



SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

**TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION**

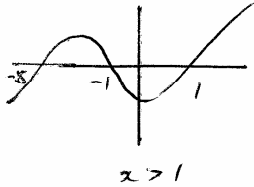
Mathematics Extension 1

Sample Solutions

Section	Marker
A	Mr Dunn
B	Ms Nesbitt
C	Mr Bigelow

Section A

1 a) $(x^2 - 1)(x + 5) > 0$



$x > 1$

AND $-5 < x < -1$ (2 marks)

b) $y = \ln \sqrt{x+1}$
 $= \frac{1}{2} \ln(x+1)$

$y' = \frac{1}{2(x+1)}$ (2 marks)

c) $\int_0^{\pi/6} \sec 2x \tan 2x \, dx$
 $= \left[\frac{1}{2} \sec 2x \right]_0^{\pi/6}$

$= \frac{1}{2} \sec \frac{\pi}{3} - \frac{1}{2} \sec 0$

$= \frac{1}{2} \times 2 - \frac{1}{2} \times 1$

$= \frac{1}{2}$ (2 marks)

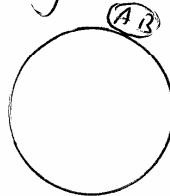
d) $\int_0^{\sqrt{3}} \frac{dx}{9+x^2} = \left[\frac{1}{3} \tan^{-1} \frac{x}{3} \right]_0^{\sqrt{3}}$
 $= \left[\frac{1}{3} \tan^{-1} \frac{\sqrt{3}}{3} \right]$

$= \frac{1}{3} \tan^{-1} \frac{1}{\sqrt{3}}$

$= \frac{1}{3} \times \frac{\pi}{6}$

$= \frac{\pi}{18}$ (2 marks)

e) Total number of arrangements = $7!$



If A and B are together

Then $2 \times 6!$

Hence not together

$= 7! - 2 \times 6!$

$= 6! (7-2)$

$= 5 \times 6!$

$= 3600$ (2 marks)

f) LHS = $\frac{1 - \cos \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta}$

$= \frac{1 - \cos^2 \theta + \sin^2 \theta}{\sin \theta (1 + \cos \theta)}$

$= \frac{2 \sin^2 \theta}{\sin \theta (1 + \cos \theta)}$

$= \frac{2 \sin \theta}{1 + \cos \theta}$

Let $t = \tan \frac{\theta}{2}$

$= 2 \times \frac{2t}{1+t^2}$

$= \frac{4t}{1+t^2}$

$= \frac{4t}{1+t^2+1-t^2} = \frac{4t}{2} = 2t$

$= 2 \tan \frac{\theta}{2} = \text{RHS}$ (2 marks)

c) $y = x^{-1} 2x$

let $x = 2z$

Then $y = n^{-1}u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}} \times 2$$

$$= \frac{2}{\sqrt{1-4x^2}} \quad (2 \text{ marks})$$

b) $y = 3 \sin^{-1} \sqrt{1-x^2}$

Consider $\sqrt{1-x^2}$

$$-1 \leq x \leq 1$$
 Range Domain

Then

$$y = 3 \sin^{-1} 0 \text{ to } 3 \sin^{-1} 1$$

ie $0 \leq y \leq \frac{3\pi}{2}$ range (2 marks)

c) $\sqrt{3} \cos x - \sin x = 2 \cos(x+d)$

$$= R \cos X \cos h - R \sin X \sin h$$

$$R_{\cos \lambda} = \sqrt{3}$$

$$2 \text{ m} \times 2 = 1$$

$$\tan \theta = \frac{1}{\sqrt{3}}$$

$$\lambda = \frac{\pi}{6}$$

$$R^2(\cos^2 \alpha + \sin^2 \alpha) = 3 + 1$$

$R = 2$

c) continued

$$2 \cos\left(x + \frac{\pi}{6}\right) = 1 \quad (2 \text{ marks})$$

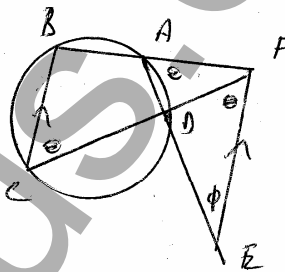
$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{6} = \pm \frac{\pi}{3}$$

$$x = 2k\pi + \frac{\pi}{3} - \frac{\pi}{2}$$

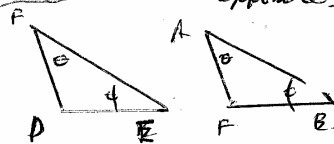
$$x = 2k\pi + \frac{\pi}{6} \quad (2 \text{ marks})$$

$$\text{or } 2k\pi - \frac{\pi}{6}$$



$$\angle FAE = \angle FBC \text{ (angle in alternate segment)}$$

$\hat{BCF} = \hat{CFE}$
(alternate opposite)



Hence $\triangle OBF \cong \triangle FBA$
(2 marks)

$$\frac{EF}{EA} = \frac{ED}{EF}$$

$$EF^2 = EA \times EO$$

(2 marks)

QUESTION THREE

- a) Prove $2^{3n} - 1$ is divisible
by 7 for $n > 1$ (integer)
Let $n=1$ Then $2^3 - 1 = 7$
is true for $n=1$

Assume

$$2^{3k} - 1 = 7K \text{ where } K \text{ is an integer}$$

Try to prove

$$2^{3k+3} - 1 = 7N \text{ where } N \text{ is an integer}$$

$$\text{LHS} = 2^3 \cdot 2^{3k} - 1$$

$$= 8(7K+1) - 1 \text{ from assumption.}$$

$$= 56K + 7$$

$$= 7(8K+1)$$

$$= 7N$$

True for $n=1$

$$n=1+1=2$$

$$n=2+1=3$$

All integers $n > 1$ (3 marks)

b) i) $y = 1 + 2\cos x - 2\cos^2 x$

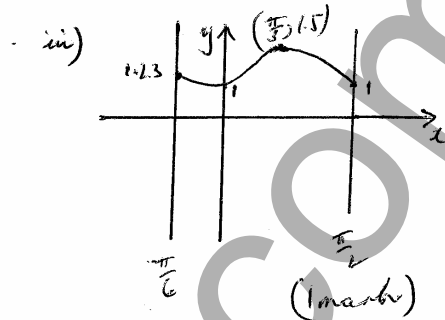
$$y' = -2\sin x + 4\cos x \sin x$$

$$= 2\sin x (2\cos x - 1) \text{ (1 mark)}$$

ii) $y' = 0$ when $\sin x = 0$
 $\cos x = 1/2$

$$\text{ie } x = 0, \pi/3$$

When $x=0, y=1$
 $x=\pi/3, y=3/2$ } 2 marks

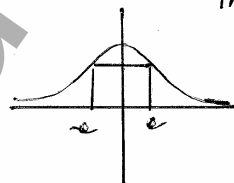


Max of 1.5 at $x = \pi/3$

Minimum of 1 at

$$x=0 \text{ or } x=\pi/2 \text{ 1 mark}$$

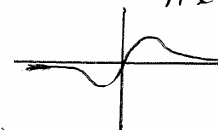
c)



$$\text{Area} = \int_{-a}^a y \, dx$$

$$= 2a \times \frac{1}{1+a^2} = \frac{2a}{1+a^2} \text{ (mark)}$$

Consider $y = \frac{2x}{1+x^2}$



$$y' = \frac{(x^2+1)2 - 2 \cdot 2x}{(1+x^2)^2}$$

$$= \frac{2-2x^2}{(1+x^2)^2}$$

$$y' = 0 \text{ when } x = \pm 1$$

THREE

c ii) continued.

$$\begin{aligned} \text{When } x &= 1+\epsilon & y' &< 0 \\ x &= 1-\epsilon & y' &> 0 \end{aligned}$$



Hence $x=1$ produces maximum

$$\text{Area} = \frac{2}{1+1} = 2 \text{ square units. (3 marks)}$$

or

$$y'' = \frac{(1+x^2)^2(-4x) - (2-2x^2)4x(1+x^2)}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)[4x^2 + (2-2x^2)]}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)(3-x^2)}{(1+x^2)^4}$$

$$\text{When } x=1 \quad y'' = \frac{-4 \times 2 \times 2}{2^4}$$

$y'' < 0$ Hence maximum.

QUESTION 4

3) $x = -2t, t = -\frac{x}{2}$

i) $y = \frac{1}{4}x^2$
 $y' = \frac{x}{2} = -t$

eqn of tangent $y - t^2 = -t(x + 2t)$

$y - t^2 + tx + 2t^2 = 0$

$tx + y + t^2 = 0$

ii) $tx + y + t^2 = 0$

at A, $y = 0$

$t(x + t) = 0$

$t(x + t) = 0, x = -t$

A. $(-t, 0)$ T. $(-2t, t^2)$

Midpoint M $(\frac{-t-2t}{2}, \frac{0+t^2}{2})$

$M = (-\frac{3t}{2}, \frac{t^2}{2})$

$x = -\frac{3t}{2}, t = -\frac{2x}{3}$

$y = \frac{t^2}{2}$

$= \frac{1}{2}(-\frac{2x}{3})^2$

$y = \frac{2x^2}{9}$

Locus of M $x^2 = \frac{9}{2}y$

$4x^3 - 12x^2 + 11x - 3 = 0$

roots $\alpha - d, \alpha, \alpha + d$ (arith series)

Sum of roots $= 3\alpha = -\frac{b}{a} = 3$

$\alpha = 3$

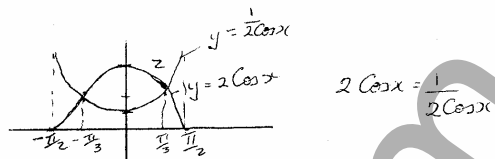
product $1(1-d) + 1(1+d) + (1-d)(1+d) = \frac{c}{a}$

$3 - d^2 = \frac{11}{4}$

$d^2 = \frac{1}{4}$

$d = \pm \frac{1}{2}$

roots $\frac{1}{2}, 1, \frac{3}{2}$.



$4 \cos^2 x = 1$

$\cos x = \frac{1}{2}$ or $\cos x = -\frac{1}{2}$

$x = -\frac{\pi}{3}, \frac{\pi}{3}$ or no soln in domain

$V = \pi \int_{-\pi/3}^{\pi/3} (4 \cos^2 x - \frac{1}{4} \sec^2 x) dx$

$2 \cos^2 x = \cos 2x + 1$

$V = 2\pi \int_0^{\pi/3} (2 \cos 2x + 2 - \frac{1}{4} \sec^2 x) dx$

$= 2\pi [\sin 2x + 2x - \frac{1}{4} \tan x]_0^{\pi/3}$

$= 2\pi (\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{4}) - 0$

$V = (\frac{4\pi^2}{3} + \frac{\sqrt{3}}{2}) \pi^3$

5) a) Find $\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}$

$\frac{dr}{dt} = -5 \text{ cm/s}$ $V = \frac{4}{3} \pi r^3$

$\frac{dv}{dr} = 4\pi r^2$

$r = 10 \text{ cm}$

$\frac{dv}{dt} = -5 \times 4 \times \pi \times 100$

$= -2000 \pi \text{ cm}^3/\text{s}$

(b) $x = 2 \cos(t + \frac{\pi}{6})$

$\dot{x} = -2 \sin(t + \frac{\pi}{6})$

$\ddot{x} = -2 \cos(t + \frac{\pi}{6})$

$\ddot{x} = -1^2 x$, in the form $-\ddot{x} = n^2 x, n=1$

$\therefore$ motion is SHM

(ii) Period $= \frac{2\pi}{n} = 2\pi$

(iii) $x = 2 \cos(t + \frac{\pi}{6}) = 0$

$t + \frac{\pi}{6} = \frac{\pi}{2} + 2n\pi$

$t = \frac{\pi}{3} \text{ sec (1st osc.)}$

(iv) $2 \cos(t + \frac{\pi}{6}) = 1$

$t + \frac{\pi}{6} = \frac{\pi}{3} + 2n\pi$

$t = \frac{\pi}{6} \text{ (1st osc.)}$

$\dot{x} = -2 \sin \frac{\pi}{3}$

$V = -2 \times \frac{\sqrt{3}}{2}$

$V = -\sqrt{3} \text{ cm/s}$

QUESTION 5(c)

$$\int \sqrt{16-x^2} \, dx$$

$$= \int \sqrt{16-16\sin^2\theta} \cdot 4\cos\theta \, d\theta$$

$$x = 4\sin\theta$$

$$\frac{dx}{d\theta} = 4\cos\theta$$

$$dx = 4\cos\theta \, d\theta$$

$$\int \sqrt{16\cos^2\theta} \cdot 4\cos\theta \, d\theta$$

$$\int 4\cos\theta \cdot 4\cos\theta \, d\theta$$

$$16 \int \cos^2\theta \, d\theta$$

$$8 \int (\cos 2\theta + 1) \, d\theta$$

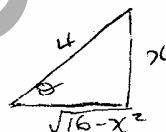
$$\cos 2\theta = 2\cos^2\theta - 1$$

$$2\cos^2\theta = \cos 2\theta + 1$$

$$8 \left(\frac{1}{2} \sin 2\theta + \theta \right)$$

$$4 \sin 2\theta + 8\theta + C$$

$$4 \cdot 2 \sin\theta \cos\theta + 8\theta$$



$$4 \times 2 \cdot \frac{x}{4} \frac{\sqrt{16-x^2}}{4} + 8 \sin^{-1} \frac{x}{4}$$

$$\theta = \sin^{-1} \frac{x}{4}$$

$$= \frac{x}{2} \sqrt{16-x^2} + 8 \sin^{-1} \frac{x}{4} + C$$

Question 6.

(a) If $y' = \frac{3x}{4+x^2}$

$$y = \frac{3}{2} \ln(4+x^2) + C.$$

(b) $P(x) = 8x^3 - 12x^2 + 6x + 13$

$$P'(x) = 24x^2 - 24x + 6 \\ = 6(2x-1)^2$$

(i) $P(x)$ is increasing where $P'(x) > 0$
i.e. $6(2x-1)^2 > 0$

$\therefore$ all Reals, except $x = \frac{1}{2}$.

(ii) Since $P(x) \rightarrow -\infty$ as $x \rightarrow -\infty$, $P(0) = 13$,
and $P(x)$ is increasing for all $x \neq \frac{1}{2}$,
it follows that there must be a
root x_1 , where $x_1 < 0$.

(iii) $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$

$$\text{if } a_1 = -1 \text{ then } a_2 = -1 - \frac{-8-12-6+13}{24+24+6} \\ = -1 - \frac{-13}{54} \\ = -\frac{41}{54} \\ = \boxed{-0.76} \text{ (2.D.P.)} \checkmark \checkmark$$

$$(c) (i) T = S + A e^{-kt} \quad \text{--- (A)}$$

$$\therefore \frac{dT}{dt} = -k A e^{-kt}$$

$$= -k(T-S) \text{ from (A) } \checkmark \checkmark$$

$$(ii) \text{ When } t=0, T=1390 \text{ and } S=30 \text{ (constant)}$$

$$\therefore 1390 = 30 + A e^0$$

$$\therefore A = 1360$$

$$\therefore T = 30 + 1360 e^{-kt}$$

$$\text{When } t=10, T=1060$$

$$\therefore 1060 = 30 + 1360 e^{-10k}$$

$$\frac{1030}{1360} = e^{-10k}$$

$$-10k = \ln \frac{103}{136}$$

$$k \doteq 0.0278$$

$$\therefore T = 30 + 1360 e^{-0.0278t}$$

$$\text{Let } T = 110$$

$$110 = 30 + 1360 e^{-0.0278t}$$

$$\frac{80}{1360} = e^{-0.0278t}$$

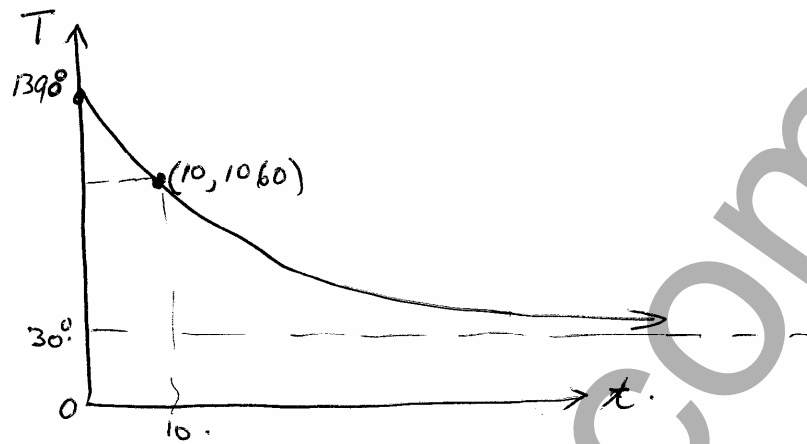
$$\ln \frac{8}{136} = -0.0278t$$

$$t = \frac{\ln \frac{1}{17}}{-0.0278}$$

$$\doteq 102 \text{ min}$$

$$\therefore \boxed{\text{it takes 92 mins longer.}}$$

(11)



Question 7.

(a) now

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n \quad \text{--- (A)}$$

(i) Differentiate both sides of (A) above -

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots + r\binom{n}{r}x^{r-1} + \dots + n\binom{n}{n}x^{n-1}.$$

Let $x=1$

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + r\binom{n}{r} + \dots + n\binom{n}{n}$$

$$\text{ie. } \boxed{\sum_{r=1}^n r\binom{n}{r} = n 2^{n-1}} \quad \checkmark \quad \left(\begin{array}{l} \text{NB This is} \\ \text{equivalent to} \\ \sum_{r=0}^n r\binom{n}{r} = n 2^{n-1} \end{array} \right)$$

$$(ii) \text{ R.T.P. } \sum_{r=0}^n (r+1)\binom{n}{r} = 2^{n-1}(n+2)$$

$$\text{LHS} = \sum_{r=0}^n r\binom{n}{r} + \sum_{r=0}^n \binom{n}{r}$$

$$= n 2^{n-1} + 2^n$$

$$= \boxed{2^{n-1}(n+2)} \quad \checkmark \checkmark$$

= R.H.S.

(if we let $x=1$ in (A))

$$2^n = \sum_{r=0}^n \binom{n}{r}$$

(b) (i) $x = vt \Rightarrow t = \frac{x}{v}$.

$\therefore y = -\frac{1}{2}gt^2 + h$ becomes

$y = -\frac{1}{2}g\left(\frac{x}{v}\right)^2 + h$

$y = h - \frac{1}{2}g\frac{x^2}{v^2}$ ✓

(ii) $x = vt \cos \alpha \Rightarrow t = \frac{x}{v \cos \alpha}$ $\therefore y = -\frac{1}{2}gt^2 + vt \sin \alpha + h$

becomes $y = -\frac{1}{2}g\left(\frac{x}{v \cos \alpha}\right)^2 + v\frac{x}{v \cos \alpha} \sin \alpha + h$

i.e. $y = x \tan \alpha - \frac{gx^2 \sec^2 \alpha}{2v^2} + h$ ✓

(iii) Substitute $(d, 0)$ in (i) $0 = h - \frac{gd^2}{2v^2}$

$\therefore h = \frac{gd^2}{2v^2}$ ✓

(iv) Substitute $(d, 0)$ in (ii)

$0 = d \tan \alpha - \frac{gd^2 \sec^2 \alpha}{2v^2} + h$

$0 = d \tan \alpha - h(1 + \tan^2 \alpha) + h$ ($h = \frac{gd^2}{2v^2}$)

$\therefore h \tan^2 \alpha - d \tan \alpha = 0$

$\tan \alpha (h \tan \alpha - d) = 0$

$\therefore \tan \alpha = 0$ or $\tan \alpha = \frac{d}{h}$

Clearly $\tan \alpha \neq 0 \therefore \tan \alpha = \frac{d}{h}$ ✓✓

(v). Substitute $(2d, 0)$ into (ii).

$$2d \tan \alpha - \frac{g \cdot 4d^2}{2v^2} \sec^2 \alpha + h = 0.$$

$$2d \tan \alpha - 4h \sec^2 \alpha + h = 0.$$

$$2d \tan \alpha - 4h(1 + \tan^2 \alpha) + h = 0$$

$$2d \tan \alpha - 4h - 4h \tan^2 \alpha + h = 0$$

$$4h \tan^2 \alpha - 2d \tan \alpha + 3h = 0$$

$$\text{For } \tan \alpha \text{ to be real } 4d^2 - 4 \times 4h \times 3h \geq 0.$$

$$\text{ie } 4d^2 - 48h^2 \geq 0.$$

$$4d^2 \geq 48h^2$$

$$d^2 \geq 12h^2$$

$$\boxed{d \geq 2h\sqrt{3}}$$

✓✓✓