

Question 1:

- (a)(i) Find the derivative of $x^2 \cos x$. 2
- (ii) Evaluate $\int_1^6 \frac{x}{x^2 + 4} dx$. 2
- (b)(i) Sketch $y = |x + 1|$. 2
- (ii) Hence or otherwise solve $|x + 1| = 3x$. 1
- (c) If $f(x) = 2 \sin^{-1}(3x)$, find
- (i) the domain and range of $f(x)$, 2
- (ii) $f\left(\frac{1}{6}\right)$, 1
- (iii) $f'\left(\frac{1}{6}\right)$. 2

QUESTION 2:

(START A NEW PAGE)

- (a) P(-7,3), Q(9,15) and B(14,0) are three points and A divides the interval PQ in the ratio 3:1. Prove that PQ is perpendicular to AB. 3
- (b) By using the substitution $u^2 = x + 1$ evaluate $\int_0^3 \frac{x+2}{\sqrt{x+1}} dx$. 3
- (c) Water flows from a hole in the base of a cylindrical vessel at a rate given by 6

$$\frac{dh}{dt} = -k\sqrt{h}$$

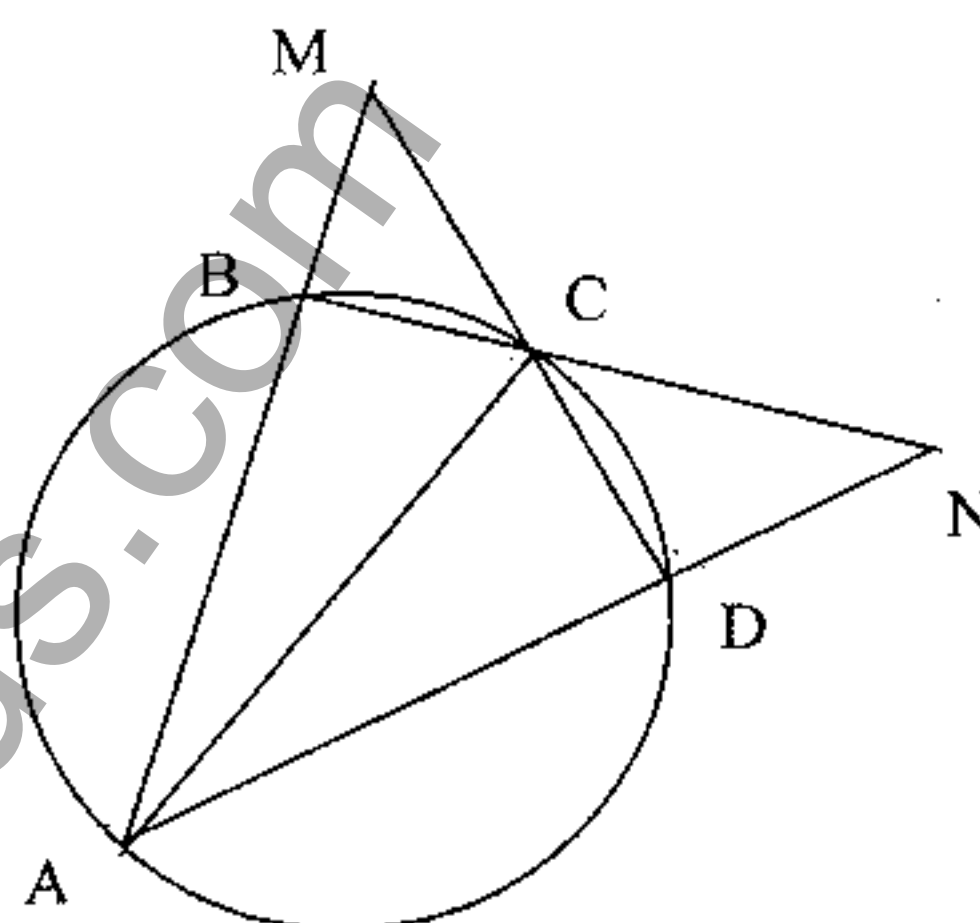
where k is a constant and h mm is the depth of water at time t minutes.
If the depth of water falls from 2500mm to 900mm in 5 minutes, find how much longer it will take to empty the vessel.

QUESTION 3:**(START A NEW PAGE)**

- (a) Find the value of the constant term in the expansion of $\left(3x + \frac{2}{\sqrt{x}}\right)^6$. 3
- (b) Three boys (Adam, Bruce, Chris) and three girls (Debra, Emma, Fay) form a single queue at random in front of the school canteen window. Find the probability that:
- (i) the first two to be served are Emma and Adam in that order, 2
- (ii) a boy is at each end of the queue, 1
- (iii) no two girls stand next to each other. 1

- (c) In the figure ABM , DCM , BCN and ADN are straight lines and $\angle AMD = \angle BNA$.

- (i) Copy the diagram onto your answer sheet and prove that $\angle ABC = \angle ADC$.
- (ii) Hence prove that AC is a diameter.

**QUESTION 4:****(START A NEW PAGE)**

- (a)(i) Given that $\sin^2 A + \cos^2 A = 1$, prove that $\tan^2 A = \sec^2 A - 1$. 2
- (ii) Sketch the curve $y = 4 \tan^{-1} x$ clearly showing its range. 2
- (iii) Find the volume of the solid formed when the area bounded by the curve $y = 4 \tan^{-1} x$, the y -axis and the line $y = \pi$ is rotated one revolution about the y -axis. 2
- (b)(i) An object has velocity $v \text{ ms}^{-1}$ and acceleration $\ddot{x} \text{ ms}^{-2}$ at position $x \text{ m}$ from the origin, show that $\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \ddot{x}$. 2
- (ii) The acceleration (in ms^{-2}) of an object is given by $\ddot{x} = 2x^3 + 4x$.
- (α) If the object is initially 2 m to the right of the origin traveling with velocity 6 ms^{-1} , find an expression for v^2 (the square of its velocity) in terms of x . 2
- (β) What is the minimum speed of the object? (Give a reason for your answer) 2

QUESTION 5:**(START A NEW PAGE)**

- (a) The curves $y = e^{-2x}$ and $y = 3x + 1$ meet on the y-axis. Find the size of the acute angle between these curves at the point where they meet. 3
- (b)(i) Sketch the function $y = f(x)$ where $f(x) = (x - 1)^2 - 4$ clearly showing all intercepts with the co-ordinate axes. (Use the same scale on both axes) 2
- (ii) What is the largest positive domain of f for which $f(x)$ has an inverse $f^{-1}(x)$? 1
- (iii) Sketch the graph of $y = f^{-1}(x)$ on the same axes as (i). 1
- (c) In tennis a player is allowed a maximum of two serves when attempting to win a point. If the first serve is not legal it is called a fault and the server is allowed a second serve. If the second serve is also illegal then it is called a double fault and the server loses the point. The probability that Pat Smash's first serve will be legal is 0.4. If Pat Smash needs to make a second serve then the probability that it will be legal is 0.7.
- (i) Find the probability that Pat Smash will serve a double fault when trying to win a point. 2
- (ii) If Pat Smash attempts to win six points, what is the probability that he will serve at least two double faults? (Give answer correct to 2 decimal places) 3

QUESTION 6:**(START A NEW PAGE)**

- (a) A spherical bubble is expanding so that its volume is increasing at $10 \text{ cm}^3 \text{ s}^{-1}$. Find the rate of increase of its radius when the surface area is 500 cm^2 . 3
(Volume = $\frac{4}{3}\pi r^3$, Surface area = $4\pi r^2$)
- (b) Prove by Mathematical Induction that: 4
 $2(1!) + 5(2!) + 10(3!) + \dots + (n^2 + 1)n! = n(n + 1)!$ for positive integers $n \geq 1$.
- (c) If $y = \frac{\log_e x}{x}$ find $\frac{dy}{dx}$ and hence show that $\int_e^{e^2} \frac{1 - \log_e x}{x \log_e x} dx = \log_e 2 - 1$. 5

QUESTION 7:**(START A NEW PAGE)**

- (i) By considering the expansion of $\sin(X + Y) - \sin(X - Y)$ prove that

3

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right).$$

- (ii) Also given that $\cos A - \cos B = 2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{B-A}{2}\right)$ prove that

2

$$\frac{\sin A - \sin B}{\cos A - \cos B} = -\cot\left(\frac{A+B}{2}\right).$$

- (iii) Prove that the position of a projectile t seconds after projection from ground level with initial horizontal and vertical velocity components of $V \cos \alpha$ and $V \sin \alpha$ respectively is given by $x = Vt \cos \alpha$ and $y = -\frac{1}{2}gt^2 + Vt \sin \alpha$.
(Assume that there is no air resistance)

2

- (iv) Two objects P and Q are projected from the same ground position at the same time with initial speed $V \text{ ms}^{-1}$ at angles α and β respectively ($\beta > \alpha$).

- (α) If at time t seconds the line joining P and Q makes an acute angle θ with the

horizontal prove that $\tan \theta = \left| \frac{\sin \beta - \sin \alpha}{\cos \beta - \cos \alpha} \right|.$

3

- (β) Hence show that $\theta = \frac{1}{2}(\pi - \alpha - \beta).$

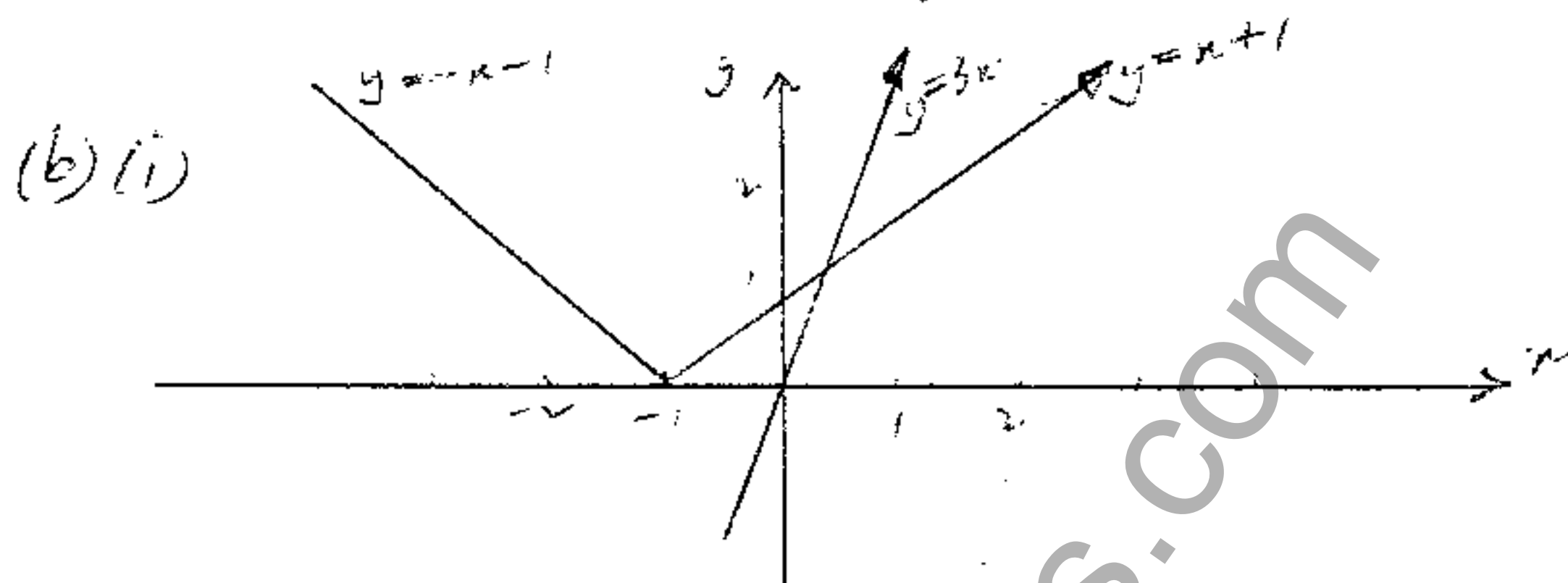
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QUESTION 1

(a)(i) $y' = 2x \cos x - x^2 \sin x$

(ii) $\int_1^6 \frac{x}{x^2+4} dx = \frac{1}{2} [\ln(x^2+4)]_1^6$
 $= \frac{1}{2} (\ln 40 - \ln 5)$
 $= \frac{1}{2} \ln 8$



(ii) $3x = x + 1$ (from graph)
 $2x = 1$
 $x = \frac{1}{2}$

(c)(i) Domain $-\frac{1}{3} \leq x \leq \frac{1}{3}$
 Range $-\pi \leq y \leq \pi$

(ii) $f(\frac{1}{6}) = 2 \sin^{-1}(\frac{1}{6})$
 $= \pi/3$

(iii) $f(x) = 2 \cdot \frac{3}{\sqrt{1-9x^2}}$
 $f'(\frac{1}{6}) = \frac{6}{\sqrt{1-9/36}}$
 $= \frac{6}{\sqrt{3/4}}$
 $= \frac{12}{\sqrt{3}} \text{ or } 4\sqrt{3}$

QUESTION 2.

(a) $P(-7, 3)$ $Q(9, 15)$ $B(14, 0)$

$\times$
 $3:1$

$$A\left(\frac{-7+27}{4}, \frac{3+4 \cdot 5}{4}\right) = A(5, 12)$$

$$m(PQ) = \frac{15-3}{9+7}$$
$$= \frac{3}{4}$$

$$m(AB) = \frac{12-0}{5-14}$$
$$= -\frac{4}{3}$$

$$m(PQ) \cdot m(AB) = \frac{3}{4} \cdot -\frac{4}{3}$$
$$= -1$$

$\therefore PQ \perp AB$ (prod. of slopes $= -1$)

(b) $x=0$ $u^2=1$
 $u=1$ (take $u>0$)

$x=3$ $u^2=4$
 $u=2$ (take $u>0$)

$$u^2 - 1 = x$$

$$\frac{dx}{du} = 2u$$

$$dx = 2u du$$

$$\therefore \int_0^3 \frac{x+2}{\sqrt{x+1}} dx = \int_1^2 \frac{u^2+1}{\sqrt{u^2}} 2u du$$

$$= 2 \int_1^2 \frac{u^2+1}{u} du$$

$$= 2 \int_1^2 \left(u + \frac{1}{u}\right) du$$

$$= 2 \left[\frac{1}{2}u^2 + \ln u \right]_1^2$$

$$= 2 \left\{ \left(\frac{4}{2} + \ln 2\right) - \left(\frac{1}{2} + \ln 1\right) \right\}$$

$$= 3 + 2\ln 2 \quad 6\frac{2}{3}$$

$$2(c) \quad \frac{dt}{dh} = -\frac{1}{k} \cdot h^{-1/2}$$

$$t = -\frac{1}{k} \cdot 2h^{1/2} + c$$

$$t = -\frac{2\sqrt{h}}{k} + c$$

$$t = 0, h = 2500$$

$$0 = -\frac{100}{k} + c \quad \text{--- (1)}$$

$$t = 5, h = 900$$

$$5 = -\frac{60}{k} + c \quad \text{--- (2)}$$

$$(2) - (1) \quad 5 = -\frac{60}{k} + \frac{100}{k}$$

$$5k = 40$$

$$k = 8$$

$$\text{from (1)} \quad c = \frac{100}{8}$$

$$\therefore t = -\frac{\sqrt{h}}{4} + 12.5$$

$$\text{when } h = 0, t = 12.5$$

$$\therefore \text{extra time} = 12.5 - 5 \\ = 7.5 \text{ min.}$$

QUESTION 3.

$$\begin{aligned} (a) \quad T_{r+1} &= {}^6C_r (3x)^{6-r} \left(\frac{2}{5x}\right)^r \\ &= {}^6C_r 3^{6-r} 2^r \cdot x^{6-r} \cdot x^{-r} \\ &= {}^6C_r 3^{6-r} 2^r x^{6-1/2r} \end{aligned}$$

for constant term degree of $x = 0$

$$\therefore 6 - \frac{1}{2}r = 0$$

$$\frac{1}{2}r = 6$$

$$r = 4$$

$$\therefore \text{constant} = \frac{6 \cdot 3 \cdot 2^4}{4!} = 2160$$

(b) (i) $\begin{matrix} \textcircled{1} & \textcircled{1} & \textcircled{4} & \textcircled{3} & \textcircled{2} & \textcircled{1} \\ E & A & & & & \end{matrix}$

$$\text{Prob} = \frac{4!}{6!}$$

$$= \frac{1}{30}$$

$$\text{or Prob} = \frac{1}{6} \cdot \frac{1}{5}$$

$$= \frac{1}{30}$$

(ii)

$$\begin{matrix} & B & & & & \\ \textcircled{3} & & \textcircled{4} & & \textcircled{3} & & \textcircled{1} & & \textcircled{2} \end{matrix}$$

$$\text{Prob} = \frac{3 \cdot 2 \cdot 4!}{6!}$$

$$= \frac{1}{5}$$

(iii)

$$\begin{matrix} 2 & \textcircled{3} & 4 & \textcircled{4} & 3 & \textcircled{1} \\ G & B & G & B & G & B \end{matrix}$$

$$\text{Prob} = \frac{3! \cdot 4 \cdot 3 \cdot 2}{6!}$$

$$= \frac{1}{5}$$

(Place Boys then fill gaps with girls)

(c) (i) Let $\hat{AMD} = \hat{ANB} = \alpha^\circ$

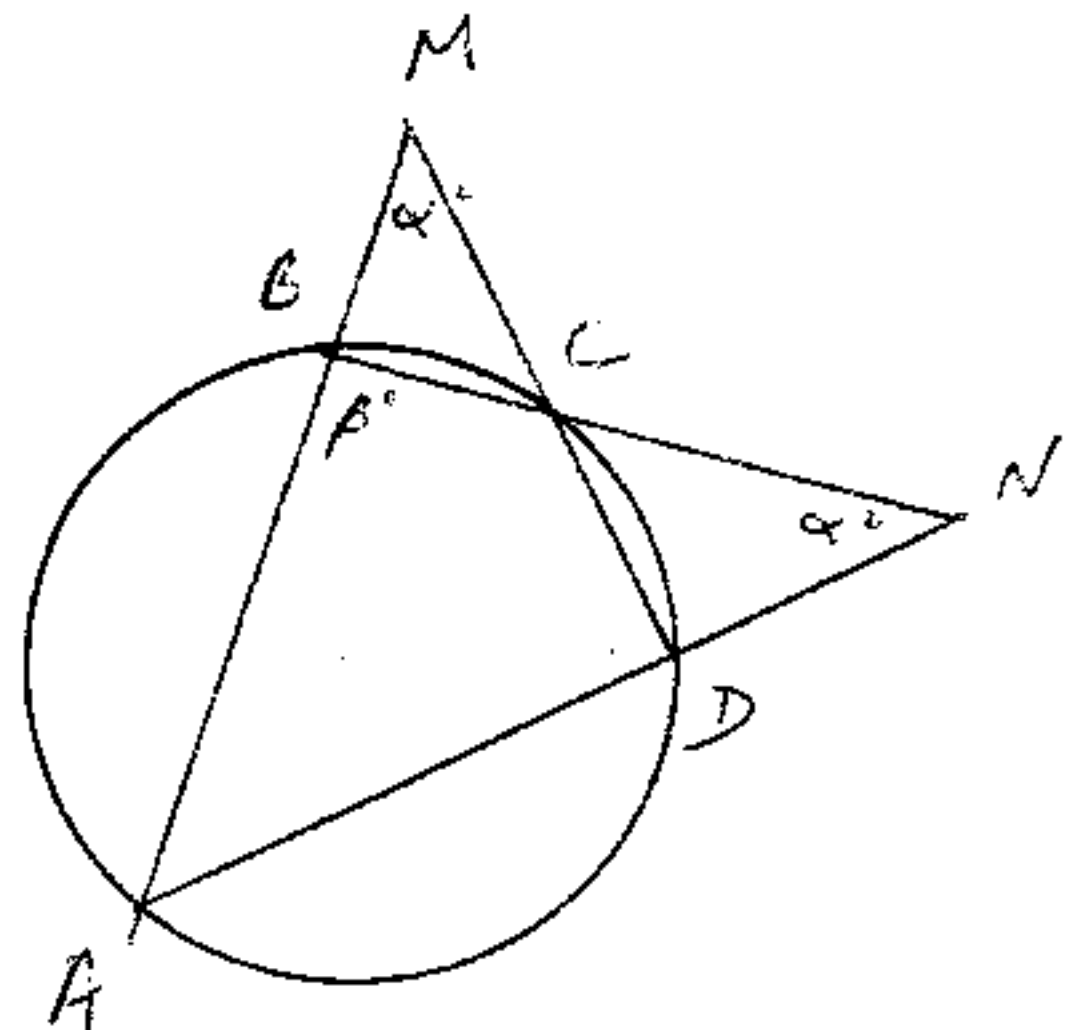
$$\text{and } \hat{ABC} = \beta^\circ$$

$$\hat{BCM} = (\beta - \alpha)^\circ \quad (\text{exterior angle of } \triangle BMC \text{ equals sum of opposite interior angles})$$

$$\hat{DCN} = (\beta - \alpha)^\circ \quad (\text{vertically opposite angles})$$

$$\hat{ADC} = \beta^\circ \quad (\text{exterior angle of } \triangle CND \text{ equals sum of opposite interior angles})$$

$$\therefore \hat{ABC} = \hat{ADC} \quad \text{Hence } AC \parallel BD$$



Q3(c)(ii) $\hat{ABC} + \hat{ADC} = 180^\circ$ (opposite angles of cyclic quadrat ABCD are supplementary)
 $2\hat{ABC} = 180^\circ$ ($\hat{AC} = \hat{DC}$, part (i))
 $\hat{ABC} = 90^\circ$
 $\therefore AC$ is a diameter (angle in semicircle)
 $\therefore 90^\circ$

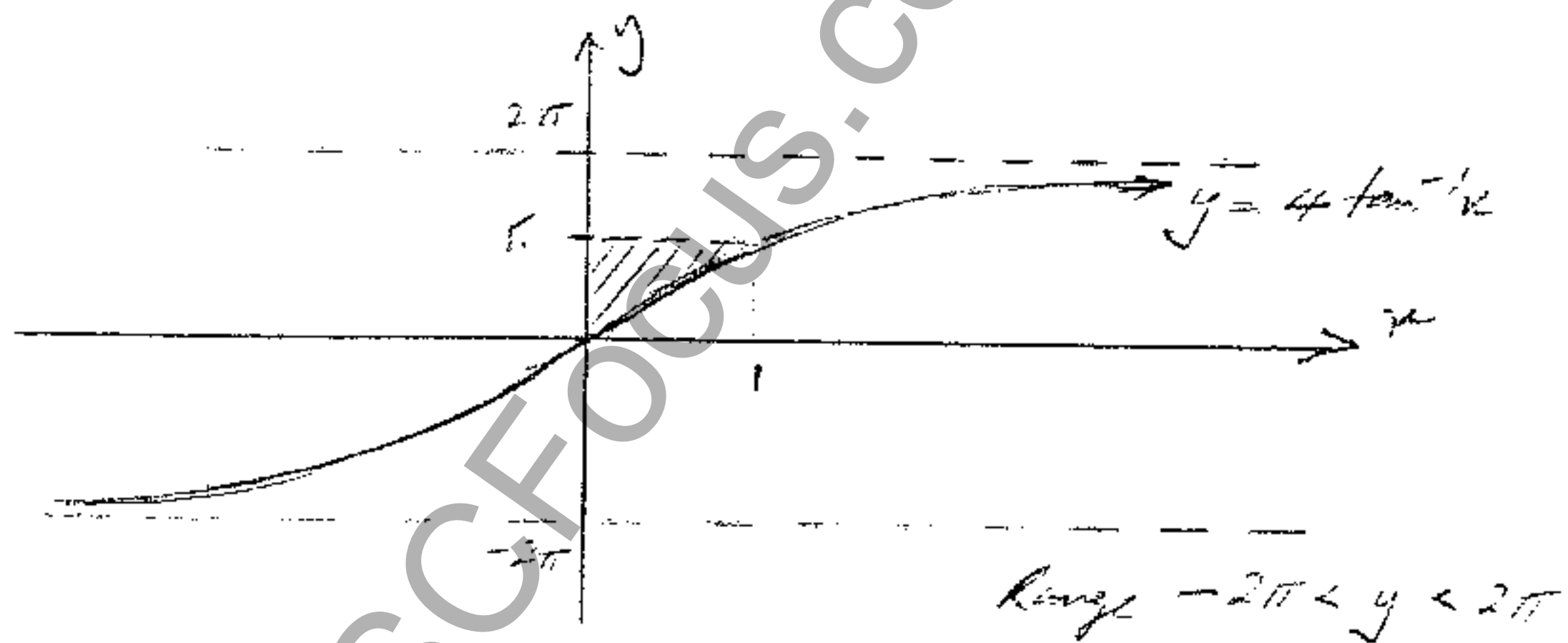
QUESTION 4

(a)(i) $\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A} = \frac{1}{\cos^2 A}$

$$\tan^2 A + 1 = \sec^2 A$$

$$\tan^2 A = \sec^2 A - 1$$

(ii)



(iii) $y/4 = \tan^{-1} x$
 $x = \tan y/4$

$$\begin{aligned} V &= \pi \int_0^\pi x^2 dy \\ &= \pi \int_0^\pi \tan^2 y/4 dy \\ &= \pi \int_0^\pi \sec^2 y/4 - 1 dy \end{aligned}$$

$$= \pi \left[4 \tan y/4 - y \right]_0^\pi$$

$$\begin{aligned} &= \pi \left\{ (4 \tan \pi/4 - \pi) - (4 \tan 0 - 0) \right\} \\ &= \pi (4 - \pi) \end{aligned}$$

$$\begin{aligned}
 (b)(i) \quad \frac{d}{dx} \left(\frac{1}{2} v^2 \right) &= \frac{d}{dv} \left(\frac{1}{2} v^2 \right) \cdot \frac{dv}{dx} \\
 &= v \frac{dv}{dx} \\
 &= \frac{dx}{dt} \cdot \frac{dv}{dx} \\
 &= \frac{dv}{dt} \\
 &= a
 \end{aligned}$$

$$(ii)(\alpha) \quad \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = 2x^3 + 4x$$

$$\frac{1}{2} v^2 = \frac{x^4}{2} + 2x^2 + C$$

$$t=0, x=2, v=6$$

$$18 = 8 + 8 + C$$

$$C = 2$$

$$v^2 = x^4 + 4x^2 + 4$$

$$(b) \quad v^2 = (x^2 + 2)^2 \quad \therefore v^2 \geq 4 \quad \therefore v \neq 0$$

$\therefore$ object never changes direction.

$\therefore$ always moves to right with increasing speed

since initial vel > 0 & accel > 0 for $x > 0$

$\therefore$ min speed is initial speed

$\therefore$ min. speed = 6 ms^{-1}

QUESTION 5

$$(a) \quad y' = -2e^{-2x}$$

$$\text{when } x=0, y' = -2e^0$$

$$m_1 = -2$$

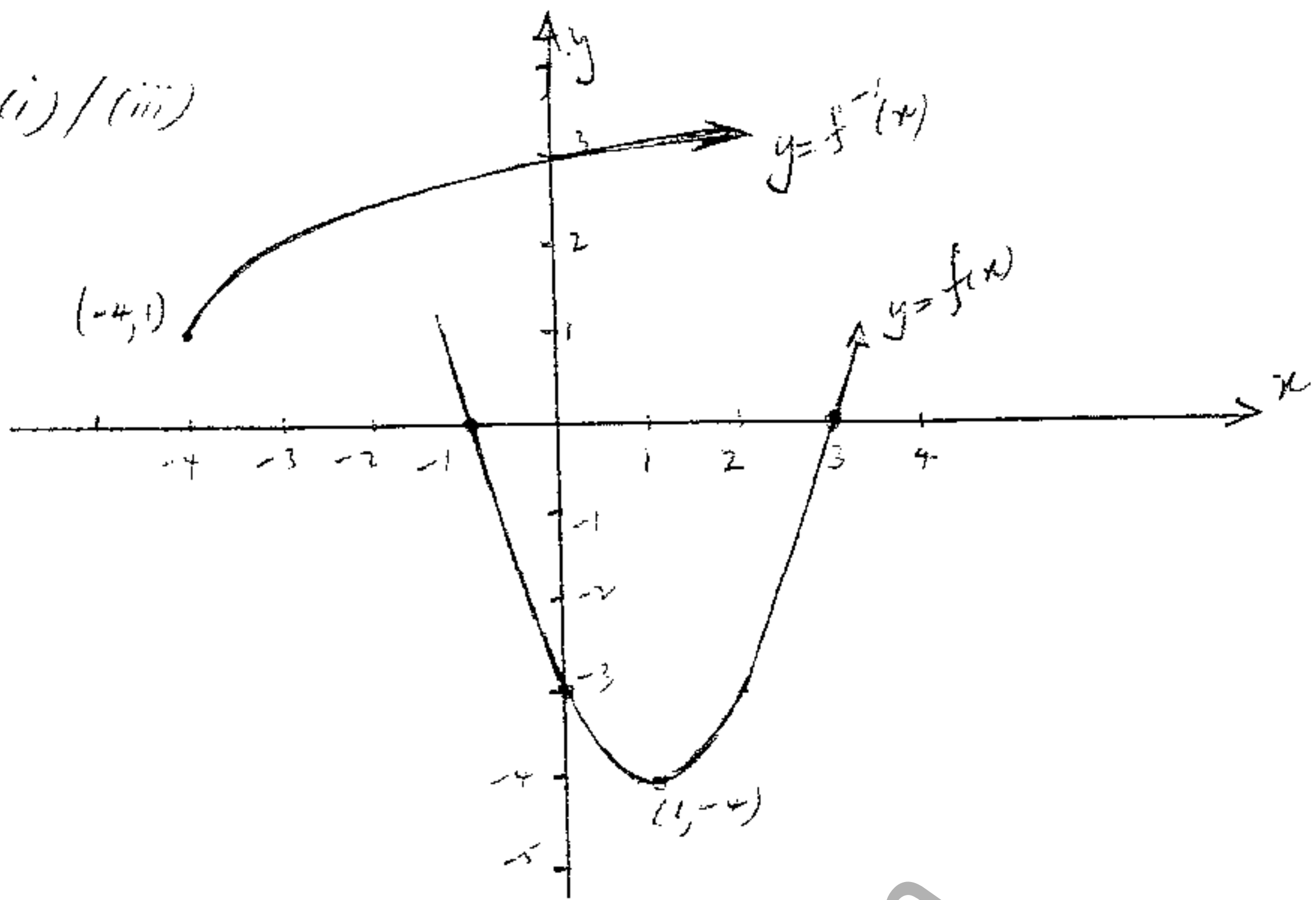
$$m_2 = 3$$

$$\tan \theta = \left| \frac{3 + 2}{1 + (3)(-2)} \right|$$

$$= 1$$

$$\theta = \pi/4, \text{ or } 45^\circ$$

(b) (i) / (iii)

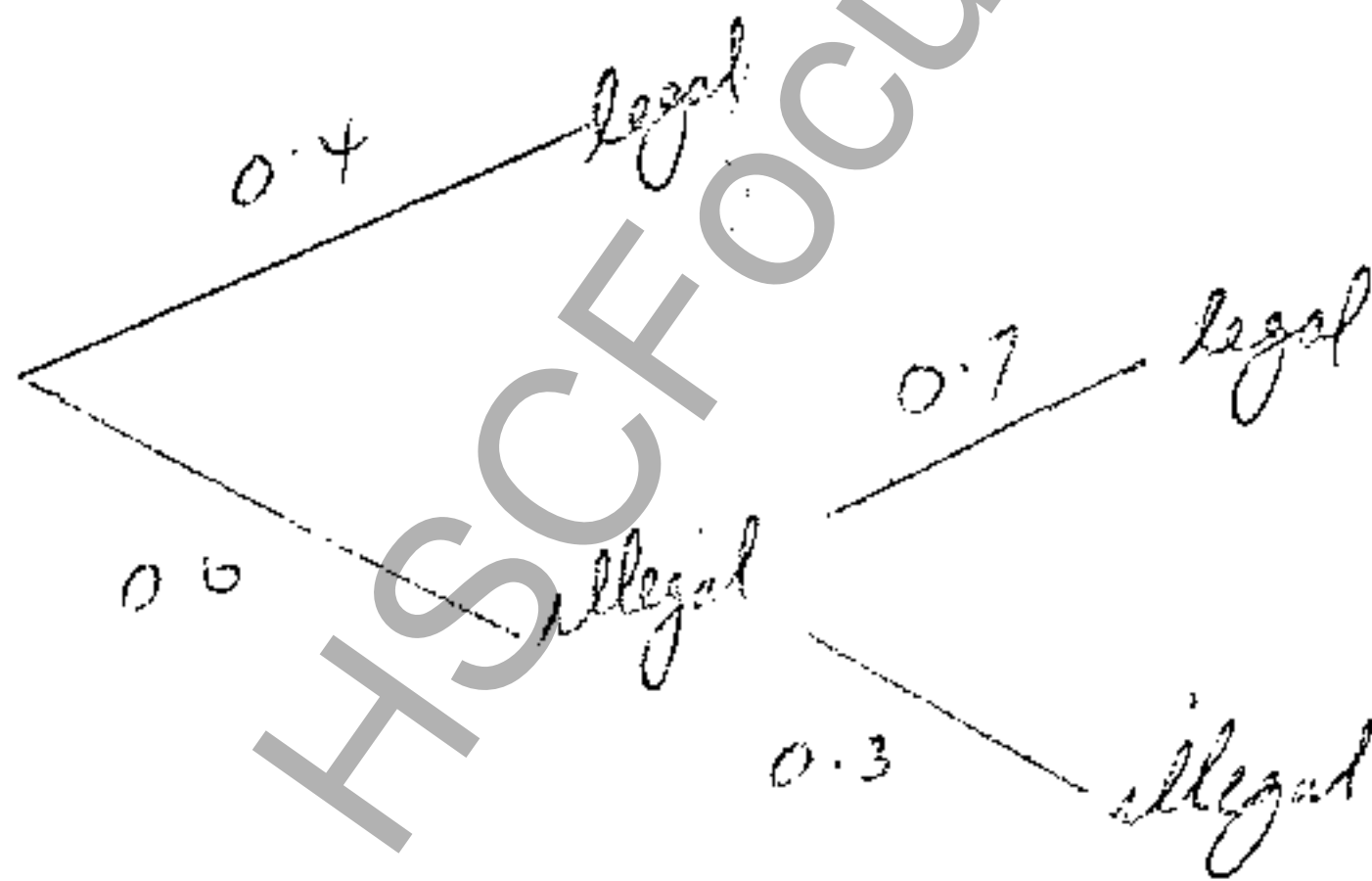


(ii) $x \geq 1$

(iii) see graph.

(c)

(i)



$$P(\text{double fault}) = 0.6 \times 0.3 \\ = 0.18$$

(ii) $(0.82 + 0.18)^6$

$$P(\text{at least 2 double fault}) = 1 - \{ P(0 \text{ double faults}) \\ + P(1 \text{ double fault}) \} \\ = 1 - \{ {}^6C_0 (0.82)^6 (0.18)^0 + {}^6C_1 (0.82)^5 (0.18)^1 \}$$

$$\approx 0.30$$

QUESTION 6.

$$\begin{aligned}(a) \quad \frac{dr}{dt} &= \frac{dr}{dv} \cdot \frac{dv}{dt} \\ &= \frac{1}{4\pi r^2} \cdot 10 \\ &= \frac{10}{4\pi r^2}\end{aligned}$$

$$\begin{aligned}V &= \frac{4}{3}\pi r^3 \\ \frac{dv}{dr} &= 4\pi r^2\end{aligned}$$

$$\text{when } SA = 500 \quad (= 4\pi r^2)$$

$$\begin{aligned}\frac{dr}{dt} &= \frac{10}{500} \\ &= \frac{1}{50} \text{ cm/s.}\end{aligned}$$

$$(b) \quad \text{when } n=1, \quad LHS = 2(1!) \quad RHS = 1(2!) \\ = 2 \quad = 2$$

$\therefore$ true for $n=1$

assume true for $n=k$

$$\text{i.e. } 2(1!) + 5(2!) + \dots + (k^2+1)k! = k(k+1)!$$

to prove true for $n=k+1$

$$\text{i.e. } 2(1!) + 5(2!) + \dots + (k^2+1)k! + [(k+1)^2+1](k+1)! \\ = (k+1)(k+2)!$$

$$\begin{aligned}\text{Now LHS} &= 2(1!) + 5(2!) + \dots + (k^2+1)k! + (k^2+2k+2)(k+1)! \\ &= k(k+1)! + (k^2+2k+2)(k+1)! \quad (\text{by assumption}) \quad (Q??) \\ &= (k+1)! \{ k + k^2 + 2k + 2 \} \\ &= (k+1)! (k^2 + 3k + 2) \\ &= (k+1)! (k+2)(k+1) \\ &= (k+2)! (k+1) \\ &= RHS\end{aligned}$$

$\therefore$ if true for $n=k$ then true for $n=k+1$ &
since true for $n=1$ then true for all
 $n \geq 1$.

Q6(c)

$$\frac{dy}{dx} = \frac{(x)(\frac{1}{x}) - (1)(\ln x)}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

$$\int_e^{e^2} \frac{1 - \ln x}{x \ln x} dx = \int_e^{e^2} \frac{\frac{1 - \ln x}{x^2}}{\frac{\ln x}{x}} dx$$

$$= \left[\ln\left(\frac{\ln x}{x}\right) \right]_e^{e^2}$$

$$= \ln\left(\frac{\ln e^2}{e^2}\right) - \ln\left(\frac{\ln e}{e}\right)$$

$$= \ln\left(\frac{2}{e^2}\right) - \ln\left(\frac{1}{e}\right)$$

$$= \ln\left(\frac{2}{e^2} \times \frac{e}{1}\right)$$

$$= \ln\left(\frac{2}{e}\right)$$

$$= \ln 2 - \ln e$$

$$= \ln 2 - 1$$

QUESTION 7

$$(i) \sin(x+y) - \sin(x-y) = (\sin x \cos y + \cos x \sin y) - (\sin x \cos y - \cos x \sin y)$$

$$= 2 \cos x \sin y \quad (1)$$

$$\text{Let } x+y = A \quad \& \quad x-y = B$$

$$\therefore 2x = A+B$$

$$2y = A-B$$

$$x = \frac{A+B}{2} \quad (1)$$

$$y = \frac{A-B}{2} \quad (1)$$

$$\therefore \sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$(ii) \frac{\sin A - \sin B}{\cos A - \cos B} = \frac{2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)}{2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{B-A}{2}\right)} \quad (1)$$

$$= \frac{\cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)}{-\sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)} \quad (1)$$

$$= -\cot\left(\frac{A+B}{2}\right)$$

$$(iii) \quad \ddot{x} = 0$$

$$\dot{x} = C_1$$

$$t=0, \dot{x} = V \cos \alpha$$

$$\therefore V \cos \alpha = C_1$$

$$\dot{x} = V \cos \alpha$$

$$x = Vt \cos \alpha + C_2$$

$$t=0, x=0 \therefore C_2=0$$

$$x = Vt \cos \alpha \quad (1)$$

$$\ddot{y} = -g$$

$$\dot{y} = -gt + C_3$$

$$t=0, \dot{y} = V \sin \alpha$$

$$V \sin \alpha = C_3$$

$$\dot{y} = -gt + V \sin \alpha$$

$$y = -\frac{1}{2}gt^2 + Vt \sin \alpha + C_4$$

$$t=0, y=0 \therefore C_4=0$$

$$y = -\frac{1}{2}gt^2 + Vt \sin \alpha \quad (2)$$

(iv) (a) Particle P

$$x_P = Vt \cos \alpha$$

$$y_P = -\frac{1}{2}gt^2 + Vt \sin \alpha$$

Particle Q

$$x_Q = Vt \cos \beta$$

$$y_Q = -\frac{1}{2}gt^2 + Vt \sin \beta$$

$$\tan \theta = \text{slope PR}$$

$$= \left| \frac{(-\frac{1}{2}gt^2 + Vt \sin \beta) - (-\frac{1}{2}gt^2 + Vt \sin \alpha)}{Vt \cos \beta - Vt \cos \alpha} \right| \quad (3)$$

$$= \left| \frac{Vt (\sin \beta - \sin \alpha)}{Vt (\cos \beta - \cos \alpha)} \right| \quad (4)$$

$$= \left| \frac{\sin \beta - \sin \alpha}{\cos \beta - \cos \alpha} \right| \quad (5)$$

$$(b) \quad \tan \theta = \left| -\cot\left(\frac{\alpha+\beta}{2}\right) \right| \quad \text{from (ii)} \quad (6)$$

$$= \tan\left(\frac{\pi}{2} - \left(\frac{\alpha+\beta}{2}\right)\right) \quad (\because \alpha, \beta \text{ are acute})$$

$$\therefore \theta = \frac{\pi}{2} - \left(\frac{\alpha+\beta}{2}\right) \quad (7)$$

$$\theta = \frac{1}{2}(\pi - \alpha - \beta)$$