



2011 HSC Course Assessment Task 3
(Trial HSC Examination)

- Working time – 2 hours (plus 5 minutes reading time)
- **Commence each new question on a new page.**
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

- ☐ 12M3A – Mr Lam
- ☐ 12M3B – Mr Weiss
- ☐ 12M3C – Mr Lin
- ☐ 12M4A – Mr Fletcher/Mrs Collins
- ☐ 12M4B – Mr Ireland
- ☐ 12M4C – Mrs Collins/Mr Rezcallah

STUDENT NUMBER # BOOKLETS USED:

[illegible]

- Question 1** (12 Marks) Commence a NEW page. **Marks**
- (a) Find $\int \frac{3x^2 - 2}{x^3 - 2x + 1} dx$. **1**
- (b) If $y = \sqrt{\cos^3 2x}$, find $\frac{dy}{dx}$ in simplest form. **3**
- (c) Find $\int \frac{t}{\sqrt{1+t}} dt$ using the substitution $u = 1 + t$. **3**
- (d) i. Show that $\frac{d}{dx} \left[\tan^{-1} \left(\frac{x}{2} \right) + \frac{2x}{x^2 + 4} \right] = \frac{16}{(x^2 + 4)^2}$ **3**
 ii. Hence or otherwise, evaluate **2**

$$\int_{-2}^2 \frac{dx}{(x^2 + 4)^2}$$

Leave your answer in exact form.

- Question 2** (12 Marks) Commence a NEW page. **Marks**
- (a) Solve $\frac{4}{2x - 1} < 1$ for x and sketch the solution on a number line. **3**
- (b) If α , β and γ are the roots of $8x^3 - 6x^2 + 1 = 0$, evaluate
- i. $\alpha + \beta + \gamma$ **1**
 ii. $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**
 iii. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **1**
- (c) Solve $\sin \theta - 2 \cos \theta = 1$ for $0^\circ \leq \theta \leq 360^\circ$. **3**
- (d) Find the second derivative of $y = \sin(x^3)$. Draw a sketch of the curve in the immediate neighbourhood of $x = 0$. **3**

Question 3 (12 Marks)

Commence a NEW page.

Marks

- (a) Prove that $\frac{\sin 2A}{\sin A} - \frac{\cos 2A}{\cos A} = \sec A$. **2**
- (b) Show that $\frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2}$. **2**
- (c) Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta$. **3**
- (d) Find the values of a and b in the expression $f(x) = 2x^4 + ax^3 - 2x^2 + bx + 6$ so that it is exactly divisible by $2x + 1$ and has a remainder of 12 when it is divided by $x - 1$. **3**
- (e) Sketch $y = \sin(\sin^{-1} x)$. **2**

Question 4 (12 Marks)

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Marks

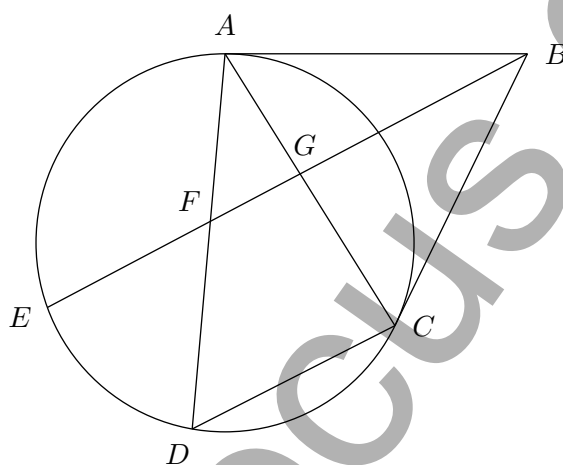
- (a) Consider the curve $y = \frac{\log_e x}{x}$.
- Show that this curve cuts the x axis at one point only. **1**
 - Discuss the behaviour of $y = \frac{\log_e x}{x}$ as $x \rightarrow 0$. **1**
 - Show the maximum value of y is $\frac{1}{e}$. **3**
 - Sketch the curve. **1**
- (b) A bowl of water, heated to 100°C is placed in a cool room maintained at -5°C . After t minutes, the temperature T of the water is changing so that
- $$\frac{dT}{dt} = -k(T - B)$$
- Show that $T = B + Ae^{-kt}$, where A and B are constants, satisfies this condition and find the values of A and B . **2**
 - After 20 minutes in the cool room, the water is 40°C . How long after being put in the cool room until the water reaches 10°C ? Answer to the nearest minute. **2**
- (c) Sketch the function $y = \frac{1}{2} \sin^{-1} \left(\frac{x}{2} \right)$. **2**

Question 5 (12 Marks)

Commence a NEW page.

Marks

- (a) Divide the interval $A(3, -2)$ and $B(-4, 5)$ externally in the ratio $3 : 1$. **2**
- (b) Let $f(x) = 2x^3 + 2x - 1$.
- i. Show that $f(x)$ has one root between $x = 0$ and $x = 1$. **1**
 - ii. By considering $f'(x)$, explain why there is only one root for $f(x)$. **1**
 - iii. Taking $x = 0$ as an initial approximation, use Newton's method to find a closer approximation. **2**
- (c) In the diagram EB is parallel to DC . Tangents from B meet the circle at A and C . Copy the diagram into your writing booklet.



Prove that

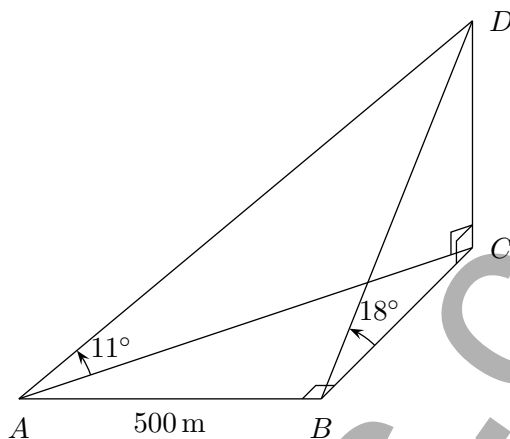
- i. $\angle BCA = \angle BFA$. **3**
- ii. $ABCF$ is a cyclic quadrilateral. **1**
- iii. $DF = CF$. **2**

Question 6 (12 Marks)

Commence a NEW page.

Marks

- (a) Prove using mathematical induction that $3^{2n+4} - 2^{2n}$ is divisible by 5. **3**
- (b) The angle of elevation of the summit of a mountain due north is 18° . On walking 500 m due west the angle of elevation is found to be 11° . **3**



Find the height of the mountain.

- (c) Prove that

$$\frac{d^2x}{dt^2} = \frac{d}{dx} \left(\frac{1}{2}v^2 \right)$$

2

where $\frac{d^2x}{dt^2}$ is the acceleration and v is the velocity.

- (d) A particle is moving in a straight line and its acceleration is given by

$$\frac{d^2x}{dt^2} = \frac{1}{36 + x^2}$$

and is initially at rest at the origin.

- i. Find v^2 as a function of x and explain why v is always positive for $t > 0$. **2**

- ii. Find

A. The velocity at $x = 6$. **1**

B. The velocity as $t \rightarrow \infty$. **1**

Question 7 (12 Marks)		Commence a NEW page.	Marks
(a)	i. Prove that if the displacement x of a particle P is related to the time t by the equation $x = 3 \cos(2\pi t)$ then the motion is simple harmonic.		1
	ii. Find the initial velocity.		1
	iii. Determine the greatest acceleration.		1
	iv. Determine when the particle is first at $x = \frac{3}{2}$.		1
	v. Express v^2 in terms of x and state the interval within which P is restricted.		2
(b)	i. Show that the equation of the normal at $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ is $x + py = 2ap + ap^3$		2
	ii. The normal at P meets the y axis at N and M is the midpoint of PN . Find the coordinates of M .		2
	iii. Show that the locus of M is another parabola with its vertex being the focus of the original parabola.		2

End of paper.

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x + C, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax + C, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C, \quad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right) + C$$

NOTE: $\ln x = \log_e x$, $x > 0$

Suggested Solutions

Question 1 (Fletcher)

(a) (1 mark)

$$\int \frac{3x^2 - 2}{x^3 - 2x + 1} dx = \log_e(x^3 - 2x + 1) + C$$

(b) (3 marks)

- ✓ [1] for changing into index form.
- ✓ [1] for correct differentiation via chain rule.
- ✓ [1] for final answer in simplest form.

$$\begin{aligned} y &= (\cos 2x)^{\frac{3}{2}} \\ \frac{dy}{dx} &= \frac{3}{2} \times (-2 \sin 2x) \times (\cos 2x)^{\frac{1}{2}} \\ &= -3 \sin 2x (\cos 2x)^{\frac{1}{2}} \\ &= -3 \sin 2x \sqrt{\cos 2x} \end{aligned}$$

(c) (3 marks)

- ✓ [1] for correctly changing variable of the integrand to u
- ✓ [1] for correct primitive in u .
- ✓ [1] for final answer in simplest form.

$$\int \frac{t}{\sqrt{1+t}} dt$$

Let $u = 1 + t$; then $du = dt$:

$$\begin{aligned} \int \frac{t}{\sqrt{1+t}} dt &= \int \frac{u-1}{u^{\frac{1}{2}}} du \\ &= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du \\ &= \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\ &= \frac{2}{3} (1+t)^{\frac{3}{2}} - 2(1+t)^{\frac{1}{2}} + C \end{aligned}$$

(d) i. (3 marks)

- ✓ [1] for derivative of $\tan^{-1} \frac{x}{2}$.
- ✓ [1] for derivative of $\frac{2x}{x^2+4}$.
- ✓ [1] for final answer.

$$\begin{aligned} \frac{d}{dx} \left[\tan^{-1} \left(\frac{x}{2} \right) + \frac{2x}{x^2+4} \right] \\ &= \frac{\frac{1}{2}}{1 + \left(\frac{x}{2} \right)^2} + \frac{2x(x^2+4) - 2x(2x)}{(x^2+4)^2} \\ &= \frac{2}{4+x^2} + \frac{8-2x^2}{(x^2+4)^2} \\ &= \frac{2(x^2+4) + 8-2x^2}{(x^2+4)^2} \\ &= \frac{16}{(x^2+4)^2} \end{aligned}$$

ii. (2 marks)

- ✓ [1] for correct primitive.
- ✓ [1] for final answer.

$$\begin{aligned} \int_{-2}^2 \frac{dx}{(x^2+4)^2} \\ &= \frac{1}{16} \left[\tan^{-1} \frac{x}{2} + \frac{2x}{x^2+4} \right]_{-2}^2 \\ &= \frac{1}{16} \left[\frac{\pi}{4} + \frac{1}{2} - \left(-\frac{\pi}{4} - \frac{1}{2} \right) \right] \\ &= \frac{1}{16} \left(\frac{\pi}{2} + 1 \right) \\ &= \frac{\pi+2}{32} \end{aligned}$$

Question 2 (Lin)

(a) (3 marks)

- ✓ [1] for multiplying both sides by square of denominator.
- ✓ [1] for obtaining correct roots for $(2x-1)(2x-5)$.
- ✓ [1] for final answer.

$$\begin{aligned} \frac{4}{2x-1} &< \frac{1}{x(2x-1)^2} \\ (2x-1)^2 &> 4(2x-1) \\ (2x-1)^2 - 4(2x-1) &> 0 \\ (2x-1)(2x-1-4) &> 0 \\ (2x-1)(2x-5) &> 0 \\ \therefore x &< \frac{1}{2} \text{ or } x > \frac{5}{2} \end{aligned}$$

(b) $8x^3 - 6x^2 + 0x + 1 = 0.$

i. (1 mark)

$$\alpha + \beta + \gamma = -\frac{b}{a} = \frac{3}{4}$$

ii. (1 mark)

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 0$$

iii. (1 mark)

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} \\ &= \frac{\frac{c}{a}}{-\frac{d}{a}} = -\frac{c}{d} = 0\end{aligned}$$

(c) (3 marks)

✓ [1] for expressing

$$\sin \theta - 2 \cos \theta = \sqrt{5} \sin(\theta - 63^\circ 26')$$

✓ [1] for obtaining $\sin(\theta - 63^\circ 26') = \frac{1}{\sqrt{5}}.$

✓ [1] for final answers.

$$\begin{aligned}\sin \theta - 2 \cos \theta &\equiv R \sin(\theta - \phi) \\ &= R \sin \theta \cos \phi - R \cos \theta \sin \phi\end{aligned}$$

Equating coefficients,

$$\begin{cases} R \cos \phi = 1 & (1) \\ R \sin \phi = 2 & (2) \end{cases}$$

(2) $\div$ (1):

$$\begin{aligned}\tan \phi &= 2 \\ \therefore \phi &= 63^\circ 26'\end{aligned}$$

(1)² + (2)²:

$$\begin{aligned}R^2 \cos^2 \phi + R^2 \sin^2 \phi &= 2^2 + 1^2 \\ \therefore R &= \sqrt{5}\end{aligned}$$

$$\therefore \sin \theta - 2 \cos \theta = \sqrt{5} \sin(\theta - 63^\circ 26') = 1$$

$$\sin(\theta - 63^\circ 26') = \frac{1}{\sqrt{5}}$$

$$\theta - 63^\circ 26' = 26^\circ 34' \text{ or } 153^\circ 26'$$

$$\therefore \theta = 90^\circ \text{ or } 216^\circ 52'$$

(d) (3 marks)

$$y = \sin(x^3)$$

$$\frac{dy}{dx} = 3x^2 \cos(x^3)$$

Apply product rule to find 2nd derivative,

$$u = 3x^2 \quad v = \cos(x^3)$$

$$u' = 6x \quad v' = -3x^2 \sin(x^3)$$

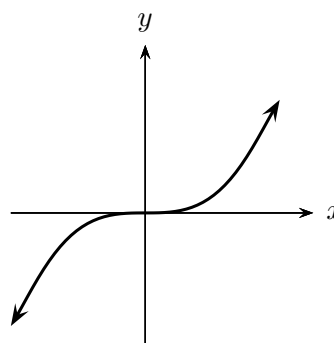
$$\begin{aligned}\frac{d^2y}{dx^2} &= -9x^4 \sin(x^3) + 6x \cos(x^3) \\ &= 3x(2 \cos(x^3) - 3x^2 \sin(x^3))\end{aligned}$$

$$\left. \frac{dy}{dx} \right|_{x=0} = 0 \quad \left. \frac{d^2y}{dx^2} \right|_{x=0} = 0$$

From the information, $\exists$ possible horizontal point of inflexion around $x = 0$.

Test sign of 2nd derivative:

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	$-$	0	$+$
	$\cap$		$\cup$

Hence concavity changes around $x = 0$.**Question 3** (Weiss)

(a) (2 marks)

✓ [1] for obtaining $\frac{2 \cos^2 A - \cos^2 A + \sin^2 A}{\cos A}.$

✓ [1] for final answer.

$$\begin{aligned}\frac{\sin 2A}{\sin A} &= \frac{\cos 2A}{\cos A} \\ &= \frac{2 \sin A \cos A}{\sin A} - \frac{\cos^2 A - \sin^2 A}{\cos A} \\ &= \frac{2 \cos^2 A - \cos^2 A + \sin^2 A}{\cos A} \\ &= \frac{\cos^2 A + \sin^2 A}{\cos A} = \sec A\end{aligned}$$

(b) (2 marks)

- ✓ [1] for transforming $\sin \theta$ and $\cos \theta$ to their t formulae.
- ✓ [1] for final answer.

$$\frac{1 - \cos \theta}{\sin \theta} = \frac{1 - \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{\frac{1+t^2-(1-t^2)}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{2t^2}{2t} = t = \tan \frac{\theta}{2}$$

(c) (3 marks)

- ✓ [1] for transforming $\cos^2 \theta$ to $\frac{1}{2} + \frac{1}{2} \cos 2\theta$.
- ✓ [1] for evaluating primitive.
- ✓ [1] for final answer.

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta &= \int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2} \cos 2\theta \, d\theta \\ &= \left[\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2} \left(\frac{\pi}{4} \right) + \frac{1}{4} \sin \frac{\pi}{2} \\ &= \frac{\pi}{8} + \frac{1}{4} = \frac{\pi+2}{8} \end{aligned}$$

(d) (3 marks)

$$f(x) = 2x^4 + ax^3 - 2x^2 + bx + 6$$

By the remainder theorem, $f(1) = 12$:

$$\begin{aligned} 2 + a - 2 + b + 6 &= 12 \\ a + b &= 6 \end{aligned}$$

By the factor theorem, $f\left(-\frac{1}{2}\right) = 0$:

$$\begin{aligned} 2\left(-\frac{1}{2}\right)^4 + a\left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right)^2 - \frac{1}{2}b + 6 &= 0 \\ \underbrace{\frac{1}{8} - \frac{a}{8} - \frac{1}{2} - \frac{b}{2}}_{\times 8} &= -6 \end{aligned}$$

$$1 - a - 4 - 4b = -48$$

$$a + 4b = 45$$

$$\begin{cases} a + b = 6 & (1) \\ a + 4b = 45 & (2) \end{cases}$$

(2) - (1):

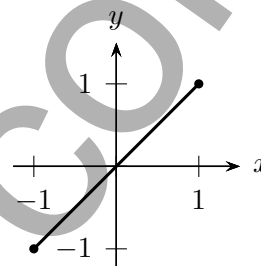
$$3b = 39$$

$$\therefore b = 13$$

$$\therefore a = -7$$

(e) (2 marks)

- ✓ [1] for line.
- ✓ [1] for boundary points.

**Question 4** (Lam)

(a) i. (1 mark)

$$y = \frac{\log_e x}{x} = 0$$

As $x \neq 0$,

$$\therefore \log_e x = 0$$

$$\therefore x = 1$$

ii. (1 mark)

 $\log_e x \ll x$ when $x < 1$

$$\therefore \lim_{x \rightarrow 0} \frac{\log_e x}{x} = -\infty$$

iii. (3 marks)

- ✓ [1] for correctly applying quotient rule.
- ✓ [1] for $x = e$.
- ✓ [1] testing for local maximum.

$$u = \log_e x \quad v = x$$

$$u' = \frac{1}{x} \quad v' = 1$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{x \times \frac{1}{x} - \log_e x}{x^2} \\ &= \frac{1 - \log_e x}{x^2} \end{aligned}$$

Stationary pts occur when $\frac{dy}{dx} = 0$,

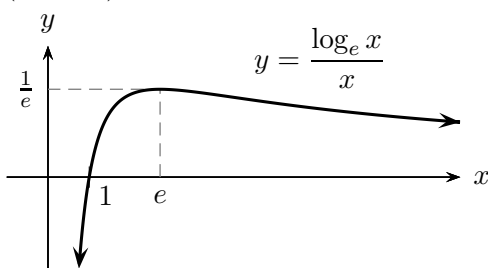
$$1 - \log_e x = 0$$

$$\log_e x = 1$$

$$\therefore x = e$$

x	e
$\frac{dy}{dx}$	$\begin{array}{ccc} + & 0 & - \end{array}$
y	$(e, \frac{1}{e})$

iv. (1 mark)



(b) i. (2 marks)

✓ [1] for $A = 105$

✓ [1] for $B = -5$

$$\frac{dT}{dt} = -k(T - B)$$

$$\frac{dT}{T - B} = -k dt$$

Integrating both sides,

$$\ln(T - B) = -kt + C$$

$$T - B = e^{-kt+C} = Ae^{-kt}$$

$$\therefore T = B + Ae^{-kt}$$

(Alternatively, differentiate and verify.)

By inspection,

$$B = -5$$

When $t = 0$, $T = 100^\circ\text{C}$

$$100 = -5 + Ae^0$$

$$\therefore A = 105$$

ii. (2 marks)

✓ [1] for finding the value of k .

✓ [1] for final answer.

When $t = 20$, $T = 40$

$$40 = -5 + 105e^{-20k}$$

$$45 = 105e^{-20k}$$

$$e^{-20k} = \frac{3}{7}$$

$$-20k = \log_e \frac{3}{7}$$

$$k = -\frac{1}{20} \log_e \frac{3}{7}$$

$$\therefore T = -5 + 105e^{\frac{1}{20}t \log_e \frac{3}{7}}$$

At $T = 10^\circ\text{C}$,

$$10 = -5 + 105e^{\frac{1}{20}t \log_e \frac{3}{7}}$$

$$15 = 105e^{\frac{1}{20}t \log_e \frac{3}{7}}$$

$$e^{\frac{1}{20}t \log_e \frac{3}{7}} = \frac{1}{7}$$

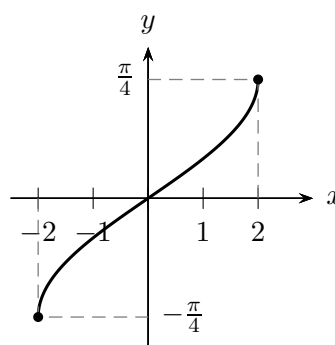
$$\frac{1}{20}t \log_e \frac{3}{7} = -\log_e 7$$

$$t = \frac{-20 \log_e 7}{\log_e \frac{3}{7}} = 45.93 \dots = 46 \text{ min}$$

(c) (2 marks)

✓ [1] for shape.

✓ [1] for endpoints and passing through origin.



Question 5 (Collins)

(a) (2 marks)

- ✓ [1] for correct method.
 ✓ [1] for correct final answer.

$$\begin{array}{ccc} A(3, -2) & & B(-4, 5) \\ & \swarrow \quad \searrow & \\ & 3 \quad \quad -1 & \end{array}$$

$$\begin{aligned} & \left(\frac{(3)(-1) + (-4)(3)}{3 - 1}, \frac{(-2)(-1) + (3)(5)}{2} \right) \\ &= \left(\frac{-3 - 12}{2}, \frac{2 + 15}{2} \right) = \left(-\frac{15}{2}, \frac{17}{2} \right) \end{aligned}$$

(b) i. (1 mark)

$$\begin{aligned} f(x) &= 2x^3 + 2x - 1 \\ f(0) &= -1 \quad f(1) = 3 \end{aligned}$$

As there is a change of sign in the y value between $x = 0$ and $x = 1$ and $f(x)$ is continuous between $x = 0$ and $x = 1$, therefore $f(x)$ has a root between $x = 0$ and $x = 1$.

ii. (1 mark)

✗ do not accept $f'(x) > 0$.

$$f'(x) = 6x^2 + 2$$

As $6x^2 + 2 > 0 \forall x$, therefore $f'(x)$ has no stationary points (therefore no turning points) at all. Hence there is only one root for $f(x)$.

iii. (2 marks)

- ✓ [1] for $f(0) = -1, f'(0) = 2$.
 ✓ [1] for final answer.

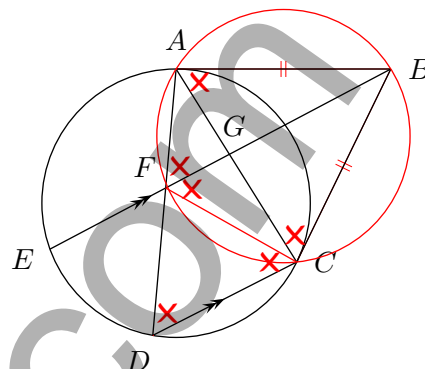
$$f(0) = -1 \quad f'(0) = 2$$

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 0 - \frac{-1}{2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{8}\right) + 2\left(\frac{1}{2}\right) - 1 \\ &= \frac{1}{4} \end{aligned}$$

(c) i. (3 marks)

- ✓ [1] for $\angle$ in isos $\triangle$.
 ✓ [1] for $\angle$ alternate segment.
 ✓ [1] for corresponding $\angle$.



- $AB = CB$
(tangents from external point)
 - $\angle BCA = \angle BAC$
($\angle$ opposite equal sides, isos $\triangle BCA$)
 - $\angle BAC = \angle ADC$
($\angle$ in alternate segment)
 - $\angle ADC = \angle AFB$
(corresponding $\angle$, $EB \parallel DC$)
- $\therefore \angle BCA = \angle BFA$.

ii. (1 mark)

From the previous part, $\angle BCA = \angle BFA$. These angles represent $\angle$ at the circumference standing on arc AB . Hence C, F, A and B lie on the circumference of another circle, and $CFAB$ is a cyclic quadrilateral.

iii. (2 marks)

- ✓ [1] ONLY if incorrect working from (i) leads to proof that $\triangle DFC$ is isosceles.
 ✓ [1] ONLY if a property of cyclic quadrilaterals is correctly used. Otherwise,
 ✓ [1] for each dot point.
- $\angle CAB = \angle CFB$
($\angle$ standing on the same arc CB)
 - $\angle CFB = \angle FCD$
(alternate $\angle$, $EB \parallel DC$)
- $\therefore FD = FC$
(sides opposite equal $\angle$, isos $\triangle DFC$)

Question 6 (Ireland)

(a) (3 marks)

✓ [1] for testing base case.

✓ [2] for correct inductive step proof.

To prove $3^{2n+4} - 2^{2n}$ is divisible by 5 for all $n \geq 1$.**Base case:** $n = 1$

$$3^{2(1)+4} - 2^{2(1)} = 3^6 - 2^2 = 725$$

which is divisible by 5. Hence the base case is true.

Inductive step: Assume $3^{2k+4} - 2^{2k}$ is divisible by 5, i.e.

$$3^{2k+4} - 2^{2k} = 5M$$

$$3^{2k+4} = 5M + 2^{2k}$$

for k and $M \in \mathbb{Z}^+$. Examine the $(k+1)$ -th term:

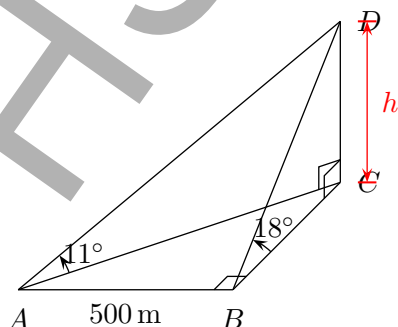
$$\begin{aligned} & 3^{2(k+1)+4} - 2^{2(k+1)} \\ &= 3^{2+2k+4} - 2^{2k+2} \\ &= 3^2 \cdot 3^{2k+4} - 2^2 \cdot 2^{2k} \\ &= 9(5M + 2^{2k}) - 4(2^{2k}) \\ &= 9 \cdot 5M + 9 \cdot 2^{2k} - 4 \cdot 2^{2k} \\ &= 9 \cdot 5M + 5 \cdot 2^{2k} \\ &= 5(9M + 2^{2k}) = 5P \end{aligned}$$

where $P(= 9M + 2^{2k}) \in \mathbb{Z}^+$.By the principle of mathematical induction, $3^{2n+4} - 2^{2n}$ is divisible by 5 for all positive integers n .

(b) (3 marks)

✓ [1] for BC and AC in terms of h .✓ [1] for numerical expression in h^2 .

✓ [1] for final answer.

In $\triangle BCD$

$$\frac{h}{BC} = \tan 18^\circ \Rightarrow BC = \frac{h}{\tan 18^\circ}$$

Likewise in $\triangle ACD$,

$$AC = \frac{h}{\tan 11^\circ}$$

As ABC is right angled, apply Pythagoras' Theorem:

$$\begin{aligned} AB^2 + BC^2 &= AC^2 \\ 500^2 + \left(\frac{h}{\tan 18^\circ}\right)^2 &= \left(\frac{h}{\tan 11^\circ}\right)^2 \\ 500^2 &= h^2 \left(\frac{1}{\tan^2 11^\circ} - \frac{1}{\tan^2 18^\circ}\right) \\ \therefore h^2 &= \frac{500^2}{\frac{1}{\tan^2 11^\circ} - \frac{1}{\tan^2 18^\circ}} \\ &\approx \frac{500^2}{16.9943 \dots} \\ \therefore h &= 121.3 \text{ m (1 d.p.)} \end{aligned}$$

(c) (2 marks)

✓ [1] for obtaining $\frac{d}{dx} \left(\frac{1}{2}v^2\right) = \frac{dx}{dt} \cdot \frac{dv}{dx}$.

✓ [1] for correct proof.

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{2}v^2\right) &= \frac{d}{dv} \left(\frac{1}{2}v^2\right) \frac{dv}{dx} \quad (\text{chain rule}) \\ &= v \frac{dv}{dx} = \frac{dx}{dt} \cdot \frac{dv}{dx} \\ &= \frac{dv}{dt} = \frac{d^2x}{dt^2} = a \end{aligned}$$

(d) i. (2 marks)

✓ [1] for $v^2 = \frac{1}{3} \tan^{-1} \frac{x}{6}$.

✓ [1] for justification.

$$\begin{aligned} \frac{d}{dx} \left(\frac{1}{2}v^2\right) &= \frac{1}{36+x^2} \\ \frac{1}{2}v^2 &= \int \frac{1}{36+x^2} dx = \frac{1}{6} \tan^{-1} \frac{x}{6} + C \end{aligned}$$

When $t = 0$, $x = 0$ and $v = 0$:

$$\therefore 0 = \frac{1}{6}(0) + C$$

$$\therefore C = 0$$

$$\therefore v^2 = \frac{1}{3} \tan^{-1} \frac{x}{6}$$

As $\frac{d^2x}{dt^2} > 0$ and velocity is initially 0, so for $t > 0$, the velocity will always be positive.

ii. A. (1 mark)

When $x = 6$,

$$\begin{aligned} v^2 &= \frac{1}{3} \tan^{-1} 1 \\ &= \frac{1}{3} \times \frac{\pi}{4} = \frac{\pi}{12} \\ \therefore v &= \pm \sqrt{\frac{\pi}{12}} \end{aligned}$$

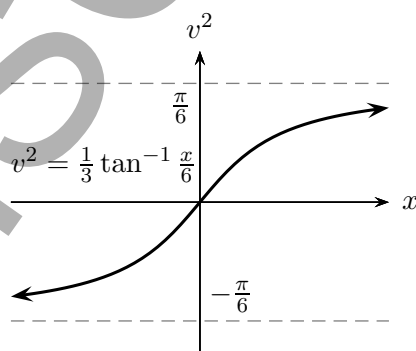
But from previous part, $v > 0$.

$$\therefore v = \sqrt{\frac{\pi}{12}}$$

B. (1 mark)

As $t \rightarrow \infty$, $x \rightarrow \infty$. Hence

$$\lim_{x \rightarrow \infty} \left(\frac{1}{3} \tan^{-1} \frac{x}{6} \right) = \frac{1}{3} \times \frac{\pi}{2} = \frac{\pi}{6}$$



$$\therefore v \rightarrow \sqrt{\frac{\pi}{6}} \text{ as } t \rightarrow \infty.$$

Question 7 (Collins)

(a) i. (1 mark)

✓ [1] only if 2nd derivative is correct and presented in SHM form.

$$x = 3 \cos(2\pi t)$$

$$\dot{x} = -6\pi \sin(2\pi t)$$

$$\ddot{x} = -12\pi^2 \cos(2\pi t)$$

$$= -4\pi^2 \times 3 \cos(2\pi t)$$

$$= -(2\pi)^2 \times 3 \cos(2\pi t) = -n^2 x$$

As acceleration is proportional to displacement and directed to its opposite direction, therefore motion is SHM.

ii. (1 mark)

When $t = 0$, $\dot{x} = 0$.

iii. (1 mark)

Maximum acceleration when $\cos 2\pi t = -1$. $\therefore a_{\max} = 12\pi^2 \text{ ms}^{-2}$

iv. (1 mark)

When $x = \frac{3}{2}$,

$$\frac{x}{3} = \cos(2\pi t)$$

$$\cos(2\pi t) = \frac{1}{2}$$

$$2\pi t = \frac{\pi}{3}$$

$$\therefore t = \frac{1}{6}$$

v. (2 marks)

✓ [1] for correct v^2 .

✓ [1] for correct interval $x \in [-3, 3]$.

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = -4\pi^2 x$$

Integrating,

$$\frac{1}{2} v^2 = -4\pi^2 \times \frac{1}{2} x^2 = -2\pi^2 x^2 + C$$

When $t = 0$, $v = 0$ and $x = 3$.

$$0 = -2\pi^2 \times (3^2) + C$$

$$\therefore C = 18\pi^2$$

$$\frac{1}{2}v^2 = -2\pi^2 x^2 + 18\pi^2$$

$$v^2 = 36\pi^2 - 4\pi^2 x^2$$

$$= 4\pi^2 (9 - x^2)$$

$$(= n^2(a^2 - x^2))$$

Restriction on P :

$$9 - x^2 \geq 0$$

$$\therefore -3 \leq x \leq 3$$

(b) i. (2 marks)

✓ [1] for $m_{\perp} = -\frac{1}{p}$.

✓ [1] for final answer.

$$\begin{aligned} x^2 &= 4ay \Rightarrow y = \frac{x^2}{4a} \\ \frac{dy}{dx} &= \frac{2x}{4a} \Big|_{x=2ap} = \frac{4ap}{4a} = p \\ \therefore m_{\perp} &= -\frac{1}{p} \end{aligned}$$

Using the point-gradient formula,

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap$$

$$\therefore x + py = ap^3 + 2ap$$

ii. (2 marks)

✓ [1] for coordinates of M .

✓ [1] for coordinates of N .

When the normal meets the y axis,

$$x = 0.$$

$$0 + py = ap^3 + 2ap$$

$$\therefore y = ap^2 + 2a$$

$$\therefore N(0, ap^2 + 2a)$$

M is the midpoint of PN :

$$M = \left(\frac{2ap + 0}{2}, \frac{ap^2 + (ap^2 + 2a)}{2} \right)$$

$$= \left(ap, \frac{2ap^2 + 2a}{2} \right) = (ap, ap^2 + a)$$

iii. (2 marks)

✓ [1] for locus.

✓ [1] for stating vertex of parabola.

$$\begin{cases} x = ap & (1) \\ y = a(p^2 + 1) & (2) \end{cases}$$

From (1),

$$\therefore p = \frac{x}{a}$$

Substitute to $y = a(p^2 + 1)$:

$$\therefore y = a \left(\left(\frac{x}{a} \right)^2 + 1 \right)$$

$$= a \left(\frac{x^2}{a^2} + 1 \right)$$

$$= \frac{x^2}{a} + a$$

$$ay = x^2 + a^2$$

$$x^2 = ay - a^2 = a(y - a)$$

which is a parabola with vertex $(0, a)$ – the focus of the original parabola.