Mathematics Extension 1 2010 Trial HSC

Question 1

a)

$$\lim_{x \to 0} \frac{5x}{\sin 2x} = \frac{5}{2} \lim_{x \to 0} \frac{2x}{\sin 2x}$$

$$= \frac{5}{2} \times 1$$

$$= 2.5$$

b)

$$\int \frac{dx}{16 + x^2} = \frac{1}{4} \tan^{-1} \frac{x}{4} + C$$

c)

$$\int_{\pi/9}^{\pi/6} \sin 3x \cos 3x \, dx = \frac{1}{3} \int_{\pi/9}^{\pi/6} (3\cos 3x) \sin 3x \, dx$$

$$= \frac{1}{3} \left[\frac{\sin^2 3x}{2} \right]_{\pi/9}^{\pi/6}$$

$$= \frac{1}{3} \left[\frac{\sin^2 \frac{\pi}{2}}{2} - \frac{\sin^2 \frac{\pi}{3}}{2} \right]$$

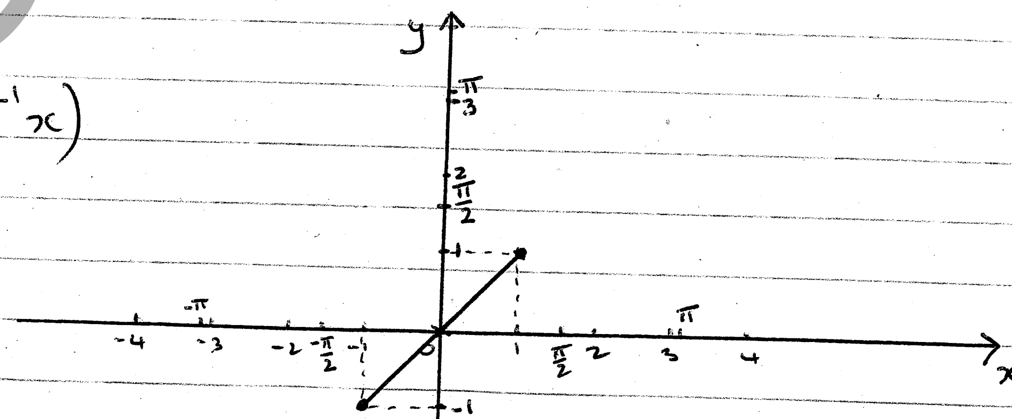
$$= \frac{1}{6} \left[1^2 - \left(\frac{\sqrt{3}}{2} \right)^2 \right]$$

$$= \frac{1}{6} \left[1 - \frac{3}{4} \right]$$

$$= \frac{1}{6} \times \frac{1}{4}$$

$$= \frac{1}{24}$$

d) $y = \cos(\cos^{-1} x)$



$x = 0$, $\cos^{-1} 0 = \frac{\pi}{2}$

$\cos \frac{\pi}{2} = 0$

$x = 1$, $\cos^{-1} 1 = 0$

$\cos 0 = 1$

$x = -1$, $\cos^{-1}(-1) =$

$\cos \pi = -1$

e) $\int x^2 e^{x^3-1}\,dx = \int (u+1)^{2/3} e^{u} \cdot \frac{1}{3}(u+1)^{-2/3}\,du$

$u = x^3 - 1$

$u+1 = x^3$

$(u+1)^{1/3} = x$

$\frac{1}{3}(u+1)^{-2/3}du = dx$

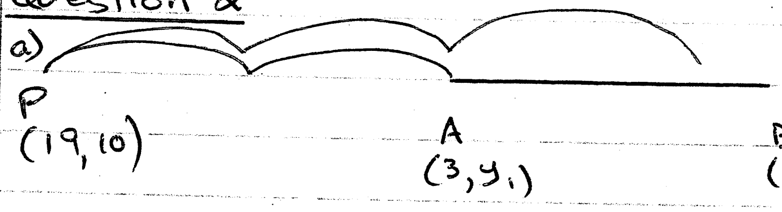
$= \frac{1}{3}\int e^{u}\,du$

$= \frac{1}{3}e^{u} + A$

$= \frac{1}{3}e^{x^3-1} + A$

Question 2

a)



$$AP : PB = -2 : 3$$

$$19 = \frac{3 \times 3 - 2 \times x_2}{-2+3} \qquad\qquad 10 = \frac{3 \times y_1 - 2 \times 1}{-2+3}$$

$$19 = 9 - 2x_2 \qquad\qquad 10 = 3y_1 - 2$$

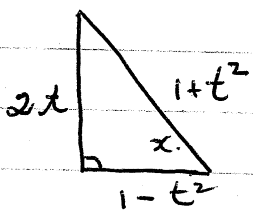
$$10 = -2x_2 \qquad\qquad 12 = 3y_1$$

$$-5 = x_2 \qquad\qquad 4 = y_1$$

b)

$$5\sin x - 7\cos x = 3 \qquad 0^\circ \leq x \leq 360^\circ$$

Let $\tan\frac{x}{2} = t \quad \therefore \tan x = \frac{2t}{1-t^2}$



$$\therefore 5 \times \frac{2t}{1+t^2} - 7 \times \frac{1-t^2}{1+t^2} = 3$$

$$10t - 7(1-t^2) = 3(1+t^2)$$

$$10t - 7 + 7t^2 = 3 + 3t^2$$

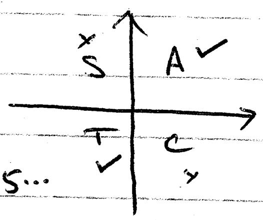
$$4t^2 + 10t - 10 = 0$$

$$2t^2 + 5t - 5 = 0$$

$$t = \frac{-5 \pm \sqrt{25 - 4 \times 2 \times -5}}{4}$$

$$= \frac{-5 \pm \sqrt{25+40}}{4}$$

$$= \frac{-5 \pm \sqrt{65}}{4}$$



ie $t = 0.7655\ldots$ OR $t = -3.2655\ldots$

ie $\tan\frac{x}{2} = 0.7655\ldots$ $\qquad$ $\tan\frac{x}{2} = -3.2655\ldots$

$\frac{x}{2} = 37^\circ 26'$ or $180^\circ + 37^\circ 26'\ldots$ outside domain $\qquad$ $\frac{x}{2} = 180^\circ - 72^\circ 58'\ldots$

$\frac{x}{2} = 107^\circ 2'\ldots$

ie $x = 74^\circ 52'$ $\qquad$ $x = 214^\circ 3'$

Now test $x = 180^\circ$; LHS $= 5\sin 180^\circ - 7\cos 180^\circ$ $\quad$ RHS $= 3$

$= 0 - 7$

$= 1$

LHS $\neq 3 \therefore x = 180^\circ$ not a solution

Solution is
$x = 74^\circ 52'$ OR $107^\circ 2'$

c) $2x - 3y + 8 = 0$ $\qquad$ $5x - 2y - 9 = 0$

$2x + 8 = 3y$ $\qquad$ $5x - 9 = 2y$

$\frac{2}{3}x + \frac{8}{3} = y$ $\qquad$ $\frac{5}{2}x - \frac{9}{2} = y$

$m_1 = \frac{2}{3}$ $\qquad$ $m_2 = \frac{5}{2}$

Acute angle given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{2}{3} - \frac{5}{2}}{1 + \frac{2}{3} \times \frac{5}{2}}\right|$$

$$= \left|-\frac{11}{16}\right|$$

$$= \frac{11}{16}$$

$$\therefore \theta = 34^\circ 31' \text{ (nearest minute)}$$

d)

$$y = \cos^{-1} 2x$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(2x)^2}} \cdot 2$$

$$= \frac{-2}{\sqrt{1-4x^2}}$$

when $x = 0$, $\frac{dy}{dx} = \frac{-2}{\sqrt{1-0}} = -2$, $y = \cos^{-1} 0 = \frac{\pi}{2}$

Thus, gradient tangent is -2

$\therefore$ Equation of normal at $x = 0$ is

$$y - \frac{\pi}{2} = -2(x - 0)$$

$$y = -2x + \frac{\pi}{2}$$

Question 3.

a) $\ddot{x} = -4x$

(i) For $x = 3\sin 2t$

$v = \frac{dx}{dt} = 3\cos 2t \times 2$

$= 6\cos 2t$

$\ddot{x} = \frac{d^2x}{dt^2} = 6 \times -\sin 2t \times 2$

$= -12\sin 2t$

$= -4(3\sin 2t)$

$= -4x$, since $x = 3\sin 2t$

(ii) Particle returns to the Origin when $\ddot{x} = 0$

ie $-12\sin 2t = \ddot{x} = 0$

$\sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Thus after starting from the origin when $t=0$, particle returns at $\frac{\pi}{2}$ seconds, π seconds during the first 4 seconds

(iii) when $t=0$, $v = 6\cos 2t$

$= 6\cos 0$

$= 6$

when $t=\frac{\pi}{2}$, $v = 6\cos\pi$

$= -6$

when $t=\pi$, $v = 6\cos 2\pi$

$= 6\times 1$

$= 6$

ie initially velocity is 6 units/s then at $\frac{\pi}{2}$ secs is -6 units/s and at π secs is 6 units/s

(iv)

$$\ddot{x} = -4x$$

$$\frac{d(\frac{1}{2}v^2)}{dx} = -4x$$

$$\frac{1}{2}v^2 = \frac{-4x^2}{2} + c$$

when $x = 0,\ v = 6$ $\therefore$ $\frac{1}{2} \times 36 = 0 + c$

$$18 = c$$

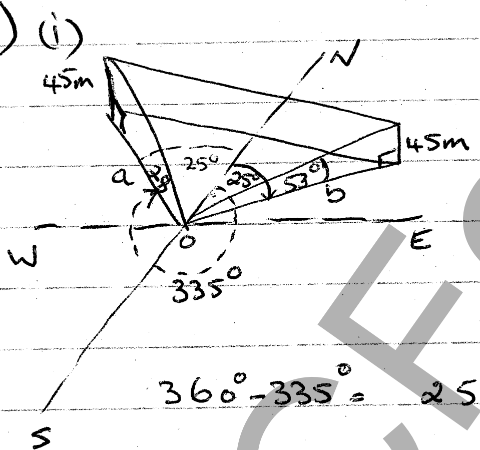
$$\therefore \frac{1}{2}v^2 = -2x^2 + 18$$

$$v^2 = -4x^2 + 36$$

$$v = \pm\sqrt{36 - 4x^2}$$

$$v = \pm 2\sqrt{9 - x^2}$$

b) (i)



(i) $\tan 28° = \frac{45}{a}$ $\qquad$ $\tan 53° = \frac{45}{b}$

$a = \frac{45}{\tan 28°}$ $\qquad$ $b = \frac{45}{\tan 53°}$

In ground triangle

$360° - 335° = 25°$

$$d^2 = a^2 + b^2 - 2ab\cos(25° + 25°)$$

$$d^2 = \left(\frac{45}{\tan 28°}\right)^2 + \left(\frac{45}{\tan 53°}\right)^2 - 2\left(\frac{45^2}{\tan 28° \tan 53°}\right)\cos 50°$$

$$= 4623.117\ldots$$

$$\therefore d = 67.993\ldots$$

distance is 68 m (correct to nearest m)

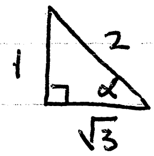
(ii)

$$\text{Speed} = \frac{67.993\ldots\ m}{1\ \text{min}}$$

$$= \frac{67.993\ldots\ m}{(1 \times 60)\ \text{sec}}$$

$= 1.133\ldots$ m/sec OR 1.13 m/sec (2dp)

c) (i) $\sqrt{3}\sin 2x + \cos 2x = 2\left[\sin 2x \times \frac{\sqrt{3}}{2} + \cos 2x \times \frac{1}{2}\right]$



$$= 2\sin(2x+\alpha)$$

$\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$$= 2\sin\left(2x+\frac{\pi}{6}\right)$$

ie $R = 2$ and $\alpha = \frac{\pi}{6}$

(ii) $\sqrt{3}\sin 2x + \cos 2x = 0$

$$2\sin\left(2x+\frac{\pi}{6}\right) = 0$$

$$\sin\left(2x+\frac{\pi}{6}\right) = 0.$$

$2x + \frac{\pi}{6} = 0 + n\pi$, $n \in$ Integers

$2x = n\pi - \frac{\pi}{6}$

$x = \frac{n\pi}{2} - \frac{\pi}{12}$, $n \in$ Integers

Question 4.

a) (i)

$$A_0 = 200$$
$$A_1 = 200\left(1 + \frac{9}{1200}\right)^1$$
$$A_1 = 200(1.0075)^1$$
$$A_2 = 200(1.0075)^2 + 200(1.0075)^1$$
$$A_3 = 200(1.0075)^3 + 200(1.0075)^2 + 200(1.0075)^1$$
$$A_n = 200(1.0075)^n + 200(1.0075)^{n-1} + \cdots + 200(1.0075)$$
$$A_n = 200(1.0075)\left[1.0075^{n-1} + 1.0075^{n-2} + \cdots + 1\right]$$

G.P. since $\frac{T_2}{T_1} = \frac{T_3}{T_2} = 1.0075$

$a = 1,\ r = 1.0075$

$$\therefore A_n = 200(1.0075)\left[\frac{1(1 - 1.0075^n)}{1 - 1.0075}\right]$$

(ii) For $A_n = 90000$,

$$90000 = 200(1.0075)\left[\frac{1 - 1.0075^n}{1 - 1.0075}\right]$$
$$\frac{90000 \times (1 - 1.0075)}{200 \times 1.0075} = 1 - 1.0075^n$$
$$1.0075^n = 1 - \frac{90000 \times (1 - 1.0075)}{200 \times 1.0075}$$
$$1.0075^n = 4.3498\ldots$$
$$n \log_e 1.0075 = \log_e 4.3498\ldots$$
$$n = \frac{\log_e 4.3498\ldots}{\log_e 1.0075}$$
$$n = 196.7538\ldots$$

It will take $196.753\ldots$ months

$16.396\ldots$ years $\approx$ 16 yrs 4.75 month

b) $$P(x) = x^3 + (2-k)x^2 + (4-2k)x + 8$$

(i) $$P(-2) = (-2)^3 + (2-k)(-2)^2 + (4-2k)(-2) + 8$$
$$= -8 + 8 - 4k - 8 + 4k + 8$$
$$= 0$$
$\therefore$ $x = -2$ is a root of $P(x) = 0$

(ii) Now, $P(x) = (x+2)(x^2 - kx + 4)$
$x = -2$ is already a root of $P(x) = 0$
Thus $x^2 - kx + 4 = 0$ must have two roots.
For 2 distinct roots $\Delta > 0$
$$\text{ie } b^2 - 4ac > 0$$
$$(-k)^2 - 4 \times 1 \times 4 > 0$$
$$k^2 - 16 > 0$$
$$k^2 > 16$$
For 3 distinct roots; $k < -4$ OR $k > 4$

c) $${}^nC_r + {}^nC_{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-\overline{r+1})!}$$
$$= \frac{n!}{r!(n-r)(n-\overline{r+1})!} + \frac{n!}{(r+1)r!(n-\overline{r+1})!}$$
$$= \frac{n!}{r!(n-\overline{r+1})!}\left[\frac{1}{n-r} + \frac{1}{r+1}\right]$$
$$= \frac{n!}{r!(n-\overline{r+1})!}\left[\frac{r+1+n-r}{(n-r)(r+1)}\right]$$
$$= \frac{n!}{r!(n-\overline{r+1})!}\left[\frac{n+1}{(n-r)(r+1)}\right]$$
$$= \frac{(n+1)\,n!}{(r+1)\,r!\,(n-r)(n-\overline{r+1})!}$$
$$= \frac{(n+1)!}{(r+1)!\,(n-r)!}$$
$$= \frac{(n+1)!}{(r+1)!\,(n+1-\overline{r+1})!} = {}^{n+1}C_{r+1}$$

d) $2x^3 - 6x^2 + x + 3 = 0$

α, β, γ are roots.

$\alpha+\beta+\gamma = \frac{-b}{a}$
$= \frac{--6}{2}$
$= 3$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$
$= \frac{1}{2}$

$\alpha\beta\gamma = \frac{-d}{a}$
$= \frac{-3}{2}$

$\therefore \alpha\beta + \alpha\gamma = \frac{1}{2} - \beta\gamma$

$2(\alpha\beta+\alpha\gamma) = 2\left(\frac{1}{2} - \beta\gamma\right)$

$2(\alpha\beta+\alpha\gamma) = 1 - 2\beta\gamma$

Question 5

a) To prove that $5^n - 1$ is divisible by 4, $n \geq 1$

ie $5^n - 1 = 4A$, where $A \in$ Integers

For $n = 1$, $5^n - 1 = 5^1 - 1$

$= 4 = 4 \times 1$ where $1 \in$ Integers

$\therefore$ True for $n = 1$

Assume true for some integer $n = k$

ie $5^k - 1 = 4B$, $B \in$ Integers

Prove true for $n = k+1$,

$$5(5^k - 1) = 4B \times 5$$

$$5^1 \times 5^k - 5 \times 1 = 20B$$

$$5^{k+1} - 5 = 20B$$

$$5^{k+1} - 1 - 4 = 20B$$

$$5^{k+1} - 1 = 20B + 4$$

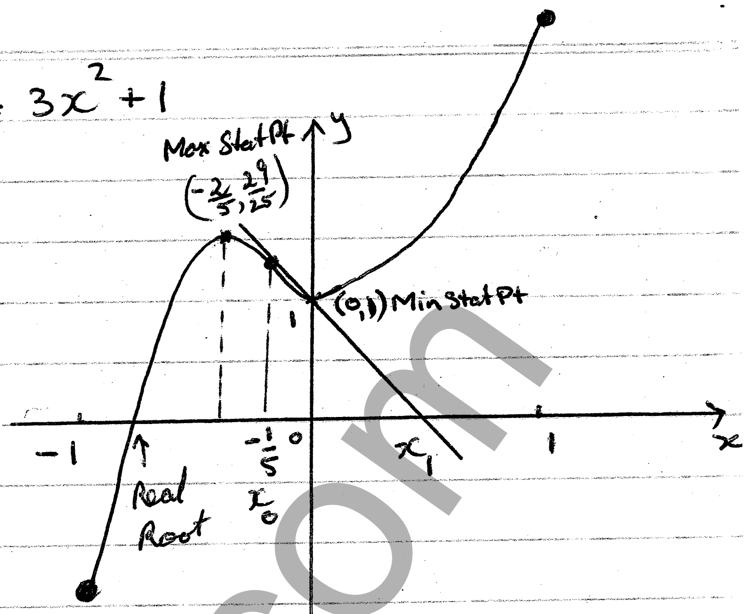
$$5^{k+1} - 1 = 4(5B+1)$$

$= 4C$, where $C \in$ Integers

Thus true for all $n \geq 1$

by the Principle of Mathematical Induction

b) $P(x) = 5x^3 + 3x^2 + 1$



(i) $P'(x) = 15x^2 + 6x$
Stationary points when $P'(x) = 0$

$3x(5x + 2) = 0$

$\begin{cases} x = 0 \\ y = 1 \end{cases}$ or $\begin{cases} x = -\frac{2}{5} \\ y = \frac{29}{25} \end{cases}$

(ii) let $x_0 = -\frac{1}{5}$ be 1st approximation for root.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = -\frac{1}{5} - \frac{5 \times \left(-\frac{1}{5}\right)^3 + 3\left(-\frac{1}{5}\right)^2 + 1}{15\left(-\frac{1}{5}\right)^2 + 6\left(-\frac{1}{5}\right)}$$

$$= -\frac{1}{5} - \frac{-9}{5}$$

$$= -\frac{1}{5} + \frac{9}{5}$$

$$= \frac{8}{5}$$

(iii) Since the 1st approximation was taken on the other side of a stationary point to the real root the gradient of the tangent $(f'(x))$ at the point when $x = -\frac{1}{5}$, is opposite to the direction of the tangent on the side where $x = x_0$. Thus the new approximation $x = x_1$ goes further from the real solution (see diagram)

c) (i) $\left(x-\frac{1}{x}\right)^{2n}$ has general term

$$T_{k+1} = {}^{2n}C_k \left(x\right)^{2n-k}\left(-\frac{1}{x}\right)^k$$

$$= {}^{2n}C_k \, x^{2n-k} \times (-1)^k \frac{1}{x^k}$$

$$= {}^{2n}C_k (-1)^k x^{2n-2k}$$

To be independent of x, $2n-2k=0$

$$2k = 2n$$

$$k = n$$

Thus required term is

$${}^{2n}C_n (-1)^n$$

(ii)

$$(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n} = \left[(1+x)\left(1-\frac{1}{x}\right)\right]^{2n}$$

$$= \left[1-\frac{1}{x}+x-1\right]^{2n}$$

$$= \left(x-\frac{1}{x}\right)^{2n}$$

(ii) Now term independent of x in $\left(x-\frac{1}{x}\right)^{2n}$ is $(-1)^n \, {}^{2n}C_n$

Alternatively,

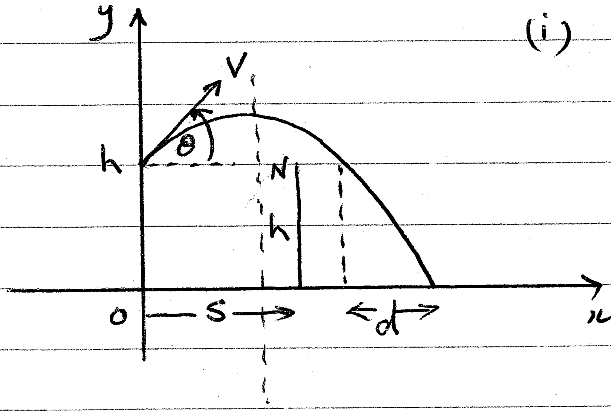
$$(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n} = \left[{}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots {}^{2n}C_r x^r + \dots {}^{2n}C_{2n} x^{2n}\right]\left[{}^{2n}C_0 + {}^{2n}C_1\left(-\frac{1}{x}\right) + {}^{2n}C_2\left(-\frac{1}{x}\right)^2 + {}^{2n}C_r\left(-\frac{1}{x}\right)^r + \dots {}^{2n}C_{2n}\right]$$

$$= \left({}^{2n}C_0\right)^2 + {}^{2n}C_1 \cdot {}^{2n}C_1(-1) + {}^{2n}C_2 \cdot {}^{2n}C_2 + {}^{2n}C_3 \, {}^{2n}C_3(-1) + \dots + {}^{2n}C_{2n} \cdot {}^{2n}C_{2n}$$

$$= \left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 - \left({}^{2n}C_3\right)^2 + \dots + \left({}^{2n}C_{2n}\right)^2$$

Question 6

a)



(i) Horizontal

$\ddot{x} = 0$

$\dot{x} = A$

when $t=0$, $\dot{x} = V\cos\theta$

$\therefore A = V\cos\theta$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta + B$

when $t=0$, $x=0$, $\therefore B=0$

$\therefore x = Vt\cos\theta$

Vertical

$\ddot{y} = -g$

$\dot{y} = -gt + C$

when $t=0$, $\dot{y} = V\sin\theta$

$\therefore C = V\sin\theta$

$\dot{y} = -gt + V\sin\theta$

$y = -\frac{gt^2}{2} + Vt\sin\theta + D$

when $t=0$, $y=h$, $D=h$

$\therefore y = -\frac{1}{2}gt^2 + Vt\sin\theta + h$

or $y = Vt\sin\theta - \frac{1}{2}gt^2 + h$

(ii) $\frac{x}{V\cos\theta} = t \Rightarrow y = V\sin\theta\left(\frac{x}{V\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + h$

$$y = h + \frac{xV\sin\theta}{V\cos\theta} - \frac{gx^2}{2V^2\cos^2\theta}$$

$$\text{ie } y = h + x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

(iii) For the ball to clear the net just

distance to the net $= 2 \times$ distance start to highest point

when $\dot{y} = 0$, $-gt + V\sin\theta = 0$

$$t = \frac{V\sin\theta}{g}$$

when $t = \frac{V\sin\theta}{g}$, $x = V\cos\theta\left(\frac{V\sin\theta}{g}\right)$

$$= \frac{V^2\sin\theta\cos\theta}{g}$$

Thus distance to net $\leq \frac{2V^2\sin\theta\cos\theta}{g}$

ie $S \leq \frac{V^2\sin 2\theta}{g}$, $\sin 2\theta = 2\sin\theta\cos\theta$

Thus $\frac{gS}{\sin 2\theta} \leq V^2$ OR $V^2 \geq \frac{gS}{\sin 2\theta}$

(iv) Ball hits ground when $y=0$ and $x=s+d$

ie $h+(s+d)\tan\theta - \dfrac{g(s+d)^2}{2V^2\cos^2\theta} = 0$

$$h+(s+d)\tan\theta = \frac{g(s+d)^2}{2V^2\cos^2\theta}$$

$$\frac{1}{h+(s+d)\tan\theta} = \frac{2V^2\cos^2\theta}{g(s+d)^2}$$

$$\frac{g(s+d)^2}{[h+(s+d)\tan\theta]2\cos^2\theta} = V^2$$

But $V^2 \geq \dfrac{Sg}{\sin 2\theta}$

$$\therefore \frac{g(s+d)^2}{[h+(s+d)\tan\theta]2\cos^2\theta} \geq \frac{Sg}{\sin 2\theta}$$

$$\frac{(s+d)^2}{[h+(s+d)\tan\theta]2\cos^2\theta} \geq \frac{S}{2\sin\theta\cos\theta}$$

$$\frac{(s+d)^2}{[h+(s+d)\tan\theta]\cos\theta} \geq \frac{S}{\sin\theta}$$

$$(s+d)^2\sin\theta \geq S\cos\theta[h+(s+d)\tan\theta] \quad \theta \text{ acute}$$

$$(s+d)^2\frac{\sin\theta}{\cos\theta} \geq S[h+(s+d)\tan\theta]$$

$$(s+d)^2\tan\theta \geq Sh + S(s+d)\tan\theta$$

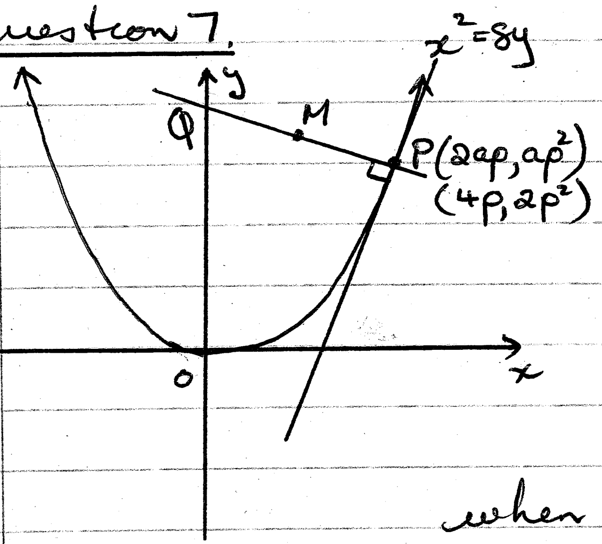
$$[(s+d)^2 - S(s+d)]\tan\theta \geq Sh$$

$$(s+d)[s+d-S]\tan\theta \geq Sh$$

$$(S+d)[d]\tan\theta \geq Sh$$

$$\therefore \tan\theta \geq \frac{Sh}{d(S+d)}$$

Question 7.

a) 

$\begin{cases} x^2 = 8y \\ x^2 = 4ay \end{cases}$ $\therefore$ $4a = 8$, $a = 2$

$P(4p, 2p^2)$

$$x^2 = 8y$$
$$y = \frac{1}{8}x^2$$
$$\frac{dy}{dx} = \frac{1}{4}x$$

when $x = 4p$, $\frac{dy}{dx} = \frac{1}{4} \times 4p$
$= p.$

$\therefore$ Gradient of normal is $-\frac{1}{p}$ – ①

$\therefore$ Equation of normal at P is

$$y - 2p^2 = -\frac{1}{p}(x - 4p)$$
$$py - 2p^3 = -x + 4p \quad – ①$$

$$x + py = 4p + 2p^3$$

Sub $x = 0$ to give ①

(ii) $M\left(\frac{4p+0}{2}, \frac{2p^2+4+2p^2}{2}\right)$ where $P(4p, 2p^2)$
$Q(0, 4+2p^2)$
– ①

ie $M(2p, 2p^2+2)$

(iii) For the locus of the midpt M

$x = 2p$ and $y = 2p^2 + 2$

$\frac{x}{2} = p$ $\quad y = 2\left(\frac{x}{2}\right)^2 + 2$ – ①

$y = \frac{x^2}{2} + 2$ – ①

ie $2y = x^2 + 4$

$2(y-2) = x^2$

Thus locus of M is a parabola with vertex $(0,2)$

b) (i)

$$V_{sand} = \pi \int_0^5 x^2 \, dy \qquad x = y^{1/3}, \quad x^2 = y^{2/3}$$

$$V_{sand} = \pi \int_0^5 y^{2/3} \, dy \quad - (1)$$

$$= \pi \left[\frac{y^{5/3}}{5/3} \right]_0^5$$

$$= \frac{3\pi}{5} \left[y^{5/3} \right]_0^5 \quad - (1)$$

$$= \frac{3\pi}{5} \left[5^{5/3} \right]$$

$$= 27.55821\ldots$$

$$\text{Volume} = 27.56 \text{ cm}^3$$

(ii) If $\frac{1}{4}$ of the sand is in the bottom half then $\frac{3}{4}$ of the sand is in the top half.

$$V_{Top} = \pi \int_0^{h_1} y^{2/3} \, dy$$

$$= \frac{3\pi}{5} \left[y^{5/3} \right]_0^{h_1}$$

$$= \frac{3\pi}{5} \left[h_1^{5/3} \right]$$

ie $\frac{3}{4}$ of $27.56 = \frac{3\pi}{5} h_1^{5/3}$ - (1)

$$\frac{27.56 \times 5}{4\pi} = h_1^{5/3}$$

$$\left(\frac{27.56 \times 5}{4\pi} \right)^{3/5} = h_1$$

$$h_1 = 4.2074\ldots \quad - (1)$$

$$= 4.2 \text{ cm (nearest mm)}$$

Since $\frac{3}{4}$ of the sand in the top half reaches a height of 4.2 cm then the $\frac{1}{4}$ sand in bottom half leave $\frac{3}{4}$ of bottom empty to a depth -4.2 cm. Thus

$$h_2 = |-5 - {}^-4.2|$$

$$= |0.8| \quad - (1)$$

ie h_2 is 0.8 cm (nearest mm)

$$\frac{3}{4} \times \frac{3\pi}{5} \left[5^{5/3} \right] = \frac{3\pi}{5} \left[h_1^{5/3} \right]$$

$$h_1 = 5 \times \left(\frac{3}{4} \right)^{3/5}$$

(iii) Rate of flow is

$\frac{dv}{dt} = -b$, since Volume in top half is decreasing over time.

$$\frac{dh_1}{dt} = \frac{dh_1}{dV} \cdot \frac{dV}{dt} \quad -(1)$$

$$= \frac{1}{\pi h_1^{2/3}} \times -b$$

where $V_{Top} = \frac{3\pi}{5} h_1^{5/3}$

(1)

$$\frac{dV}{dh_1} = \frac{5}{3} \times \frac{3\pi}{5} h_1^{2/3}$$

$$= \pi h_1^{2/3}$$

$$= \frac{-b}{\pi h_1^{2/3}}$$