SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

Sample Solutions

Section	Marker
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B	Mr Hespe
C	Mr Choy
D	Mr Fuller
E	Mr Gainford

QUESTION 1.

(a) $\log(\tan \frac{5}{11}) = \boxed{1 \cdot 32.}$

(b) $y = (5x)^{\frac{1}{2}}$

$y' = \frac{1}{2}(5x)^{-\frac{1}{2}} \times 5$

$= \boxed{\frac{5}{2\sqrt{5x}}} \quad \left(\frac{\sqrt{5}}{2} x^{-\frac{1}{2}} \right)$

(c) $2t^2 - t - 15 = 0$

$(2t+5)(t-3) = 0$

$\boxed{t = 3, -\frac{5}{2}}$

(d) $y' = 3 - 2x$

$\boxed{y = 3x - x^2 + c}$

(e) $y = 2x$ — ①

$3x + 2y = 14$ — ②

Sub ① into ②

$3x + 4x = 14$

$7x = 14$

$x = 2$

Sub in ①

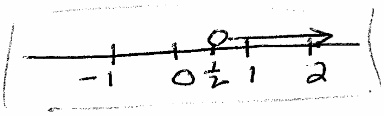
$y = 4$

$\therefore$ Soln. is $\boxed{(2, 4)}$

(f) $3 - 4x < 1$

$-4x < -2$

$\boxed{x > \frac{1}{2}}$



QUESTION 2.

(a)(i) $f(x) = (1+\cos 2x)^3$

$\therefore f'(x) = 3(1+\cos 2x)^2 \times -2\sin 2x$.

$= \boxed{-6\sin 2x(1+\cos 2x)^2}$. ✓✓

(ii) $f(x) = x^2 e^{x+2}$

$f'(x) = x^2 e^{x+2} + 2x e^{x+2}$

$= \boxed{x e^{x+2}(x+2)}$ ✓✓

(b)(i) $\int \frac{\cos x}{\sin x}\, dx = \boxed{\ln(\sin x) + C}$ ✓

(ii) $\int_{\frac{1}{2}}^{2} \left(1 - \frac{1}{x^2}\right) dx = \int_{\frac{1}{2}}^{2} (1 - x^{-2})\, dx$

$= \left[x + x^{-1}\right]_{\frac{1}{2}}^{2}$

$= (2 + \frac{1}{2}) - (\frac{1}{2} + 2)$.

$\boxed{= 0}$ ✓✓

(c)(i) $N\hat{B}Y = M\hat{X}C$ (corresponding angles).

$\hat{Y}$ is common

$\therefore$ $\triangle NBY \parallel\!\mid \triangle MXY$ (equiangular). ✓✓

$\therefore \boxed{\frac{MX}{NB} = \frac{1}{2}}$ (ratio of corresponding sides equal)

$\left[\frac{NB}{}\ \frac{MY}{NY} = \frac{1}{2}\right.$ (M is the midpt of NY)$\left.\right]$

(ii) $M\hat{C}X = A\hat{B}C$ (base angles of an isosceles triangle)

$M\hat{X}C = A\hat{B}C$ (corresponding angles). ✓

$\therefore M\hat{C}X = M\hat{X}C$

$\therefore \triangle XMC$ is isosceles $\therefore MX = MC$ $\therefore \boxed{\frac{MX}{NB} = \frac{MC}{NB} = \frac{1}{2}}$

(d) $g'(x) = 4 - 3x^2$

$\therefore g(x) = 4x - x^3 + c.$

now $4 = 4\times 2 - 8 + c.$

$\therefore c = 4$

$\therefore \boxed{g(x) = 4x - x^3 + 4}$ ✓✓

3(i) $m_{AD} = \dfrac{-3-5}{0-{}^-5}$

$= -8/5$ ✓

$\theta = \tan^{-1}(^-8/5)$

$= 180° - 58°$

$= 122°$ ✓

(ii) $M = \left(\dfrac{-5+3}{2}, \dfrac{1+5}{2}\right)$

$= (-1, 3)$ ✓

(iii) $C = (3 + (-5-0),\ 1 + (5 - {}^-3))$

$= (-2, 9)$ ✓

(iv) $m_{AB} = \dfrac{1 - {}^-3}{3-0}$

$= 4/3$ ✓

$\therefore$ Eqn of AB is $y = \dfrac{4x}{3} - 3$ ✓

or $4x - 3y - 9 = 0.$

(v) Distance $= \dfrac{|4\times(-5) - 3\times 5 - 9|}{\sqrt{4^2+3^2}}$

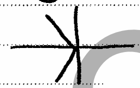
$= \dfrac{44}{5}$ ✓ (or 8.8).

3 (vi) Length $AB = \sqrt{9+16}$
$= 5$ ✓

$\therefore$ Area $= \frac{44}{5} \times 5$
$= 44$ ✓

(vii) $x + 2y < 5$ ✓ $\cap$ $x > 0$ ✓ $\cap$ $4x - 3y < 9$ ✓

4(a) $2x° = 120°, 240°, 480°, 600°$ ✓
$x° = 60°, 120°, 240°, 300°$

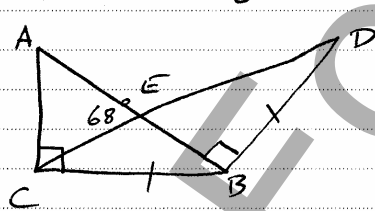


(b)(i) $\sqrt{x-5} = \dfrac{x-5}{5}$

$25x - 125 = x^2 - 10x + 25$
$x^2 - 35x + 150 = 0$
$(x-5)(x-30) = 0$ ✓
$x = 5, 30$
So P is $(30, 5)$ ✓

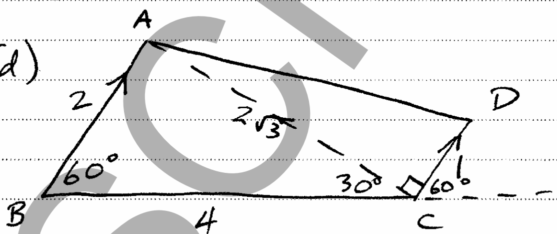
(ii) Area $= \int_5^{30} \left\{ (x-5)^{1/2} - \left(\dfrac{x-5}{5}\right) \right\} dx$ ✓
$= \left[\dfrac{2(x-5)^{3/2}}{3} - \dfrac{x^2}{10} + x \right]_5^{30}$ ✓
$= \frac{2}{3}(125 - 0) - \left(\frac{900}{10} - \frac{25}{10}\right) + 30 - 5$
$= \dfrac{125}{6}$ ✓ or $20\frac{5}{6}$ or $20.8\dot{3}$

(c)



$B\hat{E}D = 68°$ (vert. opp. $\angle$s) ✓
$B\hat{D}E = 22°$ (angle sum of Δ)
$D\hat{C}B = 22°$ ✓ (base angle of isosceles Δ).

(d)



(i) $AC^2 = 4^2 + 2^2 - 2 \times 4 \times 2 \times \cos 60°$ m^2
$= 16 + 4 - 16 \times \frac{1}{2}$ m^2
$= 12$ m^2
$AC = 2\sqrt{3}$ m. ✓ (3.464101615 calc).

4(d)(ii) $\dfrac{\sin B\hat{C}A}{2} = \dfrac{\sin 60^\circ}{2\sqrt{3}}$

$\sin B\hat{C}A = 1/2$

$B\hat{C}A = 30^\circ$ ✓

$A\hat{C}D = 180^\circ - 60^\circ - 30^\circ$

$= 90^\circ$ ✓

$\therefore AD^2 = 12 + 1$

$\therefore AD = \sqrt{13}$ ✓ (3.605551275 calculator)

$= 3.61$ m (3 sig. fig.)

5

$y = x^3 - 5x^2 + 7x - 14$

(a) (i) $\frac{dy}{dx} = 3x^2 - 10x + 7$ [1]

1

$= (3x-7)(x-1)$

(ii) $\frac{dy}{dx} = 0, \quad x = \frac{7}{3}, 1$

When $x = 1, \quad y = -11$ [1]

$x = \frac{7}{3}, \quad y = -12\frac{5}{27}$ (-12.185) [1]

3

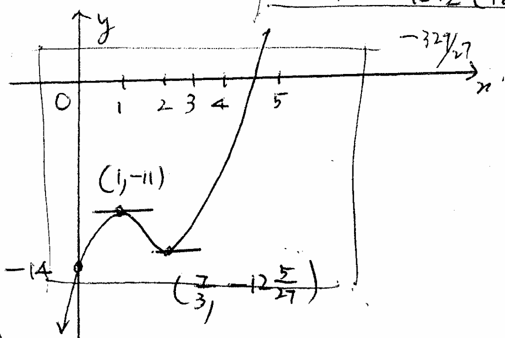
$\frac{d^2y}{dx^2} = 6x - 10$

$f''(1) = -4 < 0$ $\quad (1, -11)$ max [1]

$f''(\frac{7}{3}) = 4 > 0$ $\quad (\frac{7}{3}, -12\frac{5}{27})$ min $\quad -12.2$ (1dp)

(iii)

2



1

(iv) $f''(x) < 0, \quad 6x - 10 < 0, \quad 6x < 10$

$x < 5/3$.

(b) $a, a+d, a+2d, \ldots \ldots a+39d$.

$S_{40} = \frac{40}{2}[2a + 39d]$

$\therefore 495 = 20(2a + 39d)$.

$99 = 4(2a + 39d)$. —— 1

$a + 39d = 2a$ —— 2

$a = 39d \quad \therefore d = \frac{a}{39}$ —— 3

Subst 3 into 1

$99 = 4(2a + 39 \times \frac{a}{39})$

$\therefore 99 = 4(3a) \Longrightarrow$ $a = \frac{33}{4}$ (8.25) 3

$\Longrightarrow d = \frac{11}{52}$ cm (0.212 cm). 2

$\frac{495}{2340}$

Q(6). (a) $l = 10\theta$ $\quad \theta = 1.2^c$

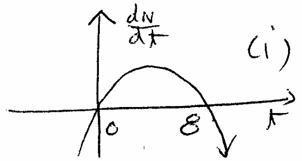
$\therefore 12 = 10\theta$

$1^c = \frac{180}{\pi}$ $\therefore 1.2^c = \boxed{\frac{180 \times 1.2}{\pi} \doteqdot 69^\circ}$ 2

(b) (i) $\boxed{r = \cos^2 x \ (<1).}$ 1

(ii) $S = \frac{\cos^2 x}{1 - \cos^2 x} = \frac{\cos^2 x}{\sin^2 x} = \boxed{\cot^2 x.}$ 2

(c). $\frac{dN}{dt} = 450t(8-t).$



(i) $\frac{dN}{dt}$ (max) $\boxed{\text{When } t=4}$

$= 450 \times 4 \times 4$

$= 7200$ 1

$\frac{dN}{dt} = 3600t - 450t^2$

$N = 1800t^2 - 150t^3 + c$ ✓

When $t=0$, $N=0$, $\Rightarrow c = 0$

(ii) $\therefore \boxed{N = 150t^2(12-t).}$ 2

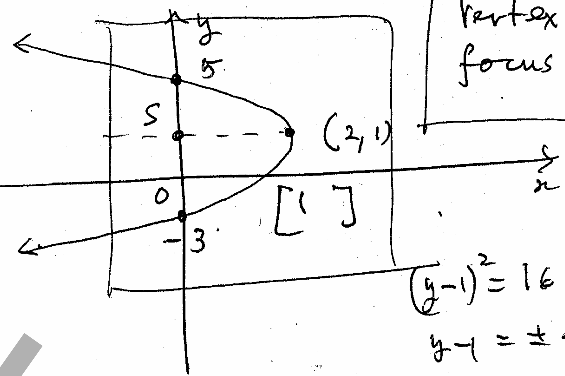
(iii) $N = 0$, $\boxed{\text{When } t=0, \text{ or } 12.}$ 1

$\therefore$ festival last for 12 hrs.

(d) $(y-1)^2 = -8(x-2)$

$4a = 8$, $a = 2$.

$\boxed{\begin{array}{l} \text{vertex } (2,1). \ [1] \\ \text{focus } (0,1) \ [1] \end{array}}$



$(y-1)^2 = 16$

$y - 1 = \pm 4$

SECTION D

Question 7

(a) when $x=0$ $\quad x=1$

$\quad y=3 \quad y=4$

$$V=\pi r^2h-\pi\int_3^4 x^2dy$$

$$=\pi(1)^2(4)-\pi\int_3^4(y-3)dy$$

$$=4\pi-\pi\left[\tfrac{1}{2}y^2-3y\right]_3^4$$

$$=4\pi-\pi\left\{\left[\tfrac{1}{2}(4)^2-3(4)\right]-\left[\tfrac{1}{2}(3)^2-3(3)\right]\right\}$$

$$=4\pi-\frac{\pi}{2}$$

$$=\frac{7\pi}{2}\text{units}^3$$

(b)(i) when $t=t_1,\ M=M_1,$ ie $\quad M_1=5e^{-0\cdot1t_1}$

when $t=t_2,\ M=\tfrac{1}{2}M_1,$ ie $\quad \tfrac{1}{2}M_1=5e^{-0\cdot1t_2}$

$\quad M_1=10e^{-0\cdot1t_2}$

$$5e^{-0\cdot1t_1}=10e^{-0\cdot1t_2}$$
$$e^{-0\cdot1t_1}=2e^{-0\cdot1t_2}$$
$$-0\cdot1t_1=\ln\left(2e^{-0\cdot1t_2}\right)$$
$$-0\cdot1t_1=\ln(2)+\ln\left(e^{-0\cdot1t_2}\right)$$
$$-0\cdot1t_1=\ln(2)+-0\cdot1t_2$$
$$t_1=-10\ln(2)+t_2$$
$$t_2-t_1=10\ln(2)$$

(ii) $\dfrac{5}{32}=5e^{-0\cdot1t}$

$$e^{-0\cdot1t}=\frac{1}{32}$$
$$-0\cdot1t=\ln\left(\tfrac{1}{32}\right)$$
$$t=-10\ln\left(\tfrac{1}{32}\right)$$

$t=34\cdot657$ correct to 3 decimal places

(c)(i) P(same letter twice)= $1\times\dfrac{1}{5}$

(ii) P(E at least once) = 1 − P(no E)

$$=1-\left(\tfrac{4}{5}\right)^n$$

P(E at least once) $\geq\dfrac{99}{100}$

$$1-\left(\frac{4}{5}\right)^n\geq\frac{99}{100}$$
$$\left(\frac{4}{5}\right)^n\leq\frac{1}{100}$$
$$n\ln\left(\frac{4}{5}\right)\leq\ln\left(\frac{1}{100}\right)$$
$$n\geq\frac{\ln\left(\frac{1}{100}\right)}{\ln\left(\frac{4}{5}\right)}$$
$$n\geq20\cdot6377....$$
$$\therefore n=21$$

Question 8

(a) $v=20t^2(3-t)$ $\qquad v=60t^2-20t^3$

(i) $a=\dfrac{dv}{dt}$

$$a=\frac{d\left(60t^2-20t^3\right)}{dt}$$
$$a=120t-60t^2$$

when $a=2,\ \ a=120(2)-60(2)^2$

$\quad a=0$

(ii) $x=\displaystyle\int\frac{dx}{dt}dt$

$$x=20t^3-5t^4+C$$

Note: $t=0,x=0\ \ \therefore C=0$

Hence, $x=20t^3-5t^4$

(iii) Let $v=0$.

$$0=20t^2(3-t)$$

$t=0$ or $t=3$

when $t=3$, $\quad x=20(3)^3-5(3)^4$

$$x = 135 \text{ km}$$

Therefore the distance travelled from station to station is 135km.

(iv) train is travelling the fastest when $a = 0$

this occurs when $t = 2$ (min) as found in (i).

when $t = 2,\ x = 20(2)^3 - 5(2)^4$

$x = 80\,\text{km}$ from Olympic Park.

(b)(i) $\displaystyle\int_0^1 \frac{dx}{1+x} = \left[\ln(1+x)\right]_0^1$

$$= \ln(1+(1)) - \ln(1+(0))$$
$$= \ln(2) - \ln(1)$$
$$= \ln(2)$$

(ii)

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

5 Function values means Simpsons Rule using 2 applications

$$\int_0^1 \frac{dx}{1+x} = \int_0^{0\cdot 5} \frac{dx}{1+x} + \int_{0\cdot 5}^1 \frac{dx}{1+x}$$

$$\approx \frac{\frac{1}{2}-0}{6}\left[1 + 4\left(\tfrac{4}{5}\right) + \tfrac{2}{3}\right] + \frac{1-\frac{1}{2}}{6}\left[\tfrac{2}{3} + 4\left(\tfrac{4}{7}\right) + \tfrac{1}{2}\right]$$
$$= \frac{1747}{2520} \quad \text{or} \quad 0\cdot 693$$

(c)

On 18th B'day deposits $500,

matures to $500(1\cdot 04)^{22}$

On 19th B'day deposits $500,

matures to $500(1\cdot 04)^{21}$

On 20th B'day deposits $500,

matures to $500(1\cdot 04)^{20}$

↓

↓

On 39th B'day deposits $500,

matures to $500(1\cdot 04)$

Yddap transferrs

$$500(1\cdot 04)^1 + 500(1\cdot 04)^2 + \ldots + 500(1\cdot 04)^{22}$$

$$a = 500(1\cdot 04)$$
$$r = 1\cdot 04$$
$$n = 22$$

$$S_n = \frac{a(r^n - 1)}{r-1}$$
$$S_{22} = \frac{500 \times 1\cdot 04(1\cdot 04^{22} - 1)}{1\cdot 04 - 1}$$

$$S_{22} = \$17808\cdot 94$$

SECTION E

Question 9

(a) (i) (α) At rest $t=0$
$t=5$
$t=9$ [1]

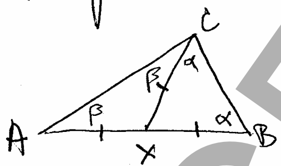
(β) $0<t<5$ [1]

(γ) $0<t<2$, $7<t<9$ [1]

* (δ) $2<t<5$, $7<t<9$ [1]

(ii) No, it has not returned. The area under the curve is the measure of distance travelled. The negative area ([5,9]) is less than the positive area ([0,5]). [2]

(b)



Aim To prove $\angle ACB = 90°$

Proof Let $\angle ABC = \alpha$
$\angle BAC = \beta$

Now $\angle BCX = \alpha$ (isos Δ)
and $\angle ACX = \beta$ (")

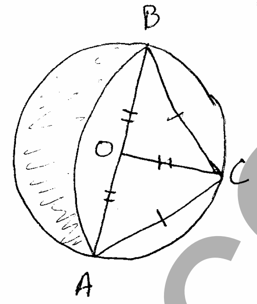
But $\angle BAC + \angle ACB + \angle CBA = 180°$ (angle sum)

$\therefore \beta + (\beta+\alpha) + \alpha = 180°$
$2(\alpha+\beta) = 180°$

$\therefore \alpha+\beta = 90°$
$\therefore \angle ACB = 90°$ [2]

QED

(ii)



(α) In ΔABC, $AO = BO = CO$ (radii)

$\therefore \angle ACB = 90°$

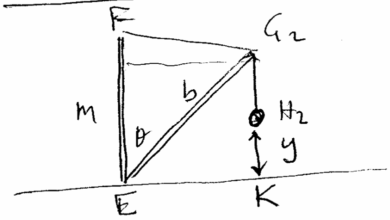
$\therefore$ Since $AO = r$, in ΔABC, using Pythagoras

$AB^2 = 2AC^2$
$(2r)^2 = 2AC^2$
$4r^2 = 2AC^2$
$AC^2 = 2r^2$
$\therefore AC = r\sqrt{2}$ [1]

(β) $\Delta_{ABC} = \frac{1}{2}(r\sqrt{2})^2$
$= r^2$

$A_{shaded} = \frac{1}{2}\pi r^2 - A_{segment}$
$= \frac{1}{2}\pi r^2 - \frac{1}{2}(r\sqrt{2})^2\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$
$= \frac{1}{2}\pi r^2 - r^2\left(\frac{\pi}{2} - 1\right)$
$= r^2$ [3]
$= \Delta_{ABC}$ QED

Question 10



(i) $R = FG_2 + G_2K - y$

$\therefore y = FG_2 + G_2K - R$

$= \sqrt{b^2+m^2-2bm\cos\theta} + b\cos\theta - R$

as required. [2]

(ii) $\dfrac{dy}{d\theta} = -b\sin\theta + \dfrac{1 \times \cancel{2}bm\sin\theta}{\cancel{2}\sqrt{b^2+m^2-2bm\cos\theta}}$

$= \dfrac{bm\sin\theta}{\sqrt{b^2+m^2-2bm\cos\theta}} - b\sin\theta$ [3]

(iii) When $\dfrac{dy}{d\theta} = 0$

$0 = \dfrac{bm\sin\theta - b\sin\theta\sqrt{b^2+m^2-2bm\cos\theta}}{\sqrt{b^2+m^2-2bm\cos\theta}}$

ie $0 = b\sin\theta\left(m - \sqrt{b^2+m^2-2bm\cos\theta}\right)$

$\therefore$ Either $\sin\theta = 0$ ie $\theta = 0$

or $m - \sqrt{b^2+m^2-2bm\cos\theta} = 0$

ie $m = \sqrt{b^2+m^2-2bm\cos\theta}$

That is $b^2 = 2bm\cos\theta$

ie $\cos\theta = \dfrac{b}{2m}$ [4]

(iv) [Clearly y is min when $\theta = 0$

$\therefore$ y is max when $\cos\theta = \dfrac{b}{2m}$]

Now lifting radius is $b\sin\theta$

$= b\sqrt{1-\cos^2\theta}$

$= b\sqrt{1-\dfrac{b^2}{4m^2}}$

$= \dfrac{b}{2m}\sqrt{4m^2-b^2}$

[3]