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NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 9/67

by James Coroneos*

1. Find expressions in $a+ib$ form for
(i) $e^{i\pi/6}$; $e^{-i\pi/4}$; $e^{i\pi/2}$; $e^{i\pi}$; $e^{-\frac{2\pi}{3}i}$; $e^{2\pi i}$; $e^{i3\pi/2}$
(ii) $e^{i\alpha}+e^{-i\alpha}$; $e^{1+i\pi/2}$; $e^{3-i\pi/4}$; $e^{3i\theta}-e^{-3i\theta}$; $e^{-i5\pi/6}+e^{i\pi/3}$

2. If $z_1=4e^{i\pi/2}, z_2=3e^{-i\pi/3}$ find **(i)** z_1z_2 **(ii)** z_1/z_2 **(iii)** z_1+z_2 **(iv)** z_1-z_2 **(v)** $\overline{z}_1+\overline{z}_2$

3. **(i)** Assuming $e^{i\theta}=\cos\theta+i\sin\theta$, find *two* expressions for $e^{2i\theta}$, and *hence* prove that $\cos 2\theta=\cos^2\theta-\sin^2\theta$ and $\sin 2\theta=2\sin\theta\cos\theta$.
(ii) Repeat **(i)**, with $e^{3i\theta}$, and hence obtain the results $\cos 3\theta=4\cos^3\theta-3\cos\theta$ and $\sin 3\theta=3\sin\theta-4\sin^3\theta$.
(iii) Obtain a result for $\cos 5\theta$ in terms of $\cos\theta$.

4. If $z=\cos\theta+i\sin\theta=e^{i\theta}$, simplify **(i)** $\frac{1}{z}$ **(ii)** $z+\frac{1}{z}$ **(iii)** $z^2+\frac{1}{z^2}$ **(iv)** $z^3+\frac{1}{z^3}$

5. **(i)** If $(1+3i)/(1-2i)=re^{i\theta}$, determine r and θ ($r>0$; $-\pi<\theta\le\pi$).
(ii) Find $\Re(z)$, $\Im(z)$, $|z|$, $\arg z$, $\overline{z}$ if $z=\frac{1+e^{i\theta}}{e^{i\theta/2}}$, where $0<\theta<\pi/2$.

6. Use the result for $e^{i\theta}$, to prove that
(a) $|z_1z_2|=|z_1||z_2|$ and $\arg(z_1z_2)=\arg z_1+\arg z_2$
(b) $|\frac{z_1z_2z_3}{z_4z_5}|=\frac{|z_1||z_2||z_3|}{|z_4||z_5|}$ and
$\arg(\frac{z_1z_2z_3}{z_4z_5})=\arg z_1+\arg z_2+\arg z_3-\arg z_4-\arg z_5$
(c) $|z^n|=|z|^n$ and $\arg(z^n)=n\arg(z)$.
(d) $(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3)$
$=\cos(\theta_1+\theta_2+\theta_3)+i\sin(\theta_1+\theta_2+\theta_3)$

*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{AMS}$-TeX.

7. Use the *principle of mathematical induction* to prove De Moivre's theorem, i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for positive integral n.

8. **(i)** Show that $1 - i\sqrt{3} = 2[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})]$, and hence that $(1 - i\sqrt{3})^9 = 2^9[\cos(-3\pi) + i\sin(-3\pi)]$ using De Moivre's theorem,

$$= 2^9[\cos\pi - i\sin\pi] = -512$$

(ii) Likewise find the value of **(a)** $(1+i)^8$ **(b)** $(-\sqrt{3}+i)^6$ **(c)** $(-2-2i)^6$

9. Find x such that $0 < x < \pi$, and $\frac{\sqrt{2}e^{-ix}}{2+i} = \frac{1-3i}{5}$.