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NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 49/67

by James Coroneos*

1. If ω is a complex cube root of unity, show that the other complex cube root is ω^2 and prove that $1+\omega+\omega^2=0$. Show that
(a) $(1-3\omega+\omega^2)(1+\omega-8\omega^2)=36$ **(b)** $(7+9\omega^4+7\omega^{-1})^6=64$

2. If $1, \omega, \omega^2$ are the 3 cube roots of unity, prove that
(a) $(1+\omega^2)^3=-1$ **(b)** $(1+\omega)^5=-\omega$ **(c)** $(\omega^2+\omega)^{-20}=1$
(d) $1+\frac{1}{\omega}+\frac{1}{\omega^2}=0$ **(e)** $\frac{1}{1+\omega}+\frac{1}{1+\omega^2}=1$ **(f)** $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}=\omega$.

3. If ω is a cube root of unity, show that
(a) $(1-\omega-\omega^2)(1-\omega+\omega^2)(1+\omega-\omega^2)=8$ **(b)** $(1+2\omega+3\omega^2)(1+2\omega^2+3\omega)=3$
(c) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$ **(d)** $(1+\omega)(1+2\omega)(1+3\omega)(1+5\omega)=21$

4. **(i)** Find the sum $1+\omega+\omega^2+\omega^3+\cdots$ to n terms, considering the cases $n=3k, n=3k+1, n=3k+2$ where k is an integer.
(ii) Show that $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)\cdots$ to $2n$ factors $=2^{2n}$
(iii) Prove that $\frac{1}{x-1}+\frac{\omega}{x-\omega}+\frac{\omega^2}{x-\omega^2}=\frac{3}{x^3-1}$

5. **(i)** If $x=a+b$, $y=a+b\omega$, $z=a+b\omega^2$, show that
(a) $x+y+z=3a$ **(b)** $xyz=a^3+b^3$ **(c)** $xy+yz+zx=3a^2$
(ii) If $p=a+b$, $q=a\omega+b\omega^2$, $r=a\omega^2+b\omega$, prove that
(a) $pqr=a^3+b^3$ **(b)** $p^2+q^2+r^2=6ab$ **(c)** $p^3+q^3+r^3=3(a^3+b^3)$

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6. **(i)** Show that $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) = a^3+b^3+c^3-3abc$
(ii) Prove that $L = (1+a\omega+a^2\omega^2)(1+a\omega^2+a^2\omega^4)$ is independent of ω, and hence show that $(1+a+a^2)L = (1-a^3)^2$

7. Form the quadratic equation with root
(a) $3+\omega, 3+\omega^2$ **(b)** $a+b\omega, a+b\omega^2$ **(c)** $a+b\omega+c\omega^2, a+b\omega^2+c\omega$

8. Form the cubic equation with roots
(a) $-1, 1+\omega, 1+\omega^2$ **(b)** $p+q, p\omega+q, p\omega^2+q$

9. If $x = a+b, y = \omega a+\omega^2 b, z = \omega^2 a+\omega b$, show that $x+y+z = 0$ and $xy+yz+zx = -3ab$. Find the values of $x^2+y^2+z^2$ and $z^4+y^4+z^4$, and show that $2(x^4+y^4+z^4) = (x^2+y^2+z^2)^2$.

10. If $\omega^3 = 1, \omega \neq 1$, show that $\frac{1+\omega^n+\omega^{2n}}{3} = 1$ or 0, according to whether n is or is not a multiple of 3. If $c_0, c_1, c_2, \ldots, c_n$ are the coefficients in the expansion of $(1+x)^n$ in ascending powers of x, show by successive substitution of $1, \omega, \omega^2$ that $c_0+c_3+c_6+\cdots = \frac{2^n-(-1)^n}{3}$ or $\frac{2^n+2(-1)^n}{3}$, according to the form of n, and state the rule which determines which of these two forms must be taken.

11. When n is a positive integer, we know that $(x^2+x+1)^n = p_0+p_1+p_2x^2+\cdots+p_{2n}x^{2n}$ where $p_0, p_1, p_2, \ldots, p_{2n}$ are certain constants. In this identity, put for x the values $1, \omega, \omega^2$ where ω is one of the complex cube roots of unity, and add the three equations thus obtained. Hence show that $p_0+p_3+p_6+p_9+\cdots = 3^{n-1}$. State and prove a corresponding theorem regarding $p_0+p_5+p_{10}+\cdots$ in the identity $(x^4+x^3+x^2+x+1)^n = p_0+p_1x+p_2x^2+\cdots+p_{4n}x^{4n}$.

12. The equation $ax^4+bx^3+cx^2+dx+e = 0$ has ω and ω^2 as two of its roots, ω being a complex cube root of unity. Prove that the other two roots are those of the quadratic equation $ax^2+(b-a)x+e = 0$, and deduce, or prove otherwise, that $c = a+d = b+e$.