Question 1.

(a) $A(\overset{x_1}{-3}, \overset{y_1}{4})$ $B(\overset{x_2}{1}, \overset{y_2}{-2})$ externally $\overset{m}{1}:\overset{n}{3}$

$$X = \frac{1 \times -1 - 3 \times -3}{1-3} = \frac{1+9}{-2} = -5$$

$$Y = \frac{1 \times -2 - 3 \times 4}{1-3} = \frac{-14}{-2} = 7$$

$(-5, 7)$ //

(b) $$\frac{d}{dx}\cos^{-1}(x^3) = \frac{-1}{\sqrt{1-x^6}} \times 3x^2 = \frac{-3x^2}{\sqrt{1-x^6}} \quad //$$

(c) $(x+1)$ is a factor if $P(-1) = 0$

$$P(-1) = (-1)^4 - (-1)^3 + k(-1) - 4$$
$$= 1 - {-1} - k - 4$$
$$= -2 - k$$
$$= 0 \text{ when } k = -2 \quad //$$

$\therefore$ $P(x) = x^4 - x^3 - 2x - 4$

if $P(x-2)$ is a factor $P(2) = 0$

$$P(2) = 2^4 - 2^3 - 2 \times 2 - 4$$
$$= 16 - 8 - 4 - 4$$
$$= 0 \quad //$$

(d) $$\frac{x+4}{x-2} \geq 3$$

$$(x+4)(x-2) \geq 3(x-2)^2$$

$$(x-2)[(x+4) - 3(x-2)] \geq 0$$

$$(x-2)[10 - 2x] \geq 0$$

$$2(x-2)(5-x) \geq 0$$

$$2 < x \leq 5 \quad //$$

(e) $$\int_0^1 \frac{4x}{2x+1}\,dx$$

$$= \int_1^3 \frac{2u-2}{u} \cdot \frac{du}{2}$$

$$= \int_1^3 \frac{u-1}{u} \cdot du$$

$$= \int_1^3 \left(1 - \frac{1}{u}\right) du$$

$$= \left[u - \ln u\right]_1^3$$

$$= (3 - \ln 3) - (1 - 0)$$

$$= 2 - \ln 3 \quad //$$

$u = 2x+1 \Rightarrow 2x = u - 1$

$4x = 2u - 2$

$\frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2}$

$x = 1 \quad u = 3$

$x = 0 \quad u = 1$

Q2. a) $3\cos x + 5\sin x = 5 \quad 0° \le x \le 360°$

$R = \sqrt{9+25} = \sqrt{34} \qquad \tan\alpha = 5/3 \quad$ ie $\alpha = 59°2'$

ie $\sqrt{34}\cos(x-\alpha) = 5$

$\cos(x-\alpha) = \frac{5}{\sqrt{34}}$

$x - 59°2' = 30°58', 329°2'$

$\therefore x = 90°, 388°$

since $0° \le x \le 360°$, $x = 90°, 388-360 = 28°$

ie $x = 90°, 28°$.

b) $y = x^2 \qquad y = 2x$

$\frac{dy}{dx} = 2x \qquad m = 2$

at $x=2$, $m_1 = 4 \qquad \therefore m_2 = 2$

$\tan\theta = \left|\frac{4-2}{1+4\times 2}\right|$

$= \frac{2}{9}$

$\therefore \theta = 12°32'$

c) $\int 2\sin^2 x = \int 1 - \cos 2x \, dx$

$= x - \frac{1}{2}\sin 2x + C$

d) i) $P(B) = 0.05 \quad P(\tilde{B}) = 0.95$

$P(1B) = {}^{10}C_1(0.05)(0.95)^9$

$= 0.31512$

ii) $P(\ge 2B) = 1 - [P(0B) + P(1B)]$

$= 1 - [0.95^{10} + 0.31512]$

$= 0.0861$

e) $x(x+5) = 6^2 \qquad PS.PR = QP^2$

$x^2 + 5x - 36 = 0$

$(x+9)(x-4) = 0$

$x = -9, 4$

$\therefore x = 4 \quad (x \ge 0)$

Question 3

(a) $V = \pi \int_0^{\frac{\pi}{2}} y^2 \, dx$

$= \pi \int_0^{\pi/2} \frac{1}{4+x^2} \, dx$

$= \frac{\pi}{2} \left[\tan^{-1} \frac{x}{2} \right]_0^{\pi/2}$

$= \frac{\pi}{2} \left[\tan^{-1} \frac{\pi}{4} - \tan^{-1} 0 \right]$

$= \frac{\pi}{2} \tan^{-1} \frac{\pi}{4} \; u^3$ (exact) 3

(1.05 2dp)

$\frac{dV}{dt} = ? \quad \frac{ds}{dt} = 0.025$ at $r = 5$

(d) $\frac{dV}{dt} = \frac{ds}{dt} \times \frac{dV}{ds}$

$= 0.025 \times \frac{r}{2}$

$= 0.025 \times \frac{5}{2}$

$= 0.0625 \text{ cm}^3/\text{s}$

Need $\frac{dV}{ds}$

$\frac{dV}{ds} = \frac{dV}{dr} \times \frac{dr}{ds}$

$= 4\pi r^2 \times \frac{1}{8\pi r}$

$= \frac{r}{2}$

$\boxed{S = 4\pi r^2 \quad \frac{ds}{dr} = 8\pi r}$

$\boxed{V = \frac{4}{3}\pi r^3 \quad \frac{dV}{dr} = 4\pi r^2}$

3

(b) $\left(3x^2 + \frac{2}{x}\right)^6$ has general term $\binom{6}{r}(3x^2)^{6-r}\left(\frac{2}{x}\right)^r$

$= \binom{6}{r} 3^{6-r} x^{12-2r} \cdot 2^r \cdot x^{-r}$

$= \binom{6}{r} 3^{6-r} \cdot 2^r \cdot x^{12-3r}$

term independent of x (power of $x = 0$)

$\therefore 12 - 3r = 0$

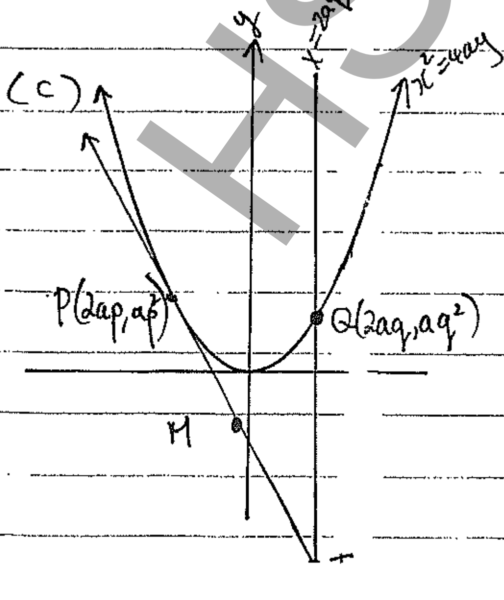
$3r = 12$

$r = 4$

$\therefore \binom{6}{4} 3^{6-4} \cdot 2^4 \cdot x^0 = 15 \times 9 \times 16$

$= 2160$ 3

(c)



(i) Finding T: $x = 2aq$ satisfies $y = px - ap^2$

$\therefore y = 2apq - ap^2$ 1

$T(2aq, 2apq - ap^2)$

(ii) $M_{PT} = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + (2apq - ap^2)}{2}\right)$ 1

$= \left(a(p+q), apq\right)$

(iii) given $pq = -1$ then $y = apq$ 1

$= -a$

$\therefore$ M lies on the directrix of $x^2 = 4a$[y]

Question 4 Solutions.

$f(x) = \frac{2x}{1-x^2} \qquad x \neq \pm 1$

$f'(x) = \frac{2(1-x^2)+4x^2}{(1-x^2)^2}$

$= \frac{2x^2+2}{(1-x^2)^2}$.

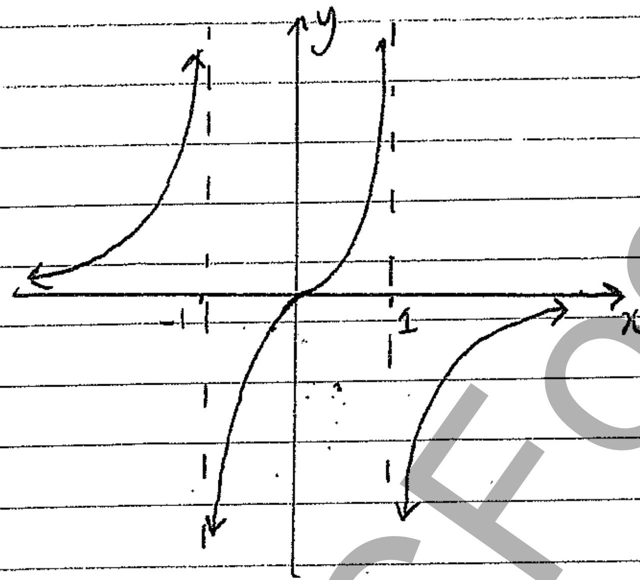
$2x^2+2 > 0$ for all x

$(1-x^2)^2 > 0$ for all x, $x \neq \pm 1$

$\therefore$ $f'(x) > 0$ for all x in domain

$f(x)$ is increasing.

ii)



ii) From graph, all k, $k \neq 0$.

b) $\lim_{x \to 0} \frac{1-\cos^2 2x}{x^2} = \lim_{x\to 0} \frac{\sin^2 2x}{x^2}$

$= 4 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)^2$

$= 4$

(c) Arrangements not together $= 1 \times 5 \times 6!$

Total arrangements $= (8-1)!$

$= 7!$

$\therefore P(\text{not together}) = \frac{1 \times 5 \times 6!}{7!}$

$= \frac{3600}{5040}$

$= \frac{5}{7}$

(d) For $0 \leq x \leq 1$,

$y = 2x(1-x) = 2x - 2x^2$

$\frac{dy}{dx} = 2 - 4x$

Let $\frac{dy}{dx} = 0$

$x = \frac{1}{2}$

when $x = \frac{1}{2}$, ~~[illegible]~~ max value is $\frac{1}{2}$

$\therefore$ Range of $f(x) = \sin^{-1}(2x(1-x))$

is $0 \leq f(x) \leq \frac{\pi}{6}$.

QUESTION 5

a) Prove true for $n=1$

$L.H.S. = 4$

$R.H.S = 2+1$

$= 3$

$\therefore L.H.S. > R.H.S.$

$\therefore$ True for $n=1$

Assume true for $n=k$

i.e. $4^k > 2k+1$

Prove true for $n=k+1$

i.e. $4^{k+1} > 2(k+1)+1$

$4^{k+1} > 2k+3$

now $4^k > 2k+1$

$4 . 4^k > 4(2k+1)$

$4^{k+1} > 8k+4$

$4^{k+1} > 2k+3+6k+1$

$\therefore 4^{k+1} > 2k+3$

$\therefore$ True for $n=k+1$

Since true for $n=1$ etc.

b) $\angle DCE = \angle CAB = x$ ($\angle$ in alternate segment)

$\angle DEC = \angle EAB = y$ ($\angle$ in alternate segment)

$\therefore \angle CAE = x+y$

$\angle CDE + x + y = 180$ (angle sum $\triangle CED$)

$\therefore \angle CDE + \angle CAE = 180$

$\therefore$ AEDC cyclic quadrilateral (opposite $\angle$'s supplementary)

c) i) $P = 1000 + Ae^{kt} \Rightarrow Ae^{kt} = P - 1000$ —— (1)

$\frac{dP}{dt} = kAe^{kt}$

$= k(P-1000)$ by substituting (1)

ii) when $t=0$, $P = 10\,000$

$10\,000 = 1000 + Ae^0$

$A = 9000$

iii) when $t=5$ $P = 8000$

$8000 = 1000 + 9000e^{5k}$

$e^{5k} = 7/9$

$5k = \ln(7/9)$

$k = -0.0503$

iv) when $P = 5000$

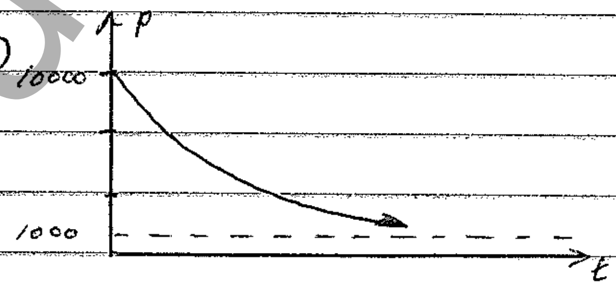
$5000 = 1000 + 9000e^{kt}$

$e^{kt} = 4/9$

$kt = \ln(4/9)$

$t = 16.12$ years

v)



Question 6

$f(x) = 3x - x^3$

$f'(x) = 3 - 3x^2$

Let $f'(x) = 0$

$0 = 3 - 3x^2$

$x^2 = 1$

$x = \pm 1$

when $x = 1$, $f(1) = 3 - 1$

$= 2$

$x = -1$, $f(-1) = -3 - (-1)^3$

$= -2$

$\therefore A = (1, 2)$; $B = (-1, -2)$

ii) Largest domain containing $(0,0)$: $-1 \le x \le 1$

iii) Range of $f(x)$ for restricted domain is $-2 \le y \le 2$

$\therefore$ Domain of $f^{-1}(x)$ is

$-2 \le x \le 2$

iv) $f'(x) = 3 - 3x^2$

$f'(0) = 3$

$\therefore \frac{dx}{dy} = \frac{1}{3}$

Gradient of inverse function at $x=0$ is $\frac{1}{3}$.

(b) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 15 - 2x$

$\frac{1}{2}v^2 = \int (15 - 2x)\,dx$

(b) $\frac{1}{2}v^2 = 15x - \frac{25x^2}{2} + c$

When $t = 0$, $x = 1.6$, $v = 0$

$\therefore 0 = 15(1.6) - \frac{25}{2}(1.6)^2 + c.$

$c = 8$

~~$v^2 = 30x - 50$~~

$v^2 = 30x - 25x^2 + 18$

$s = \sqrt{30x - 25x^2 + 18}$

(ii) Let $s = 0$

$0 = 30x - 25x^2 + 18$

$x = \frac{30 \pm \sqrt{900 + 4(25)(16)}}{50}$

$= \frac{30 \pm 50}{50}$

$x = -\frac{2}{5}, \frac{8}{5}$

Particle moves on interval

$-\frac{2}{5} \le x \le \frac{8}{5}$.

(iii) Max speed at $x = \frac{-\frac{2}{5} + \frac{8}{5}}{2}$

$= \frac{3}{5}$

$s = \sqrt{30\left(\frac{3}{5}\right) - 25\left(\frac{3}{5}\right)^2 + 16}$

$= \sqrt{25}$

$\therefore$ Max speed is $5\,cms^{-1}$ when ~~$x =$~~ $\frac{3}{5}$ cm to right of origin.

Q7 a) $x^3 - ax^2 + b = 0$

i) let roots be $\alpha, \frac{1}{\alpha}, \beta$

Product of roots: $\alpha \cdot \frac{1}{\alpha} \cdot \beta = \frac{-d}{a}$

$\therefore \beta = -b$

ii) $-b$ is a root

$\therefore P(-b) = (-b)^3 - a(-b)^2 + b = 0$

$-b^3 - ab^2 + b = 0$

$ab^2 = -b^3 + b$

$a = -b + \frac{1}{b}$

iii) $x^3 - (b - \frac{1}{b})x^2 + b = 0$

Sum of roots 2 @ a time: $\alpha \cdot \frac{1}{\alpha} + \alpha \cdot (-b) + \frac{(-b)}{\alpha} = \frac{c}{a} = 0$

$\therefore 1 - \alpha b - \frac{b}{\alpha} = 0$

$\alpha - \alpha^2 b - b = 0$

ie $b\alpha^2 - \alpha + b = 0$

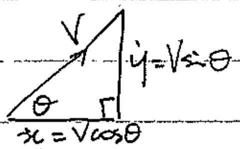
real roots $\Delta \geq 0$

$1 - 4b^2 \geq 0$

$(1-2b)(1+2b) \geq 0$

ie $-\frac{1}{2} \leq b \leq \frac{1}{2}$

Q7b) i) $\ddot{y}_P = -g$



$t = 0$

$\dot{y} = V\sin\theta$

$$\dot{y}_P = \int -g \, dt$$
$$= -gt + C_1$$
$$\dot{y}_P = -gt + V\sin\alpha$$
$$y = \int -gt + V\sin\alpha \, dt$$

$$y = -\frac{gt^2}{2} + Vt\sin\alpha + C_2$$

$t = 0$
$y = h$

$$\therefore \quad y = -\frac{gt^2}{2} + Vt\sin\alpha + h$$

ii)
$$x_Q = 2h - Ut\cos\beta$$
$$y_Q = -\frac{gt^2}{2} + Ut\sin\beta$$

iii)
$$2h - Ut\cos\beta = Vt\cos\alpha$$
$$Vt\cos\alpha + Ut\cos\beta = 2h$$
$$t = \frac{2h}{V\cos\alpha + U\cos\beta}$$

$$-\frac{gt^2}{2} + Vt\sin\alpha + h = -\frac{gt^2}{2} + Ut\sin\beta$$

$$h = Ut\sin\beta - Vt\sin\alpha$$
$$t = \frac{h}{U\sin\beta - V\sin\alpha}$$

$$\frac{2h}{V\cos\alpha + U\cos\beta} = \frac{h}{U\sin\beta - V\sin\alpha}$$
$$2U\sin\beta - 2V\sin\alpha = V\cos\alpha + U\cos\beta$$
$$V\cos\alpha + 2V\sin\alpha = 2U\sin\beta - U\cos\beta$$
$$V(\cos\alpha + 2\sin\alpha) = U(2\sin\beta - \cos\beta)$$
$$\frac{V}{U} = \frac{2\sin\beta - \cos\beta}{\cos\alpha + 2\sin\alpha}$$