

2008
Higher School Certificate
Trial Examination

Mathematics

Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- Write using black or blue pen
- Write your student number and/or name at the top of every page
- All necessary working should be shown in every question
- A table of standard integrals is provided

Total marks – 84

Attempt Questions 1 – 7

All questions are of equal value

This paper MUST NOT be removed from the examination room

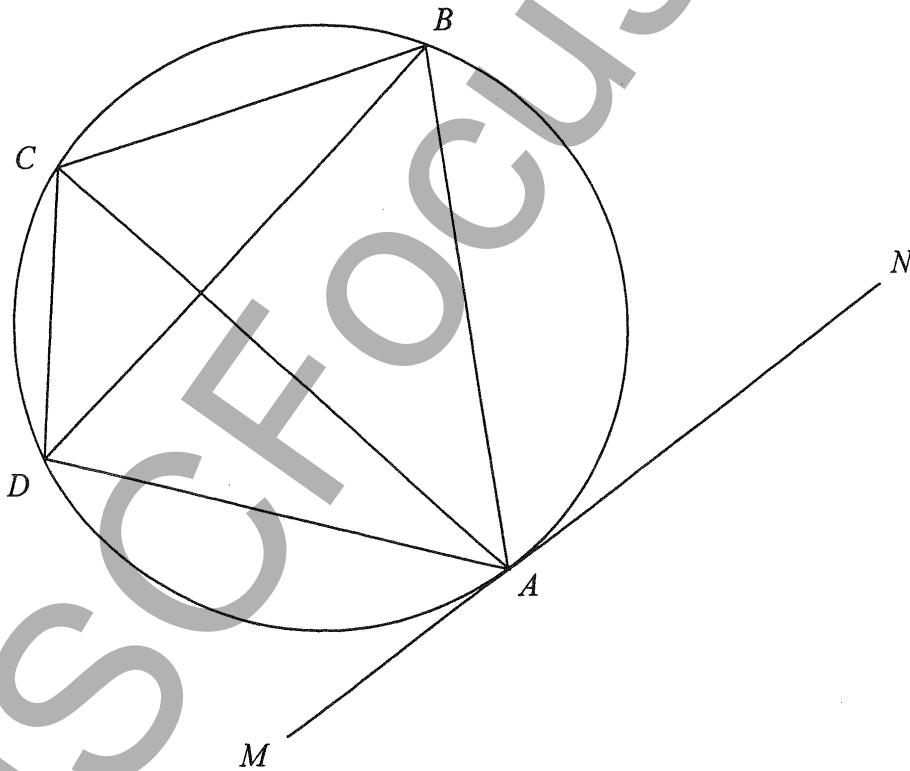
STUDENT NUMBER/NAME.....

Question 1

Begin a new booklet

- (a) Find the value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x \tan x \, dx$, giving the answer in simplest exact form. 2
- (b) Solve the inequality $\frac{x^2 - 4}{x} \geq 0$. 3
- (c)(i) Find the gradients at the point $P(1, 1)$ of the tangents to the curves $y = x^3$ and $y = 1 - \ln x$. 2
- (ii) Hence find the acute angle between these tangents, giving the answer correct to the nearest degree. 1

(d)



$ABCD$ is a cyclic quadrilateral. MAN is the tangent at A to the circle through A, B, C and D . CA bisects $\angle BCD$.

Copy the diagram. Show that $MAN \parallel DB$, giving reasons.

4

Question 2

Begin a new booklet

- (a) Find the set of values of x for which the limiting sum of the geometric series $1 + \ln x + (\ln x)^2 + \dots$ exists. 2
- (b) $A(8, \sqrt{8})$ and $B(50, \sqrt{50})$ are two points. Find the coordinates of the point $P(x, y)$ which divides the interval AB internally in the ratio $2 : 1$, giving the answer in simplest exact form. 3
- (c)(i) Express $\cos x - \sqrt{3} \sin x$ in the form $R \cos(x + \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. 2
- (ii) Hence solve $\cos x - \sqrt{3} \sin x = -2$ for $0 \leq x \leq 2\pi$. 1
- (d) $T(2t, t^2)$ is a point on the parabola $x^2 = 4y$.
- (i) Use differentiation to show that the tangent to the parabola at T has gradient t and equation $tx - y - t^2 = 0$. 2
- (ii) Hence find the values of t such that the tangent to the parabola at T passes through the point $P(1, -2)$. 2

Question 3

Begin a new booklet

- (a) Find $\frac{d}{dx}(x \sin^{-1} x)$. 2
- (b) Consider the function $f(x) = x + \frac{1}{x}$ for $x \geq 1$.
- (i) Show that the function $f(x)$ is increasing and the curve $y = f(x)$ is concave up for all values of $x > 1$. 2
- (ii) On the same diagram, sketch the graphs of $y = f(x)$ and the inverse function $y = f^{-1}(x)$ showing the coordinates of the endpoints and the equation of the asymptote. 2
- (iii) Find the equation of the inverse function $y = f^{-1}(x)$ in its simplest form. 2
- (c) Use Mathematical induction to show that for all positive integers $n \geq 1$, 4
- $$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

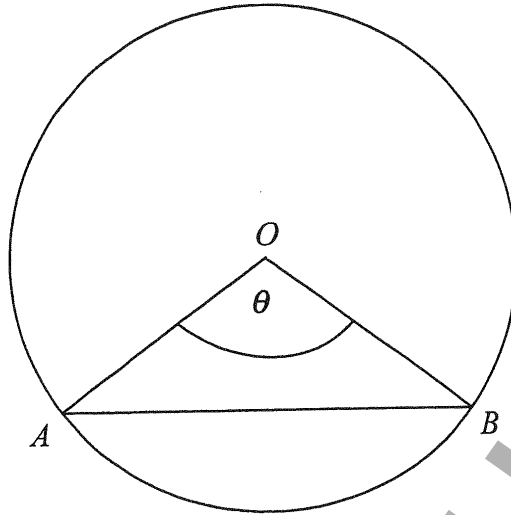
Question 4

Begin a new booklet

(a) Find $\int \cos^2 4x \, dx$.

2

(b)



AB is a chord of a circle of radius 1 metre that subtends an angle θ at the centre of the circle, where $0 < \theta < \pi$. The perimeter of the minor segment cut off by AB is equal to the diameter of the circle.

(i) Show that $\theta + 2 \sin \frac{1}{2}\theta - 2 = 0$.

2

(ii) Show that the value of θ is such that $1 < \theta < 2$.

2

(iii) Use one application of Newton's method with an initial approximation of $\theta_0 = 1$ to find the next approximation to the value of θ , giving your answer correct to 1 decimal place.

2

(c) Use the substitution $x = u^2$, $u \geq 0$, to evaluate $\int_1^{25} \frac{1}{x + \sqrt{x}} \, dx$, giving the answer in simplest exact form.

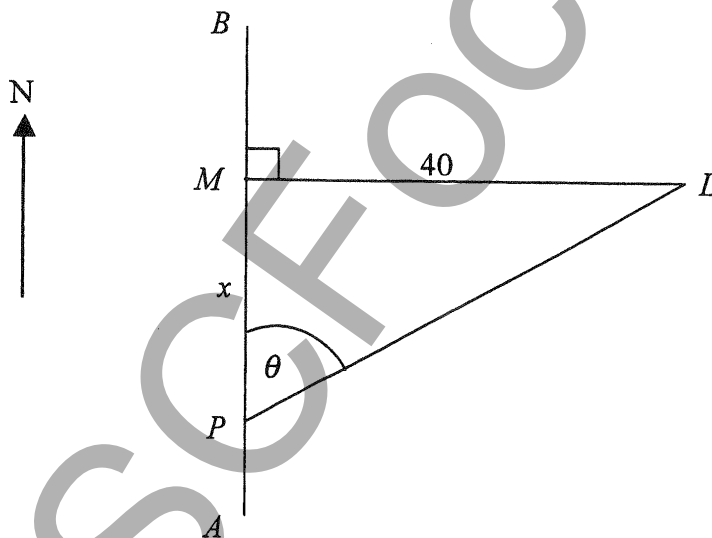
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Question 5

Begin a new booklet

- (a) Consider the polynomial $P(x) = x^3 - kx^2 + kx - 1$, where k is a real constant.
- (i) Show that 1 is a root of the equation $P(x) = 0$. 1
- (ii) Given that α , $\alpha \neq 1$, is a second root of $P(x) = 0$, show that $\frac{1}{\alpha}$ is also a root of the equation. 1
- (iii) Show that $\alpha^2 + \frac{1}{\alpha^2} = k^2 - 2k - 1$. 2
- (b) Three numerals are chosen at random from the numerals 1, 2, 3, ..., 9 to form a three digit code where order is important and repetition is allowed.
- (i) Find the probability that all three digits of the code are different. 2
- (ii) Find the probability that exactly two digits of the code are the same. 2

(c)



A boat is sailing due North from point A to point B at a steady speed of 5 ms^{-1} . A marker buoy M on its route is situated 40 metres due West of a lighthouse L . When the boat is at point P at a distance x metres from M , the bearing of the lighthouse from the boat is θ , $0 < \theta < \frac{\pi}{2}$.

- (i) Show that $\theta = \tan^{-1} \frac{40}{x}$. 1
- (ii) Hence find the rate at which θ is changing when $x = 20$. 3

Question 6

Begin a new booklet

- (a) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity $v \text{ ms}^{-1}$ given by $v = 2 - x$ and acceleration $a \text{ ms}^{-2}$. Initially the particle is 4 metres to the left of O .
- (i) Find an expression for a in terms of x . 1
- (ii) Use integration to show that $x = 2 - 6e^{-t}$. 2
- (iii) Find the exact time taken by the particle to travel 4 metres from its starting point. 1
- (iv) Sketch the graph of x against t showing the intercepts on the axes and the equations of any asymptotes. 2
- (b) A particle is performing Simple Harmonic Motion on a straight line. At time t seconds it has displacement x metres from a fixed point O on the line given by $x = 2 + 2\cos 2t$.
- (i) Sketch the graph of x against t showing the intercepts on the axes. 2
- (ii) Show that the acceleration of the particle is given by $\ddot{x} = -4(x - 2)$. 1
- (iii) Find the period of the motion. 1
- (iv) Find the distance travelled by the particle in the first 2 seconds of its motion, giving the answer correct to two significant figures. 2

Question 7

Begin a new booklet

- (a) A particle is projected from a point O with speed $V \text{ ms}^{-1}$ at an angle θ above the horizontal where $0 < \theta < \frac{\pi}{2}$. The particle moves in a vertical plane under gravity where the acceleration due to gravity is $g \text{ ms}^{-2}$. At time t seconds its horizontal and vertical displacements from O are x metres and y metres respectively.
- (i) Write down expressions for x and y in terms of V , θ , g and t . Hence show that the horizontal range R of the particle is given by $R = \frac{V^2 \sin 2\theta}{g}$. 2
- (ii) A lawn on horizontal ground is rectangular in shape with length 50 metres and breadth 20 metres. A garden sprinkler is located at one corner S of the lawn. It rotates horizontally, and delivers water at a speed of 20 ms^{-1} at angles of elevation between 15° and 45° above the horizontal. Taking $g = 10$, find the area of the lawn that can be watered by the sprinkler, giving the answer in simplest exact form. 4
- (b)(i) Write down the binomial expansion of $(1+x)^n$ in ascending powers of x . 1
- (ii) Show that $\sum_{r=1}^n {}^nC_r = 2^n - 1$. 1
- (iii) Use integration and the answer to part (i) to show that $\frac{1}{n+1} \sum_{r=1}^{n+1} {}^{n+1}C_r = \sum_{r=0}^n \frac{{}^nC_r}{r+1}$. 4

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• writes primitive and substitutes for x	1
• evaluates in simplest surd form	1

Answer

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x \tan x \, dx = \left[\sec x \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \sec \frac{\pi}{3} - \sec \frac{\pi}{4} = 2 - \sqrt{2}$$

b. Outcomes assessed : PE3

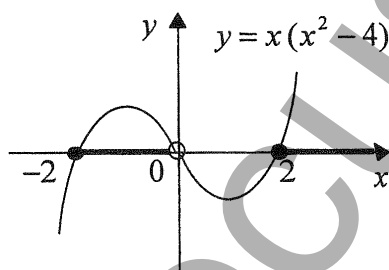
Marking Guidelines

Criteria	Marks
• writes an equivalent inequality not involving a variable denominator	1
• writes one inequality for x	1
• combines this with a second inequality for x	1

Answer

$$\frac{x^2 - 4}{x} \geq 0$$

$$x(x^2 - 4) \geq 0, \quad x \neq 0$$



By inspection of the graph,
 $-2 \leq x < 0$ or $x \geq 2$

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • finds gradient of tangent to $y = x^3$ at P	1
• finds gradient of tangent to $y = 1 - \ln x$ at P	1
ii • finds the acute angle between the lines correct to the nearest degree	1

Answer

i.

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$x = 1 \Rightarrow \frac{dy}{dx} = 3$$

Tangent at $P(1, 1)$ has gradient 3

$$y = 1 - \ln x$$

$$\frac{dy}{dx} = -\frac{1}{x}$$

$$x = 1 \Rightarrow \frac{dy}{dx} = -1$$

Tangent at $P(1, 1)$ has gradient -1

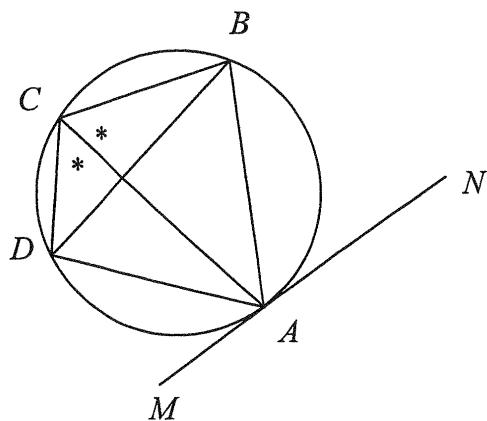
$$\text{ii. } \tan \theta = \left| \frac{3 - (-1)}{1 + 3 \times (-1)} \right| = 2 \Rightarrow \theta \approx 63^\circ \text{ (to the nearest degree)}$$

d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
• explains why angles BAN , BCA are equal	1
• explains why angles DCA , DBA are equal	1
• uses the equality of angles BCA , DCA to complete proof that angles BAN , DBA are equal	1
• quotes test for parallel lines to deduce tangent MAN is parallel to BD	1

Answer



$\angle BAN = \angle BCA$ (angle between tangent and chord drawn to point of contact is equal to angle subtended by the chord in the alternate segment)
 $\angle BCA = \angle DCA$ (given that AC bisects $\angle BCD$)
 $\angle DCA = \angle DBA$ (angles subtended at the circumference by the same arc DA are equal)
 $\therefore MAN \parallel DB$ (equal alternate angles on transversal BA since $\angle BAN = \angle DBA$)

Question 2

a. Outcomes assessed : H3, H5

Marking Guidelines

Criteria	Marks
• writes condition on common ratio $\ln x$ for existence of limiting sum	1
• solves this inequality for x	1

Answer

Limiting sum of geometric series $1 + \ln x + (\ln x)^2 + \dots$ exists for $-1 < \ln x < 1$

$\therefore$ since $f(x) = e^x$ is an increasing function,

$$e^{-1} < e^{\ln x} < e^1$$

$$\therefore \frac{1}{e} < x < e$$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• finds x coordinate of P	1
• finds y coordinate of P as the sum of two surds	1
• simplifies surd expression for y	1

Answer

$$\begin{array}{cc}
 A(8, \sqrt{8}) & B(50, \sqrt{50}) \\
 \hline
 2 & : \quad 1 \\
 \hline
 \left(\frac{100+8}{2+1}, \frac{2\sqrt{50} + \sqrt{8}}{2+1} \right)
 \end{array}$$

$$\begin{aligned}
 \therefore P \left(36, \frac{10\sqrt{2} + 2\sqrt{2}}{3} \right) \\
 P(36, 4\sqrt{2})
 \end{aligned}$$

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • finds value of R	1
• finds value of α	1
ii • solves equation for x	1

Answer

$$\begin{aligned} \text{i. } \cos x - \sqrt{3} \sin x &= 2 \left(\frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x \right) \\ &= 2 \left(\cos \frac{\pi}{3} \cos x - \sin \frac{\pi}{3} \sin x \right) \\ &= 2 \cos \left(x + \frac{\pi}{3} \right) \end{aligned}$$

$$\begin{aligned} \text{ii. } \cos x - \sqrt{3} \sin x &= -2, \quad 0 \leq x \leq 2\pi \\ \cos \left(x + \frac{\pi}{3} \right) &= -1, \quad \frac{\pi}{3} \leq x + \frac{\pi}{3} \leq 2\pi + \frac{\pi}{3} \\ x + \frac{\pi}{3} &= \pi \\ x &= \frac{2\pi}{3} \end{aligned}$$

d. Outcomes assessed : PE3, PE4

Marking Guidelines

Criteria	Marks
i • shows by differentiation that tangent has gradient t	1
• finds the equation of the tangent	1
ii • substitutes coordinates of P to write equation for t	1
• solves equation for t	1

Answer

$$\begin{aligned} \text{i. } y &= t^2 \Rightarrow \frac{dy}{dt} = 2t \\ x &= 2t \Rightarrow \frac{dx}{dt} = 2 \\ \therefore \frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt} = t \end{aligned}$$

Tangent at $T(2t, t^2)$ has gradient t

and equation $y - t^2 = t(x - 2t)$

$$y - t^2 = tx - 2t^2$$

$$tx - y - t^2 = 0$$

ii. $P(1, -2)$ lies on this tangent if

$$t + 2 - t^2 = 0$$

$$t^2 - t - 2 = 0$$

$$(t - 2)(t + 1) = 0$$

$$t = 2 \text{ or } t = -1$$

Question 3

a. Outcomes assessed : PE5, HE4

Marking Guidelines

Criteria	Marks
• applies the product rule, obtaining first term	1
• obtains second term by deriving inverse sine	1

Answer

$$\frac{d}{dx}(x \sin^{-1} x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$$

b. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
i • shows $f(x)$ is increasing for $x > 1$	1
• shows the curve $y = f(x)$ is concave up for $x > 1$	1
ii • sketches $y = f(x)$ showing endpoint and asymptote $y = x$	1
• sketches $y = f^{-1}(x)$ showing endpoint and asymptote	1
iii • makes x the subject	1
• interchanges x and y to find equation of inverse function	1

Answer

i. $f(x) = x + \frac{1}{x}$ for $x \geq 1$

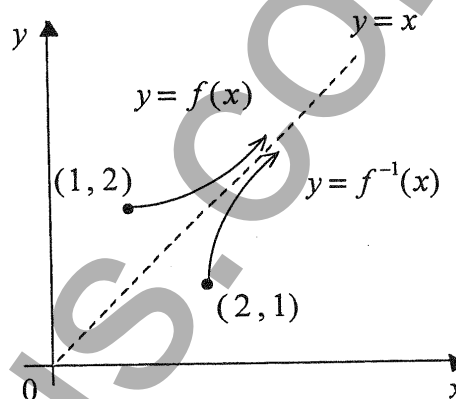
$$f'(x) = 1 - \frac{1}{x^2} > 0 \text{ for } x > 1$$

$\therefore f(x)$ is increasing for $x > 1$

$$f''(x) = \frac{2}{x^3} > 0 \text{ for } x > 1$$

$\therefore y = f(x)$ is concave up for $x > 1$

ii.



iii. $y = x + \frac{1}{x}$, $x \geq 1$ and $y \geq 2$

Rearrangement gives $x^2 - xy + 1 = 0$, $x \geq 1$ and $y \geq 2$

Considering this quadratic in x : $x = \frac{y \pm \sqrt{y^2 - 4}}{2}$, $x \geq 1$ and $y \geq 2$

Clearly the branch $x = \frac{y - \sqrt{y^2 - 4}}{2}$ contains points for which $x < 1$.

Hence expressing x as the subject of $y = f(x)$, $x = \frac{y + \sqrt{y^2 - 4}}{2}$, $y \geq 2$.

Interchanging x and y , the inverse function is $f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$, $x \geq 2$

c. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• verifies truth of statement for $n = 1$	1
• expresses LHS of $S(k+1)$ in terms of LHS of $S(k)$	1
• expresses LHS of $S(k+1)$ in terms of RHS of $S(k)$, conditional on truth of $S(k)$	1
• completes algebraic rearrangement to show $S(k+1)$ is true if $S(k)$ is true	1

Answer

Define the sequence of statements $S(n)$, $n = 1, 2, 3, \dots$ by $S(n): \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Consider $S(1)$: $LHS = \frac{1}{2!} = \frac{1}{2}$ $RHS = 1 - \frac{1}{2!} = 1 - \frac{1}{2} = \frac{1}{2}$ $\therefore S(1)$ is true

If $S(k)$ is true: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$ **

Consider $S(k+1)$: $LHS = \left\{ \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} \right\} + \frac{k+1}{(k+2)!}$
 $= \left\{ 1 - \frac{1}{(k+1)!} \right\} + \frac{k+1}{(k+2)!}$ if $S(k)$ is true, using **
 $= 1 - \frac{k+2}{(k+2)(k+1)!} + \frac{k+1}{(k+2)!}$
 $= 1 - \frac{k+2-(k+1)}{(k+2)!}$
 $= 1 - \frac{1}{(k+2)!}$
 $= RHS$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n \geq 1$.

Question 4

a. Outcomes assessed : H8

Marking Guidelines

Criteria	Marks
• expresses integrand in terms of $\cos 8x$	1
• finds primitive function	1

Answer

$$\int \cos^2 4x \, dx = \int \frac{1}{2}(1 + \cos 8x) \, dx = \frac{1}{2}x + \frac{1}{16}\sin 8x + c$$

b. Outcomes assessed : H5, PE3

Marking Guidelines

Criteria	Marks
i • uses cosine rule and trigonometric identity to find AB in terms of $\sin \frac{1}{2}\theta$	1
• adds arc length to AB , equating sum and diameter to obtain required equation	1
ii • shows $f(1)$ and $f(2)$ have opposite signs	1
• uses continuity of $f(\theta)$ to deduce equation has root between 1 and 2.	1
iii • applies Newton's rule, substituting $\theta = 1$	1
• evaluates expression to obtain next approximation, giving result correct to 1 dec. place	1

Answer

i. Using cosine rule, $AB^2 = 1^2 + 1^2 - 2 \cos \theta$

$$AB^2 = 2(1 - \cos \theta)$$

$$= 4 \sin^2 \frac{1}{2} \theta$$

$$\therefore AB = 2 \sin \frac{1}{2} \theta$$

$$\therefore \text{Perimeter} = \text{diameter} \Rightarrow \theta + 2 \sin \frac{1}{2} \theta = 2$$

$$\theta + 2 \sin \frac{1}{2} \theta - 2 = 0$$

iii. $f(\theta) = \theta + 2 \sin \frac{1}{2} \theta - 2$

$$f'(\theta) = 1 + \cos \frac{1}{2} \theta$$

$$\theta_1 = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)}$$

$$\theta_1 = 1 - \frac{-1 + 2 \sin \frac{1}{2}}{1 + \cos \frac{1}{2}}$$

$$\therefore \theta_1 \approx 1.0 \text{ (to 1 dec. place)}$$

c. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes dx in terms of du	1
• writes integrand in terms of u and changes limits to u values	1
• finds primitive function	1
• evaluates in simplest exact form	1

Answer

$$x = u^2, u \geq 0$$

$$dx = 2u du$$

$$x = 1 \Rightarrow u = 1$$

$$x = 25 \Rightarrow u = 5$$

$$\begin{aligned} \int_1^{25} \frac{1}{x + \sqrt{x}} dx &= \int_1^5 \frac{1}{u(u+1)} \cdot 2u du \\ &= 2 \left[\ln(u+1) \right]_1^5 \\ &= 2(\ln 6 - \ln 2) \\ &= 2 \ln 3 \end{aligned}$$

Question 5

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • shows $P(1) = 0$	1
ii • uses product of roots is 1 to deduce 3 rd root is reciprocal of α	1
iii • writes sum of squares of roots in terms of square of sum and sum of two-way products	1
• uses relationships between coefficients of polynomial equation and its roots	1

Answer

i. $P(x) = x^3 - kx^2 + kx - 1$

$$P(1) = 1 - k + k - 1 = 0$$

ii. Product of the roots is 1.

Hence if the roots are 1, α , β ,

then $\alpha\beta = 1$. $\therefore \frac{1}{\alpha}$ is the 3rd root.

iii. $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$\therefore \alpha^2 + \frac{1}{\alpha^2} + 1^2 = k^2 - 2k$$

$$\therefore \alpha^2 + \frac{1}{\alpha^2} = k^2 - 2k - 1$$

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • counts the number of codes with all three digits different	1
• divides by the total number of codes to find the probability	1
ii • counts the number of codes with exactly two digits the same	1
• writes the probability of such a code	1

Answer

$$i. P(\text{all different}) = \frac{9 \times 8 \times 7}{9 \times 9 \times 9} = \frac{56}{81}$$

ii. Consider code of form A, A, B or A, B, A or B, A, A
Number of such codes is $9 \times 8 \times 3$

$$P(\text{exactly two the same}) = \frac{9 \times 8 \times 3}{9 \times 9 \times 9} = \frac{8}{27}$$

c. Outcomes assessed : HE4, HE5

Marking Guidelines

Criteria	Marks
i • finds θ in terms of x	1
ii • derives θ with respect to x	1
• finds the derivative of θ with respect to t in terms of x	1
• states the rate at which θ is changing when $x = 20$	1

Answer

$$i. \tan \theta = \frac{40}{x}, \quad 0 < \theta < \frac{\pi}{2}$$

$$\therefore \theta = \tan^{-1} \frac{40}{x}$$

$$ii. \frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{1}{1 + \frac{1600}{x^2}} \cdot \frac{-40}{x^2} \cdot 5$$

$$= \frac{-200}{x^2 + 1600}$$

$$\therefore x = 20 \Rightarrow \frac{d\theta}{dt} = -\frac{1}{10}$$

θ is decreasing at a rate of 0.1 radians per second.

Question 6

a. Outcomes assessed : H3, HE3, HE5

Marking Guidelines

Criteria	Marks
i • finds a in terms of x	1
ii • finds t as a function of x by integration	1
• rearranges to find x as a function of t	1
iii • finds t when $x = 0$	1
iv • shows intercepts on the axes	1
• shows asymptote $x = 2$	1

Answer

i. $v = 2 - x$

$$a = v \frac{dv}{dx}$$

$$= (2 - x) \cdot (-1)$$

$$= x - 2$$

ii. $\frac{dx}{dt} = 2 - x$

$$\frac{dt}{dx} = \frac{1}{2 - x}$$

$$t = -\ln A(2 - x), \quad A \text{ constant}$$

$$\left. \begin{array}{l} t = 0 \\ x = -4 \end{array} \right\} \Rightarrow \begin{array}{l} 6A = 1 \\ A = \frac{1}{6} \end{array}$$

$$\therefore -t = \ln \left(\frac{2 - x}{6} \right)$$

$$e^{-t} = \frac{2 - x}{6}$$

$$6e^{-t} = 2 - x$$

$$\therefore x = 2 - 6e^{-t}$$

iii. When $t = 0$, $x = -4 \therefore v > 0$

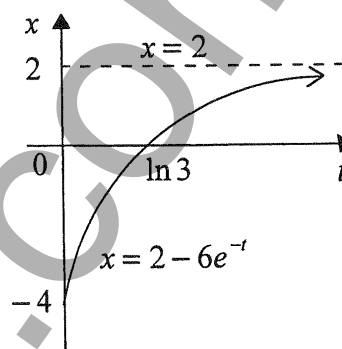
Particle is initially moving right, and it continues moving right approaching $x = 2$.

Hence particle has travelled 4 metres from its starting point when $x = 0$.

$$x = 0 \Rightarrow -t = \ln \frac{1}{3}$$

$\therefore$ particle travels first 4 metres in $\ln 3$ seconds.

iv.



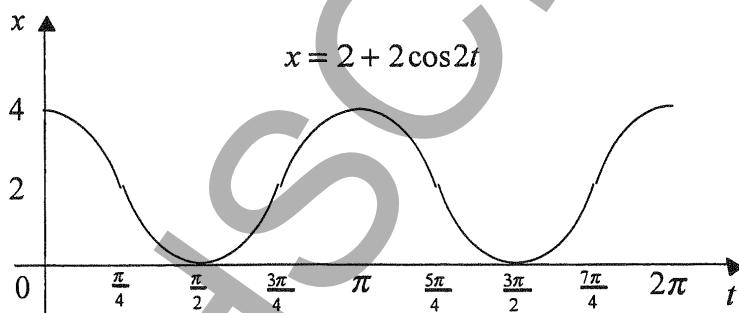
b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • sketches curve with correct shape and position showing intercept on x axis	1
• shows intercepts on t axis for at least one period	1
ii • differentiates to find $\ddot{x}$ as a function of t , and hence as a function of x	1
iii • states the period of the motion	1
iv • finds x when $t = 2$	1
• states the distance travelled in the first 2 seconds	1

Answer

i.



ii. $\dot{x} = -4 \sin 2t$

$$\begin{aligned} \ddot{x} &= -4(2 \cos 2t) \\ &= -4(x - 2) \end{aligned}$$

iii. Period is π seconds

iv. $t = 2 \Rightarrow x = 2 + 2 \cos 4 \approx 0.69$

But $\frac{\pi}{2} < 2 < \frac{3\pi}{4}$. Hence by inspection of the graph, particle has travelled 4.7 m (correct to 2 sig. fig.)

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes expressions for x and y	1
• finds x when $y = 0$ and hence required expression for R	1
ii • calculates R for $V = 20$ when $\theta = 15^\circ$, $\theta = 45^\circ$	1
• identifies region that can be watered	1
• finds the area of at least part of this region	1
• finds the total area in simplest exact form	1

Answer

i. $x = Vt \cos \theta$ $y = -\frac{1}{2}gt^2 + Vt \sin \theta$
 $x = R$ when $y = 0$ and $t \neq 0$

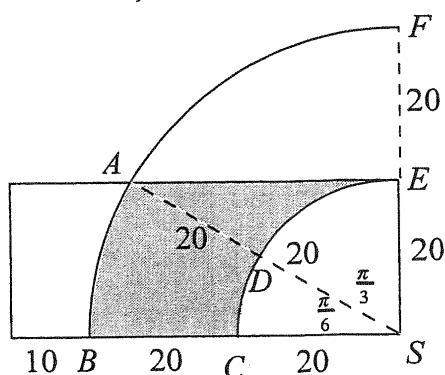
$$y = 0, t \neq 0 \Rightarrow V \sin \theta = \frac{1}{2}gt$$

$$t = \frac{2V \sin \theta}{g}$$

$$\therefore R = \frac{V^2(2 \sin \theta \cos \theta)}{g} = \frac{V^2 \sin 2\theta}{g}$$

ii. $V = 20$, $\theta = 15^\circ \Rightarrow R = 20$

$V = 20$, $\theta = 45^\circ \Rightarrow R = 40$



The area of lawn that can be watered is shaded on the diagram.

Since $\cos \angle ESA = \frac{20}{40}$, $\angle ESA = \frac{\pi}{3}$ and $\angle ASB = \frac{\pi}{6}$.

$$\begin{aligned} \text{Area} &= \text{Sector } ABS + \triangle AES - \text{Quadrant } ECS \\ &= \frac{1}{2} \times 40^2 \times \frac{\pi}{6} + \frac{1}{2} \times 40 \times 20 \sin \frac{\pi}{3} - \frac{1}{4} \times \pi \times 20^2 \\ &= 100 \times \frac{\pi}{3} + 200\sqrt{3} \end{aligned}$$

Area is $100\left(\frac{\pi}{3} + 2\sqrt{3}\right)$ square metres.

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes binomial expansion	1
ii • substitutes $x = 1$ to deduce required result	1
iii • finds primitive of LHS of i.	1
• finds primitive of RHS of i.	1
• evaluates definite integrals of LHS and RHS between limits 0 and 1	1
• uses result from ii. to deduce required result	1

Answer

i.

$$(1+x)^n \equiv {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$$

ii.

$$x = 1 \Rightarrow 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$$

But ${}^nC_0 = 1$ $\therefore \sum_{r=1}^n {}^nC_r = 2^n - 1$

iii.

$$\left[\frac{1}{n+1} (1+x)^{n+1} \right]_0^1 = \left[{}^nC_0x + {}^nC_1 \frac{1}{2}x^2 + \dots + {}^nC_n \frac{1}{n+1}x^{n+1} \right]_0^1$$

$$\frac{1}{n+1} (2^{n+1} - 1) = {}^nC_0 + \frac{1}{2} {}^nC_1 + \frac{1}{3} {}^nC_2 + \dots + \frac{1}{n+1} {}^nC_n$$

$$\therefore \frac{1}{n+1} \sum_{r=1}^{n+1} {}^{n+1}C_r = \sum_{r=0}^n \frac{{}^nC_r}{r+1} \quad (\text{using ii. with } n \rightarrow n+1)$$

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1 a	2	Trigonometric functions	H5	E2-E3
b	3	Inequalities	PE3	E2-E3
c i	2	Gradient of a tangent to a curve; Logarithmic functions	H5	E2-E3
ii	1	Angle between two lines	H5	E2-E3
d	4	Circle geometry	PE2, PE3	E2-E3
2 a	2	Series; Exponential and logarithmic functions	H3, H5	E2-E3
b	3	Division of an interval	H5	E2-E3
c i	2	Further trigonometry	H5	E2-E3
ii	1	Further trigonometry	H5	E2-E3
d i	2	Parametric representation	PE4	E2-E3
ii	2	Parametric representation	PE3	E2-E3
3 a	2	Rules of differentiation; Inverse trigonometric functions	PE5, HE4	E2-E3
b i	2	Geometrical applications of differentiation	H5	E2-E3
ii	2	Inverse functions	HE4	E2-E3
iii	2	Inverse functions	HE4	E2-E3
c	4	Mathematical induction	HE2	E3-E4
4 a	2	Integration	H8	E2-E3
b i	2	Trigonometric functions	H5	E2-E3
ii	2	Polynomials	PE3	E2-E3
iii	2	Iterative methods	PE3	E2-E3
c	4	Methods of integration	HE6	E2-E3
5 a i	1	Polynomials	PE3	E2-E3
ii	1	Polynomials	PE3	E2-E3
iii	2	Polynomials	PE3	E2-E3
b i	2	Further probability	HE3	E2-E3
ii	2	Further probability	HE3	E2-E3
c i	1	Inverse trigonometric functions	HE4	E2-E3
ii	3	Rates of change	HE5	E2-E3
6 a i	1	Velocity, acceleration as functions of x	HE5	E3-E4
ii	2	Velocity, acceleration as functions of x	HE5	E3-E4
iii	1	Exponential and logarithmic functions	H3	E3-E4
iv	2	Exponential growth and decay	HE3	E3-E4
b i	2	Simple harmonic motion	HE3	E3-E4
ii	1	Simple harmonic motion	HE3	E3-E4
iii	1	Simple harmonic motion	HE3	E3-E4
iv	2	Simple harmonic motion	HE3	E3-E4
7 a i	2	Projectile motion	HE3	E3-E4
ii	4	Projectile motion	HE3	E3-E4
b i	1	Binomial expansion	HE3	E3-E4
ii	1	Binomial expansion	HE3	E3-E4
iii	4	Binomial expansion	HE3	E3-E4