SOLUTIONS

*** MATHEMATICS EXTENSION 1 (3 Unit component)**
YEAR 12 - 2008 - HALF YEARLY EXAMINATION

Question 1

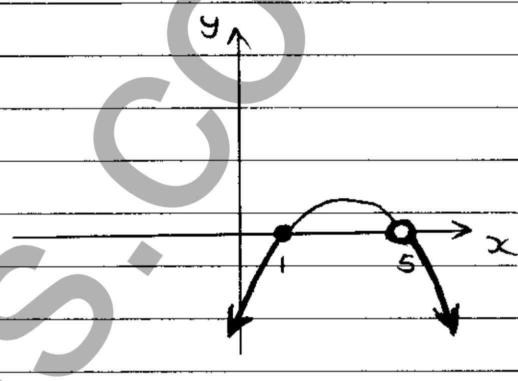
a) $\dfrac{4}{5-x} \leq 1 \rightarrow x \neq 5$ (can't ÷ by 0)

$(5-x)^{\cancel{2}} \dfrac{4}{\cancel{5-x}} \leq (5-x)^2$ ①

$(5-x)\left[4-(5-x)\right] \leq 0$

$(5-x)(x-1) \leq 0$ ①

$\therefore x \leq 1$ or $x > 5$ ①



b) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$

Let $u = 1 + \tan x$

$\therefore \dfrac{du}{dx} = \sec^2 x$

when $x = 0 \rightarrow u = 1$

$x = \frac{\pi}{4} \rightarrow u = 2$

$= \displaystyle\int_1^2 \frac{1}{\sqrt{u}}\,du$ ①

$= \left[2u^{\frac{1}{2}}\right]_1^2$ ①

$= 2\sqrt{2} - 2$ ①

c) i) $\displaystyle\int \frac{3x}{x^2+9}\,dx = \frac{3}{2}\int \frac{2x}{x^2+9}\,dx$ ①

$= \dfrac{3}{2}\ln(x^2+9) + C$ ①

c) ii) $\int \frac{3}{x^2+9}\,dx = 3\int \frac{1}{9+x^2}\,dx$

$= 3 \times \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$ (1)

$= \tan^{-1}\left(\frac{x}{3}\right) + C$ (1)

[full marks for straight to answer in one step also]

iii) $\int \frac{x^2+9}{3x}\,dx = \int \frac{x^2}{3x} + \frac{9}{3x}\,dx$

$= \int \frac{x}{3} + \frac{3}{x}\,dx \quad (x \neq 0)$ (1)

$= \frac{x^2}{6} + 3\ln x + C$ (1)

Question 2

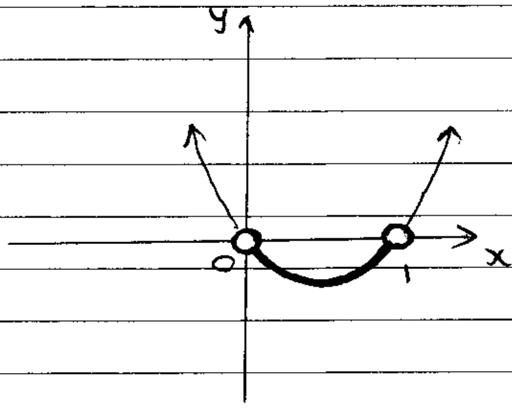
a) $\frac{x+1}{x^2+1} > 1$

$x+1 > x^2+1$ (1) (as $x^2+1>0$)

$x^2 - x < 0$

$x(x-1) < 0$

$\therefore\ 0 < x < 1$ (1)



b) $\int x(1-x^2)^5\,dx$

Let $u = 1 - x^2$

$\therefore \frac{du}{dx} = -2x$

$= -\frac{1}{2}\int -2x(1-x^2)^5\,dx$ (1)

$= -\frac{1}{2}\int u^5\,du$

$= -\frac{u^6}{12} + C$

$= -\frac{(1-x^2)^6}{12} + C$ (1)

c) $3x + y = 4 \rightarrow m_1 = -3$

$x - y = 1 \rightarrow m_2 = 1$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{-3-1}{1+(-3)(+1)} \quad 1$$

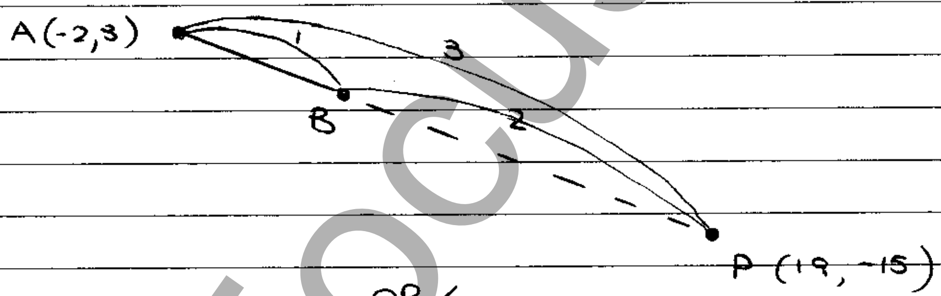
$$= \frac{-4}{-2}$$

$$= 2$$

$\therefore \theta = \tan^{-1}(2)$

$= 63°26'$ (to nearest min.) 1

d)



OR

METHOD 1

x-direction:

$x_B = -2 + \frac{1}{3}(19-(-2))$ 1

$= 5$

y-direction:

$y_B = 3 - \frac{1}{3}(3-(-15))$ 1

$= -3$

$\therefore B(5,-3)$ 1

METHOD 2

$$x = \frac{mx_2 + nx_1}{m+n}$$

$$19 = \frac{-3(x) + 2(-2)}{-3+2} \quad 1$$

$-19 = -3x - 4$

$\therefore x = 5$

$$y = \frac{my_2 + ny_1}{m+n}$$

$$-15 = \frac{-3y + 2(3)}{-3+2} \quad 1$$

$15 = -3y + 6$

$\therefore y = -3$

$\therefore B(5,-3)$ 1

c) <u>Step 1:</u> Check first case (ie. n = 1)

$$LHS = \frac{1}{1\times 5}$$
$$= \frac{1}{5}$$
$$RHS = \frac{1}{4\times 1+1}$$
$$= \frac{1}{5}$$
$$= LHS \quad \therefore \text{true for } n=1$$

<u>Step 2:</u> Assume true for $n = k$

$$\therefore \frac{1}{1\times 5} + \frac{1}{5\times 9} + \frac{1}{9\times 13} + \dots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$$

<u>Step 3:</u> Prove true for $n = k+1$

ie. required to prove $\frac{k}{4k+1} + \frac{1}{(4(k+1)-3)(4(k+1)+1)} = \frac{k+1}{4(k+1)+1}$

$$LHS = \frac{k}{4k+1} + \frac{1}{(4(k+1)-3)(4(k+1)+1)} \quad (1)$$
$$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$$
$$= \frac{k(4k+5)+1}{(4k+1)(4k+5)}$$
$$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$$
$$= \frac{\cancel{(4k+1)}(k+1)}{\cancel{(4k+1)}(4k+5)} \quad (4k+1 \neq 0)$$
$$= \frac{k+1}{4(k+1)+1} \text{ as req'd.} \quad (1)$$

<u>Step 4:</u> If true for $n = k$, then true for $n = k+1$.
Since true for $n = 1$, then true for $n = 2$, $n = 3$, $n = 4$ and so on...
$\therefore$ True for all positive integer values of n.

(1) all stages of induction completed correctly.

Question 3

a) i) $f(x) = e^{x+2}$

$$x = e^{y+2}$$ (1)

$$\ln x = y + 2$$

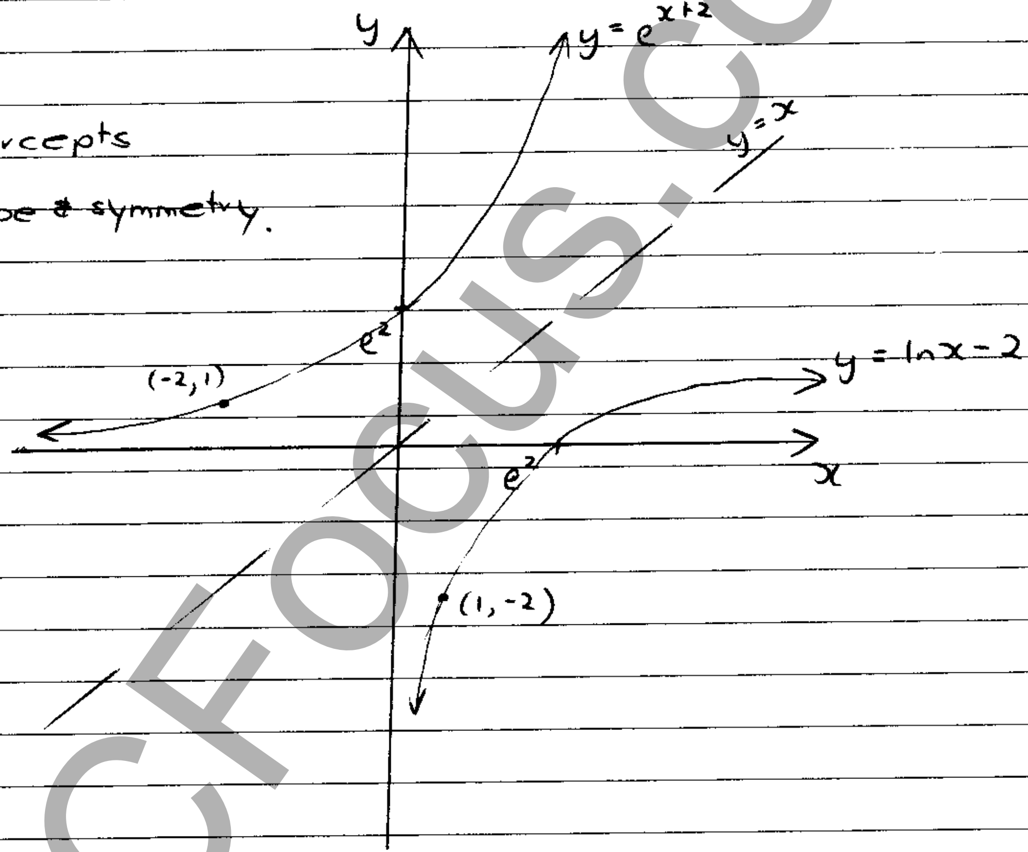
$$y = \ln x - 2$$

$$\therefore f^{-1}(x) = \ln x - 2$$ (1)

ii)

(1) intercepts

(1) ~~shape & symmetry~~.



b) $$\frac{d}{dx}\left(x^2 \cos^{-1}(3x)\right) = x^2 \cdot \frac{-1}{\sqrt{1-9x^2}} \cdot 3 + \cos^{-1}(3x) \cdot 2x$$

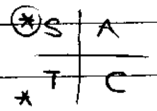
$$= 2x\cos^{-1}(3x) - \frac{3x^2}{\sqrt{1-9x^2}}$$

(1) (1)

c) Let $\alpha = \cos^{-1}\left(\sin\frac{4\pi}{3}\right)$

(where $0 \leq \alpha \leq \pi$)

$\therefore \cos\alpha = -\frac{\sqrt{3}}{2}$ (1)



$\therefore \alpha$ in 2nd quadrant (1)

Related $\angle = \frac{\pi}{6}$

$\therefore \alpha = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$ (1)

d) $y = 2\sin^{-1}\left(\frac{x}{3}\right)$

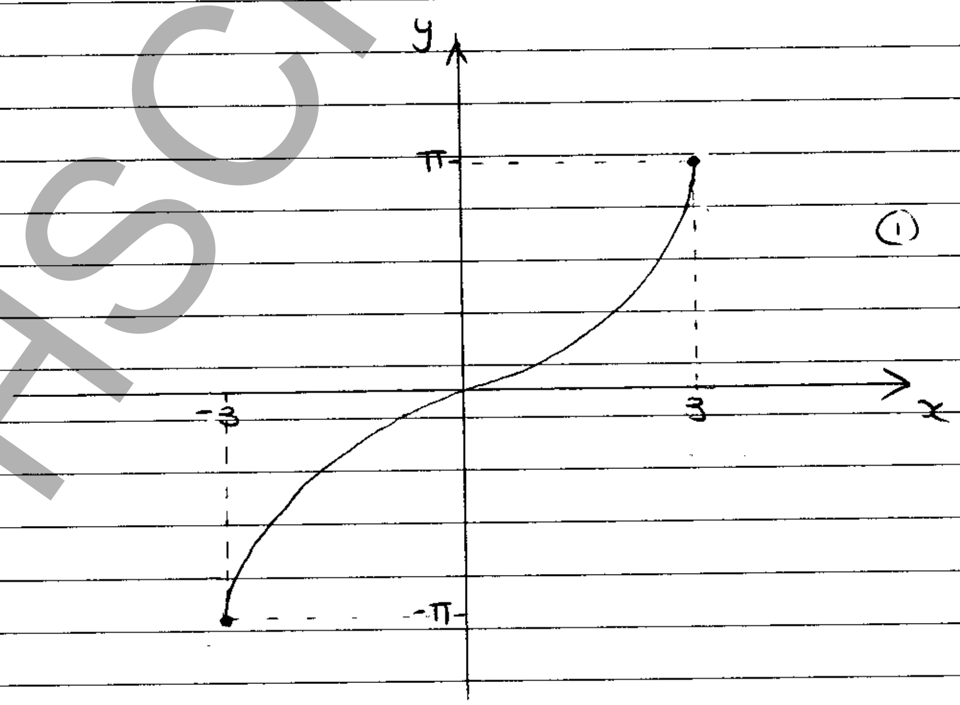
* domain: $-1 \leq \frac{x}{3} \leq 1$

$\therefore -3 \leq x \leq 3$ (1)

* range: $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x}{3}\right) \leq \frac{\pi}{2}$

$-\pi \leq 2\sin^{-1}\left(\frac{x}{3}\right) \leq \pi$

$\therefore -\pi \leq y \leq \pi$ (1)



(1)

Question 4

a) i) $\tan 45^\circ = \dfrac{h}{AO}$

$\therefore AO = \dfrac{h}{\tan 45^\circ}$

$= h$ (1) (or through isos. Δ)

ii) $\tan\alpha = \dfrac{h}{OB}$

$\therefore OB = \dfrac{h}{\tan\alpha}$

$= h\cot\alpha$ (1)

iii) $AB^2 = AO^2 + OB^2 - 2.AO.OB.\cos 60^\circ$

$50^2 = h^2 + h^2\cot^2\alpha - 2.h.h\cot\alpha.\frac{1}{2}$ (1)

$= h^2 + h^2\cot^2\alpha - h^2\cot\alpha$

Rearranging:

$h^2\cot^2\alpha - h^2\cot\alpha + h^2 = 50^2$ as req'd.

iv) $30^2\cot^2\alpha - 30^2\cot\alpha + 30^2 = 50^2$

$900\cot^2\alpha - 900\cot\alpha + 900 = 2500$

$9\cot^2\alpha - 9\cot\alpha + 16 = 0$ (1)

$\therefore \cot\alpha = \dfrac{9 \pm \sqrt{81 - 4\times 9\times -16}}{18}$

$= \dfrac{9 \pm \sqrt{657}}{18}$

$\therefore \alpha = 27^\circ$ (to nearest degree) (1)

(as $\alpha > 0$)

b) $\tan\theta = -\dfrac{1}{\sqrt{3}}$

*S	A
T	C*

related $\angle = \dfrac{\pi}{6}$

$\therefore \theta = n\pi - \dfrac{\pi}{6}$ where n is an integer

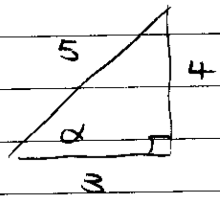
or

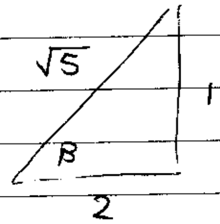
$\theta = n\pi + \dfrac{5\pi}{6}$ where n is an integer

(1) expression style ie. $n\pi + \ldots$ or $n\pi - \ldots$

(1) correct angle

c) $\cos\alpha = \frac{3}{5}$ $\qquad$ $\sin\beta = \frac{1}{\sqrt{5}}$





$\sin\alpha = \frac{4}{5}$ 1

$$\begin{aligned}\sin 2\beta &= 2\sin\beta\cos\beta \\ &= 2\times\frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}} \\ &= \frac{4}{5} \\ &= \sin\alpha\end{aligned}$$ 1

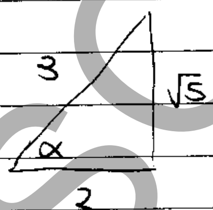
$\therefore \alpha = 2\beta$

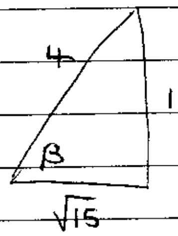
d) $\sin\left(\cos^{-1}\frac{2}{3} + \sin^{-1}\frac{1}{4}\right)$

Let $\alpha = \cos^{-1}\frac{2}{3}$ $(0 < \alpha < \pi)$ & Let $\beta = \sin^{-1}\frac{1}{4}$ $\left(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$

$\therefore \cos\alpha = \frac{2}{3}$ $\quad$ S | A* / T | C $\quad$ ∴ 1st quad.

$\therefore \sin\beta = \frac{1}{4}$ $\quad$ S | A* / T | C $\quad$ ∴ 1st quad 1





$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{\sqrt{5}}{3}\cdot\frac{\sqrt{15}}{4} + \frac{2}{3}\cdot\frac{1}{4} \\ &= \frac{\sqrt{75}+2}{12} \\ &= \frac{5\sqrt{3}+2}{12}\end{aligned}$$ 1 1

Question 5

a) i) when $t = 0$, $T = 24°C$, $R = -40°C$ 1

$\therefore 24 = -40 + Ae^{0}$

$\therefore A = 64$ 1

ii) $t = 5$, $T = 19°C$

$\therefore 19 = -40 + 64e^{5k}$ 1

$\therefore e^{5k} = \frac{59}{64}$

$\ln \frac{59}{64} = 5k$ 1

$\therefore k = \frac{1}{5} \ln \frac{59}{64}$

iii) $t = ?$ when $T = 0°C$:

$0 = -40 + 64e^{kt}$ 1

$e^{kt} = \frac{40}{64}$

$\ln\left(\frac{5}{8}\right) = kt$

$\therefore t = \frac{\ln\left(\frac{5}{8}\right)}{\frac{1}{5}\ln\left(\frac{59}{64}\right)}$ 1

$= 28.88929...$

$\doteqdot 29$ seconds.

b) $y = \frac{1}{200} t e^{-t}$

$\frac{dy}{dt} = \frac{1}{200}\left(t \cdot -e^{-t} + e^{-t} \cdot 1\right)$

$= \frac{1}{200}(1-t)e^{-t}$ 1

c) i) when $t < 1$ $\frac{dA}{dt} > 0$ (since $(1-t) > 0$) 1

$\therefore A$ is increasing for $t < 1$

when $t > 1$ $\frac{dA}{dt} < 0$ (since $(1-t) < 0$) 1

$\therefore A$ is decreasing for $t > 1$

ii) When $t = 0$, $A = 0.0005$

$$A = \int \frac{dA}{dt}\, dt$$

$$= \int \frac{1}{200}(1-t)e^{-t}\, dt$$

$$= \frac{1}{200} t e^{-t} + C \quad \text{(from part b))}$$ 1

when $t = 0$: $0.0005 = 0 + C$

$\therefore C = 0.0005$

$$\therefore A = \frac{1}{200} t e^{-t} + 0.0005$$ 1

iii) from i) the maximum occurs when $t = 1$:

$$\therefore A_{max} = \frac{1}{200} \times 1 \times e^{-1} + 0.0005$$

$$= 0.002339397\ldots$$

$$\doteqdot 0.0023 \text{ (to 4 dec. pl.)}$$ 1

Question 6

a) i) tangents to a circle from an external point are equal.

1

$\therefore TX = XA$

So ΔTXA is isosceles and $\angle XTA = \angle XAT = \theta$

(base angles of isosceles Δ)

ii) similarly, ΔAYS is isosceles with base angles $\angle YAS$ & $\angle YSA$ equal.

1

But $\angle YAS = \angle TAX = \theta$ (vertically opposite)

So $\angle YSA = \theta$ & $\angle XTA = \theta$

But these are alternate

1

So $TX \parallel YS$

b) Let $\angle ADE = \theta$

1

$\therefore \angle ABC = \theta$ (exterior angle cyclic quadrilateral)

1

ΔABC is isosceles ($AB = AC$, given)

$\therefore \angle ABC = \angle ACB = \theta$ (base angles isosceles Δ)

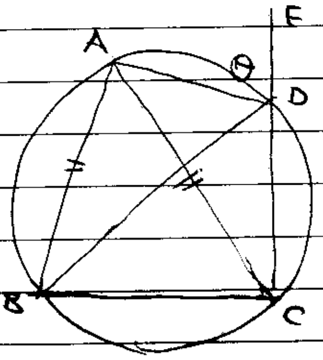
1

$\angle ACB = \angle ADB = \theta$ (angles in same segment)

1

$\therefore \angle ADE = \angle ADB$ (both equal θ)

$\therefore DB$ bisects $\angle BDE$



c) i) $\angle PAF = \angle PBF = \alpha$ (angles in same segment)

1

ii) $\angle ANB = \angle AMB = 90°$ (given)

$\therefore AB$ is a diameter of circle through A, N, M & B

(angle in a semicircle is a right $\angle$)

$\therefore A, N, M$ & B are concyclic.

1

[OR angles in same segment / standing on chord AB....]

iii) $\angle NBM = \angle MAN$ (angles standing on same arc of circle ANMB) 1

$\angle MAN = \angle PAF$ (same angle)

$= \alpha$

$\therefore \angle NBM = \alpha$

iv) In Δ's HMB + FMB:

1 side: MB is common

angle: $\angle MBH = \angle MBF = \alpha$ ($\angle MBH$ is same as $\angle NBM$, $\angle MBF$ is same as $\angle PBF$) accept as 'proven above'

1 angle: $\angle HMB = \angle FMB = 90°$ (AM is altitude of ΔAPB)

$\therefore \Delta HMB \equiv \Delta FMB$ (AAS)

1 $\therefore HM = FM$ (matching sides of congruent Δ's)

$\therefore$ M bisects HF.

~~(another option ... using tan α = ...)~~

Question 7

a) i) $m = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{\cancel{a}(p+q)\cancel{(p-q)}}{2\cancel{a}\cancel{(p-q)}}$ $(a \neq 0 + p-q \neq 0)$

$= \dfrac{p+q}{2}$ 1

$y - y_1 = m(x - x_1)$

$y - ap^2 = \dfrac{p+q}{2}(x - 2ap)$ 1

$\therefore y = \dfrac{p+q}{2}x - \cancel{\dfrac{2ap^2}{2}} - \dfrac{\cancel{2}apq}{\cancel{2}} + \cancel{ap^2}$

$\therefore y = \frac{1}{2}(p+q)x - apq$ as req'd.

ii) Focus $S(0,a)$ → satisfies equation if on chord:

$\therefore a = \frac{1}{2}(p+q) \times 0 - apq$ 1

$\therefore pq = \dfrac{a}{-a}$

$= -1$ as req'd.

iii) $PQ = PS + SQ$

$$= \sqrt{(2ap)^2 + (ap^2 - a)^2} + \sqrt{(2aq)^2 + (aq^2 - a)^2}$$

$$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2} + \sqrt{4a^2q^2 + a^2q^4 - 2a^2q^2 + a^2}$$

$$= \sqrt{a^2(2p^2 + p^4 + 1)} + \sqrt{a^2(2q^2 + q^4 + 1)}$$

$$= \sqrt{a^2(p^2+1)^2} + \sqrt{a^2(q^2+1)^2}$$

$$= a(p^2+1) + a(q^2+1) \quad 1$$

$$= a(p^2 + q^2 + 2)$$

$$= a\left(p^2 + 2 + \frac{1}{p^2}\right) \quad \left(\text{since } pq = -1 \ \ \& \ q = -\frac{1}{p} \rightarrow q^2 = \frac{1}{p^2}\right)$$

$$= a\left(p + \frac{1}{p}\right)^2 \quad 1$$

as required

b) i) $y = tx - t^2$

$A(x_a, 0)$: $\quad 0 = tx - t^2$

$\therefore x = t$

$\therefore A(t, 0) \quad 1$

gradient OT:

$$m_{OT} = \frac{t^2}{2t}$$

$$= \frac{t}{2}$$

$$\therefore m_{\perp OT} = -\frac{2}{t} \quad 1$$

$\therefore$ equation AP: $\quad y - 0 = -\frac{2}{t}(x - t)$

$\therefore y = -\frac{2}{t}(x - t)$ as required

ii) equation OT : $y - 0 = \frac{t}{2}(x - 0)$ (1)

$\therefore t = \frac{2y}{x}$ as required

iii) Locus of P → simultaneously solve AP & OT :

$y = -\frac{2}{t}(x - t)$ — (1) (1) concept of simult. solution

$t = \frac{2y}{x}$ — (2)

Sub (2) into (1) :

$y = -\frac{\cancel{2}x}{\cancel{2}y}\left(x - \frac{2y}{x}\right)$ (1)

$y^2 = -x^2 + 2y$

$x^2 + y^2 - 2y = 0$

$x^2 + y^2 - 2y + 1 = 1$ (1)

$x^2 + (y-1)^2 = 1$

$\therefore$ locus is circle : centre $(0, 1)$, radius 1 [except the point $(0,0)$]