ASCHAM SCHOOL

YEAR 12
MATHEMATICS EXTENSION 1
TRIAL EXAMINATION 2011

TIME ALLOWED: 2 HOURS PLUS 5 MINUTES' READING TIME

INSTRUCTIONS

ALL QUESTIONS MAY BE ATTEMPTED.
ALL QUESTIONS ARE OF EQUAL VALUE (12 MARKS).
ALL NECESSARY WORKING MUST BE SHOWN.
MARKS MAY NOT BE AWARDED FOR CARELESS WORK.
APPROVED CALCULATORS AND TEMPLATES MAY BE USED.

COLLECTION

START EACH QUESTION IN A NEW BOOKLET.
IF YOU USE A SECOND BOOKLET FOR A QUESTION, PLACE IT INSIDE THE FIRST.
WRITE YOUR NAME/NUMBER, TEACHER'S NAME AND QUESTION NUMBER ON EACH BOOKLET.

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1

a) Find the value of $\tan^{-1}\sqrt{3}$ in radians **(1)**

b) If α, β and γ are the roots of the equations $x^3 - 2x + 5 = 0$, find the value of $\alpha\beta\gamma$ **(1)**

c) Find $\frac{d}{dx}(\sin^{-1} 2x)$ **(2)**

d) Find the remainder when $P(x) = x^3 - 2x^2 - 2x + 1$ is divided by $x - 2$ **(2)**

e) Find $\int \cos^2 4x \, dx$ **(2)**

f) Find the co-ordinates of the point P which divides the interval joining A(-3,4) and B(2,-8) externally in the ratio 2:5. **(2)**

g) Find the acute angle between the lines $y = -x$ and $\sqrt{3}y = x$ **(2)**

Question 2 Begin a new booklet

a) Find $\int \frac{dx}{4x^2 + 1}$ **(2)**

b) Differentiate $4\sec^2 x$ **(2)**

c) Find the general solution to $\cos x = -\frac{\sqrt{3}}{2}$ **(2)**

d) Use the substitution $x = \cos\theta$ to evaluate **(4)**

$$\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\, dx$$

e) Find the Cartesian equation of the curve with parametric equations $x = \sin t$ and $y = 2 + \cos t$ **(2)**

Question 3 **Begin a new booklet**

a) Find the exact value of $\sin\left(2\sin^{-1}\frac{3}{4}\right)$ **(2)**

b) Solve $\ln(2x+3)+\ln(x-2)=2\ln(x+4)$ **(3)**

c) Solve the inequation $\frac{2x+1}{x-2}\geq 1$ **(3)**

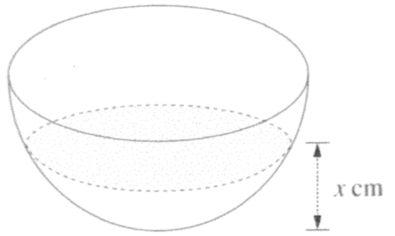
d) A particle is moving in a straight line so that its displacement x from the origin at time t, in seconds, is given by $x=\sqrt{3}\cos 2t-\sin 2t$, $t\geq 0$

i) Show that the particle moves in simple harmonic motion **(2)**

ii) Find the velocity the first time the particle is at the origin. **(2)**

Question 4 **Begin a new booklet**

a)



A hemispherical bowl of radius r cm is initially empty. Water is poured into it at a constant rate of k cm^3 per minute. When the depth of the water in the bowl is x cm, the volume, V cm^3, of the water in the bowl is given by

$$V = \frac{\pi}{3}x^2(3r - x) \qquad \textit{(do not prove this)}$$

i) Show that $\dfrac{dx}{dt} = \dfrac{k}{\pi x(2r - x)}$ **(2)**

ii) Find an expression for t as a function of x **(2)**

b) A function is defined as $f(x) = 1 - \cos\dfrac{x}{2}$ where $0 \le x \le a$

i) Find the largest value of a for which the inverse function $f^{-1}(x)$ exists. **(2)**

ii) Show that $f^{-1}(x) = 2\cos^{-1}(1 - x)$ **(2)**

iii) Sketch the graph of y = $f^{-1}(x)$ **(2)**

iv) Find the area enclosed between the curve y = $f^{-1}(x)$, the x axis and $x = 2$. **(2)**

<u>Question 5</u>

a) A ball is projected from a point 6m above the ground at an angle of 30^0 from the horizontal with velocity of V metres per second. The equations for acceleration in the horizontal and vertical direction are given by $x = 0$ *and* $y = -g$ respectively.

i) Using Calculus show that $x = \frac{Vt\sqrt{3}}{2}$ and $y = \frac{-gt^2}{2} + \frac{vt}{2} + 6$ **(3)**

ii) Hence find the Cartesian equation of the path of the ball **(2)**

iii) Assuming that $g = 9.8\ \text{m/s}^2$, will the ball clear a 50 m tall building which is 355m away if the ball is projected with a velocity of 65 m/s ? Justify your answer. **(3)**

b) Use mathematical induction to prove that for all integers $n \geq 3$ **(4)**

$$\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\ldots\ldots\ldots\left(1-\frac{2}{n}\right)=\frac{2}{n(n-1)}$$

<u>Question 6</u>

a) P$(8p,4p^2)$ and Q$(8q,4q^2)$ are variable points on the parabola $x^2 = 16y$ The tangent at P and Q meet at R.

i) Show that the equation of the chord PQ has equation

$y = \frac{1}{2}(p+q)x - 4pq$ **(2)**

ii) The chord PQ produced passes through the fixed point (4,0). Show that $p + q = 2pq$ **(2)**

iii) Find the coordinates of R **(2)**

iv) Find the Cartesian equation for the locus of R **(2)**

b) The function $f(x) = e^{2x} - x - 3$ has a zero near $x = 0.8$. Use one application of Newton's method to find a second approximation to the zero. Write your answer correct to 3 significant figures. **(2)**

c) By finding any asymptotes and intercepts draw the graph of $y = \frac{x}{(x-1)^2}$, showing significant features. *(do not use calculus)* **(2)**

Question 7

a) i) Show that $P = 4000 + Ae^{kt}$ satisfies the differential equation

$$\frac{dP}{dt} = k(P - 4000)$$ where A and k are constants. **(1)**

ii) Let P represent the approximate number of dollars needed to buy a car after time t years has elapsed. If when $t = 0$, $P = 5000$ and when $t = 10$, $P = 15000$, find A and k. **(2)**

iii) Find the year during which the cost of buying a car first exceeds \$125000 **(2)**

b) A circle centre A touches a smaller circle centre B externally at a point C. PQ is a common tangent to the two circles, touching them at P and Q. CD is a common tangent to both circles at C. R is joined to T and cuts the larger circle at S.

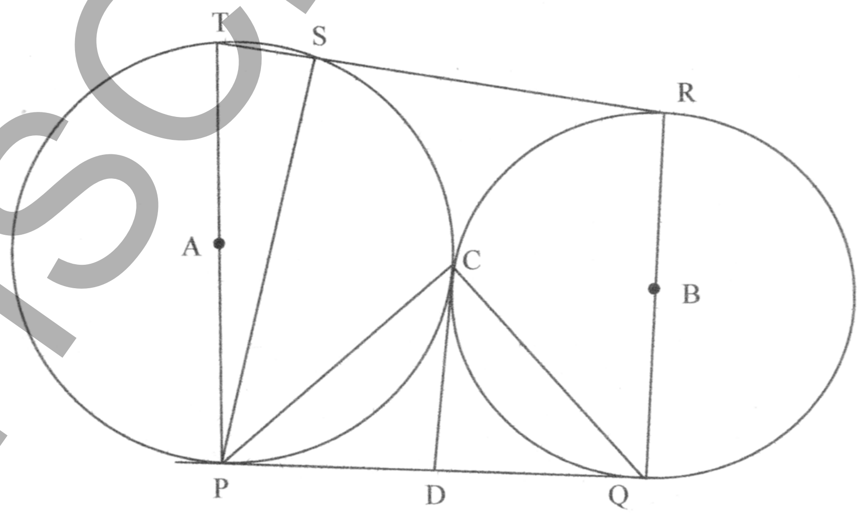
A larger diagram is drawn on the next page which is to be detached and appended to your answer booklet for Question 7.

i) Show that $\angle PCQ = 90°$. **(1)**

ii) Show that P, C and R are collinear. **(2)**

iii) Prove that BD is parallel to RCP. **(2)**

iv) Show that P, Q, R and S are concyclic points. **(2)**



(END OF EXAM)

(Attach this diagram to the end of your booklet for Question 7)