5 \times 1.8h + 144 \times 1.8^2$

$= -40.56h^2 + 140.4h + 466.56$

$\therefore 40.56h^2 - 140.4h - 466.56 = 0$.

$h = \frac{-(-140.4) \pm \sqrt{140.4^2 - 4 \times 40.56 \times -466.56}}{2 \times 40.56}$

$= \frac{140.4 \pm \sqrt{95406.854}}{81.12}$

$= \frac{140.4 \pm 308.88}{81.12}$

$\therefore h = 5.538, -2.0769$

$\therefore h = 5.538$.

$\therefore$ height of light $= 5.538 + 1.8$

$= 7.338$ m.