CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2008 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 1

Question 1 (12 marks)
(a) (2 marks)
Outcomes assessed: H5
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds the correct primitive	1
• evaluates the integral correctly	1

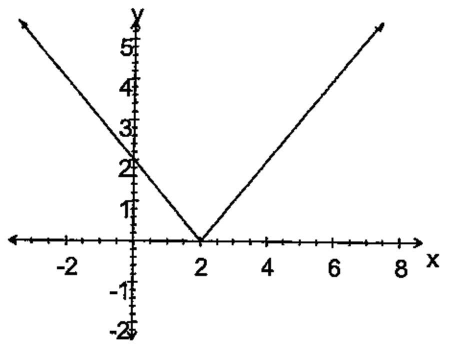
Sample Answer:

$$\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx = \tfrac{1}{2}\left[\tan 2x\right]_0^{\frac{\pi}{8}}$$
$$= \tfrac{1}{2}(\tan\tfrac{\pi}{4} - \tan 0)$$
$$= \tfrac{1}{2}(1-0)$$
$$= \frac{1}{2}$$

(b) (i) (1 mark)
Outcomes assessed: P4
Targeted Performance Bands: E2-E3

Criteria	Mark
• draws the correct graph of $y = \lvert 2-x \rvert$, including intercepts	1

Sample Answer: $y = \lvert 2-x \rvert$



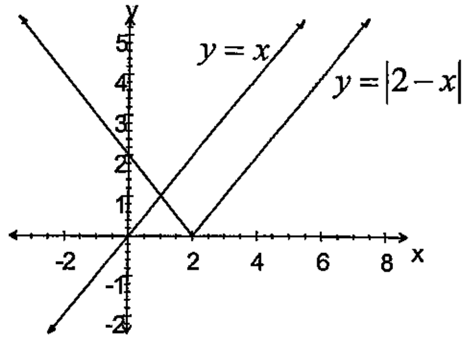
(b)(ii) (2 marks)
Outcomes assessed: PE2
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds the point of intersection or other significant step towards solution	1
• writes the correct solution	1

Sample Answer:

point of intersection of $y = x$ and $y = |2 - x|$ is (1,1)

$\therefore$ from the graph $|2 - x| < x$ when $x > 1$



(c) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• uses the factor theorem with substitution of $x = -2$	1
• solves the equation to find k	1

Sample Answer:

$$P(x) = x^2 - kx + 6$$
$$P(-2) = 4 + 2k + 6 = 0 \text{ as } (x+2) \text{ is a factor}$$
$$k = -5$$

(d) (3 marks)
Outcomes assessed: PE2
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds the TWO gradients	1
• uses the correct formula for $\tan\theta$	1
• finds the correct angle	1

Sample Answer:

From the diagram the gradients of the lines are 2 and -1.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{2+1}{1-2}\right| = 3$$

$$\therefore \theta = 71°34'$$

(e) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• establishes correct quadratic or other correct significant step towards solution	1
• finds full solution	1

Sample Answer:

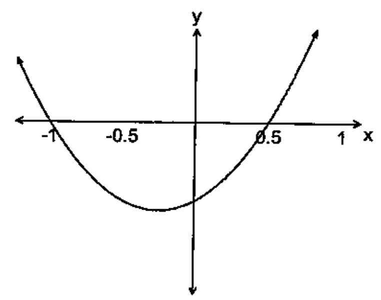
$\frac{3}{x+1} < 2$ multiply by $(x+1)^2$

$3(x+1) < 2(x+1)^2$

$2(x+1)^2 - 3(x+1) > 0$

$(x+1)(2x+2-3) > 0$

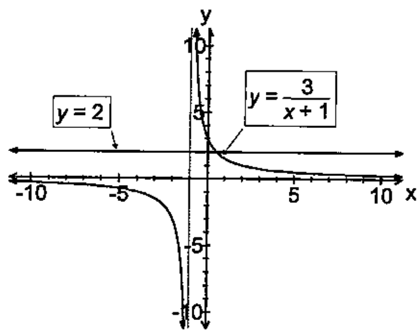
$(x+1)(2x-1) > 0$



Solution is $x < -1$ or $x > \frac{1}{2}$

$\therefore$ Jasi's solution is only partially correct.

or graphically:



ordinate of intersection $\frac{3}{x+1} = 2$

$3 = 2x + 2$

$x = \frac{1}{2}$

From the graph $x > \frac{1}{2}$ satisfies the inequality BUT the other branch of the hyperbola for $x < -1$ also satisfies the inequality.

Question 2 (12 marks)

(a) (2 marks)

Outcomes assessed: PE3

Targeted Performance Bands: E2-E3

Criteria	Mark
• finds sum and product of roots	1
• evaluates the expression correctly	1

Sample Answer:

$$2x^3 - 5x^2 + 3x - 5 = 0$$

$$\alpha + \beta + \gamma = \tfrac{5}{2} \quad \alpha\beta + \alpha\gamma + \beta\gamma = \tfrac{3}{2} \quad \alpha\beta\gamma = \tfrac{5}{2}$$

$$\begin{aligned} \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 &= \alpha\beta\gamma(\alpha + \beta + \gamma) \\ &= \frac{5}{2} \times \frac{5}{2} \\ &= \frac{25}{4} \end{aligned}$$

(b) (i) (1 mark)

Outcomes assessed: P4

Targeted Performance Bands: E2-E3

Criteria	Mark
• gives the correct solutions	1

Sample Answer:

Given $f(x) = \dfrac{2x}{\sqrt{1 - x^2}}$

$f(x)$ is undefined when $1 - x^2 \le 0$

i.e. when $x \le -1$ or $x \ge 1$

(b) (ii) (3 marks)

Outcomes assessed: HE6

Targeted Performance Bands: E2-E3

Criteria	Marks
• rewrites the integral in term of u	1
• finds the new limits	1
• evaluates the integral correctly to at least $-2\left[\frac{\sqrt{3}}{2}-1\right]$ (correct numerical equivalence)	1

Sample Answer:

$$\int_0^{\frac{1}{2}} \frac{2x}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{2\sin u}{\cos u}\cos u\,du$$

$$= 2\int_0^{\frac{\pi}{6}} \sin u\,du$$

$$= -2\left[\cos u\right]_0^{\frac{\pi}{6}}$$

$$= -2\left(\cos\frac{\pi}{6} - \cos 0\right)$$

$$= -2\left[\frac{\sqrt{3}}{2} - 1\right]$$

$$= 2 - \sqrt{3}$$

$x = \sin u$

$dx = \cos u\,du$

$x = 0 \Rightarrow u = 0$

$x = \frac{1}{2} \Rightarrow u = \frac{\pi}{6}$

(c) (i) (1 mark)

Outcomes assessed: HE4

Targeted Performance Bands: E2-E3

Criteria	Mark
• differentiates correctly	1

Sample Answer:

$$\frac{d}{dx}\left(\sin^{-1}x + \cos^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$= 0$$

(c) (ii) (2 marks)
Outcomes assessed: HE4
Targeted Performance Bands: E2-E3

Criteria	Marks
• identifies that the primitive is a constant	1
• uses a suitable substitution, or otherwise, to show that the constant is $\frac{\pi}{2}$	1

Sample Answer:

Since the derivative is zero, $\sin^{-1}x+\cos^{-1}x=C$ (C is a constant)

$$\text{Let } x=0 \Rightarrow \sin^{-1}0+\cos^{-1}0=0+\tfrac{\pi}{2}$$
$$=\tfrac{\pi}{2}$$

(d) (i) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Mark
• correct numerical expression for the answer	1

Sample Answer:

EXERCISE $\Rightarrow$ 8 letters with 3 Es

Number of arrangements $=\dfrac{8!}{3!}=6720$

(d) (ii) (2 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• significant progress towards solution	1
• correct numerical expression for answer	1

Sample Answer:

EXERCISE with C and R at the ends

Number of arrangements $=\dfrac{2!6!}{3!}=240$

Question 3 (12 marks)

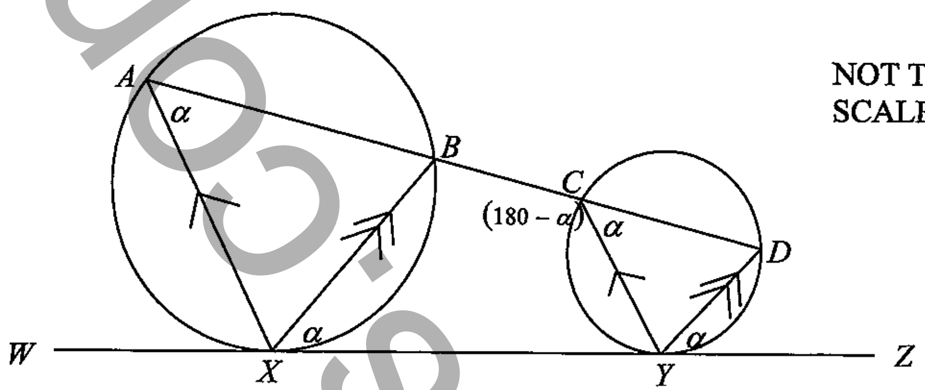
(a) (i) (3 marks)

Outcomes assessed: PE2, PE3

Targeted Performance Bands: E2-E3

Criteria	Marks
• correct application of alternate segment theorem or other correct step in explanation	1
• identifying corresponding angles or other significant progress in reasoning	1
• conclusion with reason	1

Sample Answer:



NOT TO SCALE

Given $\angle BXY = \alpha$

$\therefore \angle BAX = \alpha$ (angle between tangent and chord at the point of contact is equal to the angle in the alternate segment)

$\angle DCY = \alpha$ (corresponding to $\angle BAX$, $AX \parallel CY$)

$\angle DYZ = \alpha$ (angle between tangent and chord at the point of contact is equal to the angle in the alternate segment)

$\therefore \angle DYZ = \angle BXY$

$\therefore BX \parallel DY$ (corresponding angles are equal)

(a) (ii) (1 mark)

Outcomes assessed: PE2, PE3

Targeted Performance Bands: E2-E3

Criteria	Mark
• states the correct reason	1

Sample Answer:

$\angle BCY = 180 - \alpha$ (BCD is a straight line)

$\therefore \angle BCY + \angle BXY = 180°$

$\therefore BCYX$ is a cyclic quadrilateral as one pair of opposite angles are supplementary

(b) (3 marks)

Outcomes assessed: PE2

Targeted Performance Bands: E2-E3

Criteria	Marks
• determines the correct value of $\sin A$, including the sign	1
• determines the correct value of $\sin B$ and $\cos B$, including the sign	1
• expands $\sin(A-B)$ and gives a correct numerical expression	1

Sample Answer:

$$\cos A = \frac{3}{5} \quad \therefore \sin A = -\frac{4}{5} \quad (A \text{ is reflex and in the 4th quad})$$

$$\tan B = \frac{12}{5} \quad \therefore \sin B = -\frac{12}{13} \text{ and } \cos B = -\frac{5}{13} \quad (B \text{ is reflex and in the 3rd quad})$$

$$\begin{aligned}\sin(A-B) &= \sin A\cos B - \cos A\sin B \\ &= \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) \\ &= \frac{56}{65}\end{aligned}$$

(c) (3 marks)

Outcomes assessed: HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• writes down at least one correct term in x^6 or x^5	1
• writes down the correct equivalence	1
• finds the correct value of k (correct numerical equivalence)	1

Sample Answer:

Consider terms of the expansion of $(1-kx)^9$

Term in $x^5 = {}^9C_5(-kx)^5 = -{}^9C_5 k^5 x^5$

Term in $x^6 = {}^9C_6(-kx)^6 = {}^9C_6 k^6 x^6$

$$\therefore\ -{}^9C_5 k^5 = 2\,{}^9C_6 k^6$$

$$\begin{aligned}k &= -\frac{{}^9C_5}{2\,{}^9C_6} \\ &= -\frac{1}{2}\times\frac{9!}{5!4!}\times\frac{3!6!}{9!} \\ &= -\frac{3}{4}\end{aligned}$$

(d) (2 marks)

Outcomes assessed: PE3, HE7

Targeted Performance Bands: E2-E3

Criteria	Marks
• identifies the function	1
• uses Newton's Method to find correct approximation (correct numerical equivalence)	1

Sample Answer:

$$x = \sqrt[3]{9}$$
$$\therefore x^3 = 9 \quad \Rightarrow f(x) = x^3 - 9 \quad \therefore f'(x) = 3x^2$$
$$\text{Let } x_1 = 2 \quad \therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \Rightarrow x_2 = 2 - \frac{-1}{12}$$
$$x_2 = 2\frac{1}{12}$$

Question 4 (12 marks)

(a) (3 marks)

Outcomes assessed: HE2

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes the truth of $S(1)$	1
• establishes the correct relationship between $S(k)$ and $S(k+1)$	1
• deduces the required result	1

Sample Answer:

Let $S(n)$ be the statement $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$

Consider $S(1)$: $LHS = 1 \times 1!$; $RHS = (1+1)! - 1 = 1$.
Hence $S(1)$ is true

If $S(k)$ is true: $\sum_{r=1}^{k} r \times r! = (k+1)! - 1$ *

RTP $S(k+1)$ is true i.e. to prove $\sum_{r=1}^{k+1} r \times r! = (k+2)! - 1$

$$LHS = \sum_{r=1}^{k} r \times r! + (k+1)(k+1)!$$
$$= (k+1)! - 1 + (k+1)(k+1)! \quad \text{if } S(k)\text{, using } *$$
$$= (k+1)!\left(1 + k + 1\right) - 1$$
$$= (k+1)!\,(k+2) - 1$$

Hence if $S(k)$ then $S(k+1)$ is true. Thus since $S(1)$ is true it follows by induction that $S(n)$ is true for positive integral n.

(b) (i) (2 marks)
Outcomes assessed: HE5
Targeted Performance Bands: E2-E3

Criteria	Marks
• sets up the correct differential equation or significant progress towards result	1
• finds the desired equation	1

Sample Answer:

$$\ddot{x} = 2x - 3$$
$$\therefore \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 2x - 3$$
$$\tfrac{1}{2}v^2 = x^2 - 3x + c$$
$$\text{when } x = 0, v = 2 \quad \Rightarrow c = 2$$
$$\therefore \tfrac{1}{2}v^2 = x^2 - 3x + 2$$
$$\therefore v^2 = 2x^2 - 6x + 4$$

(b) (ii) (2 marks)
Outcomes assessed: HE5
Targeted Performance Bands: E3-E4

Criteria	Marks
• calculates the correct velocity and acceleration	1
• describes the motion	1

Sample Answer:

at $x = 1$, $v = 0$ and $\ddot{x} = -1$ m/s^2

After the object comes to rest at $x = 1$ it then moves towards the origin, and will continue moving in a negative direction.

(c) (i) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• differentiates correctly	1
• proves correct result	1

Sample Answer:

$$N=\frac{200}{1+ke^{-200t}}=200\left(1+ke^{-200t}\right)^{-1}$$

$$\frac{dN}{dt}=-200(1+ke^{-200t})^{-2}(-200ke^{-200t})$$

$$=\frac{200}{1+ke^{-200t}}\left(\frac{200ke^{-200t}}{1+ke^{-200t}}\right)$$

$$=N\left(\frac{200+200ke^{-200t}-200}{1+ke^{-200t}}\right)$$

$$=N\left(\frac{200(1+ke^{-200t})}{(1+ke^{-200t})}-\frac{200}{1+ke^{-200t}}\right)$$

$$=N(200-N)$$

(c) (ii) (2 marks)

Outcomes assessed: HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
• finds the value of k	1
• finds t (correct numerical equivalence)	1

Sample Answer:

when $t=0$, $N=1$ i.e. $1=\dfrac{200}{1+k}$ $\therefore\ k=199$

$N=\dfrac{200}{1+199e^{-200t}}$ half the colony infected i.e. $N=100$

$$100=\frac{200}{1+199e^{-200t}}$$

$$1+199e^{-200t}=2$$

$$e^{-200t}=\frac{1}{199}$$

$$-200t=\log_e\frac{1}{199}$$

$$\therefore\ t=-\frac{1}{200}\log_e\frac{1}{199}$$

$t=0.0265$ years or 9.66 days

(c) (iii) (1 mark)
Outcomes assessed: HE7
Targeted Performance Bands: E3-E4

Criteria	Mark
• shows that the limiting value of N is 200	1

Sample Answer:

as $t \to \infty \quad e^{-200t} \to 0$

$\therefore N \to \dfrac{200}{1+0} = 200$ i.e. eventually all the bees will be infected.

Question 5 (12 marks)
(a) (i) (2 marks)
Outcomes assessed: H5, HE7
Targeted Performance Bands: E2-E3

Criteria	Marks
• differentiates and determines the correct quadratic	1
• identifies that $\Delta \geq 0$ and sets up the correct inequality	1

Sample Answer:

$P(x) = -2x^3 + px^2 - qx + 5$

$P'(x) = -6x^2 + 2px - q = 0$ for stationary points

For this quadratic to have real solutions $\quad \Delta \geq 0$

i.e. $4p^2 - 24q \geq 0$

$p^2 - 6q \geq 0$

(a) (ii) (1 mark)
Outcomes assessed: H5, HE7
Targeted Performance Bands: E3-E4

Criteria	Mark
• gives the correct conclusion	1

Sample Answer:

When $p^2 - 6q = 0$ $P'(x)$ has a double root and there is only one stationary point which would be a horizontal point of inflexion.

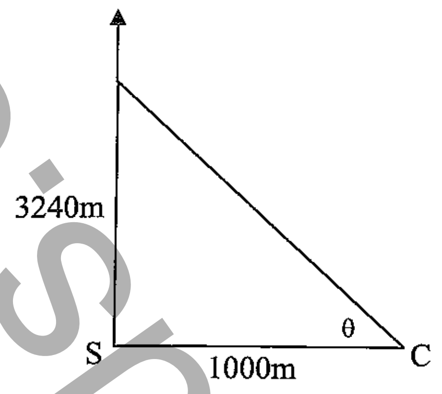
(b) (3 marks)

Outcomes assessed: HE5, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• establishes $\frac{dh}{d\theta}$	1
• evaluating θ at $t = 30$ s, or other significant progress towards the result such as correct use of chain rule	1
• correct answer	1

Sample Answer:



$\frac{dh}{dt} = 230\text{m/s}$

$\tan\theta = \frac{h}{1000}$

$h = 1000\tan\theta$

$\frac{dh}{d\theta} = 1000\sec^2\theta$

at $t = 30$ seconds, $h = 3240$ m $\Rightarrow \tan\theta = \frac{3240}{1000} \Rightarrow \theta = 1.271$ radians

$$\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt}$$
$$= \frac{\cos^2 1.271}{1000} \times 230$$
$$= 0.02 \text{ rads/sec}$$

(c) (i) (1 mark)

Outcomes assessed: HE3

Targeted Performance Bands: E2-E3

Criteria	Mark
• gives the correct answer (correct numerical equivalence)	1

Sample Answer:

$$P(\text{a basket}) = \frac{2}{5}$$

$$P(\text{ 2 points}) = {}^6C_2\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^4$$

$$= \frac{972}{3125} \quad (\text{or } 0.31104)$$

(c) (ii) (2 marks)

Outcomes assessed: HE3, HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• sets up correct inequality or other significant progress	1
• gives the correct solution, rounding to the nearest whole number	1

Sample Answer:

$$P(\text{at least one}) = 1 - P(\text{none})$$
$$= 1 - 0.6^n$$

$$\therefore 1 - 0.6^n \geq 0.9978$$

$$0.6^n \leq 0.0022$$

take logs of both sides

i.e. $n \ln 0.6 \leq \ln 0.0022$

$$n \geq \frac{\ln 0.0022}{\ln 0.6}$$

$$n \geq 11.979$$

$$n \geq 12$$

Diana would need 12 free throws.

(d) (3 marks)

Outcomes assessed: PE6, HE7

Targeted Performance Bands: E2-E3

Criteria	Marks
• some progress towards result, e.g. finds the missing angles	1
• significant progress towards result, e.g. finding BP (correct numerical equivalence)	1
• finds the angle of depression	1

Sample Answer:

$\angle BPM = 110^\circ - 22^\circ = 88^\circ$

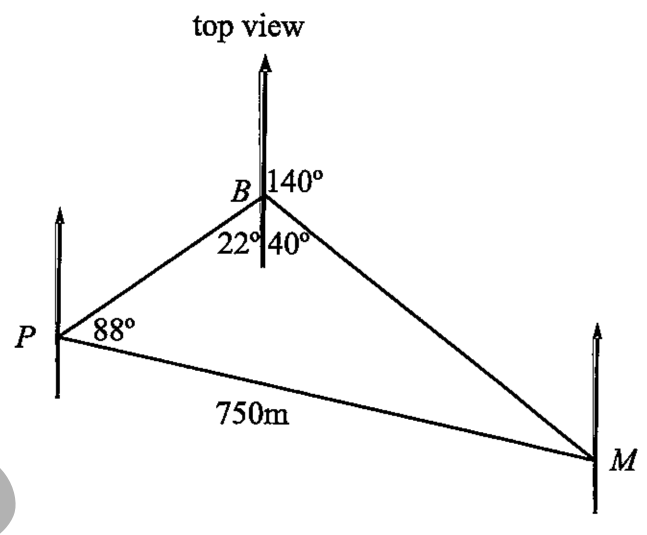
$\angle PBM = 202^\circ - 140^\circ = 62^\circ$

$\therefore \angle PMB = 30^\circ$

$$\therefore \frac{BP}{\sin 30^\circ} = \frac{750}{\sin 62^\circ}$$

$$\therefore BP = \frac{750 \sin 30^\circ}{\sin 62^\circ}$$

$= 424.71$ (2 d.p.)



angle of elevation $= \tan^{-1}\left(\frac{80}{BP}\right)$

$= 10^\circ 40'$

$\therefore$ angle of depression is $10^\circ 40'$

Question 6 (12 marks)

(a) (i) (2 marks)

Outcomes assessed: H5

Targeted Performance Bands: E3-E4

Criteria	Marks
• finds the vertical asymptote	1
• finds the horizontal asymptote	1

Sample Answer:

$$f(x) = \frac{e^x}{x-1}$$

vertical asymptote at $x = 1$

horizontal asymptote : $x \to \infty,\ f(x) \to \infty;\quad x \to -\infty,\ f(x) \to 0^-$

$\therefore\ y = 0$ is a horizontal asymptote as $x \to -\infty$

(a) (ii) (3 marks)

Outcomes assessed: PE5, PE6, H5

Targeted Performance Bands: E3-E4

Criteria	Marks
• finds the stationary point	1
• identifies intercept	1
• sketches the correct function	1

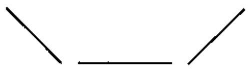
Sample Answer:

$$f'(x) = \frac{(x-1)e^x - e^x}{(x-1)^2}$$
$$= \frac{e^x(x-2)}{(x-1)^2}$$

$f'(x) = 0$ when $x = 2$ and $y = e^2$

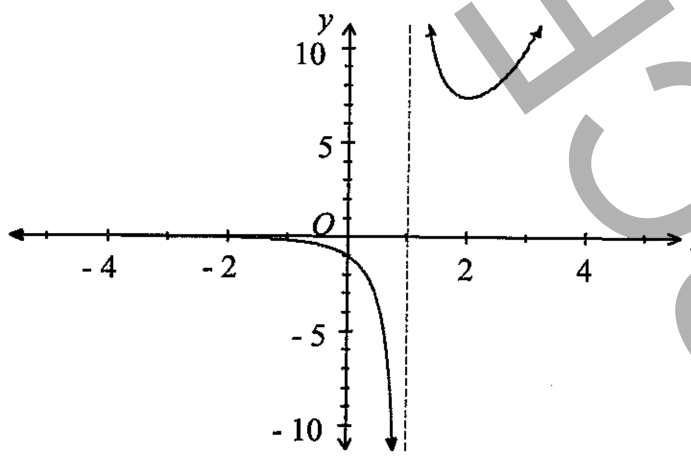
testing nature:

x	2^-	2	2^+
$f'(x)$	$-$	0	$+$



$\therefore (2, e^2)$ is a minimum point

when $x = 0, y = \dfrac{e^0}{0-1} = -1$ $\quad \therefore y$ intercept is $(0, -1)$



NOT TO SCALE

(a) (iii) (1 mark)
Outcomes assessed: HE$
Targeted Performance Bands: E2-E3

Criteria	Marks
• gives correct domain	1

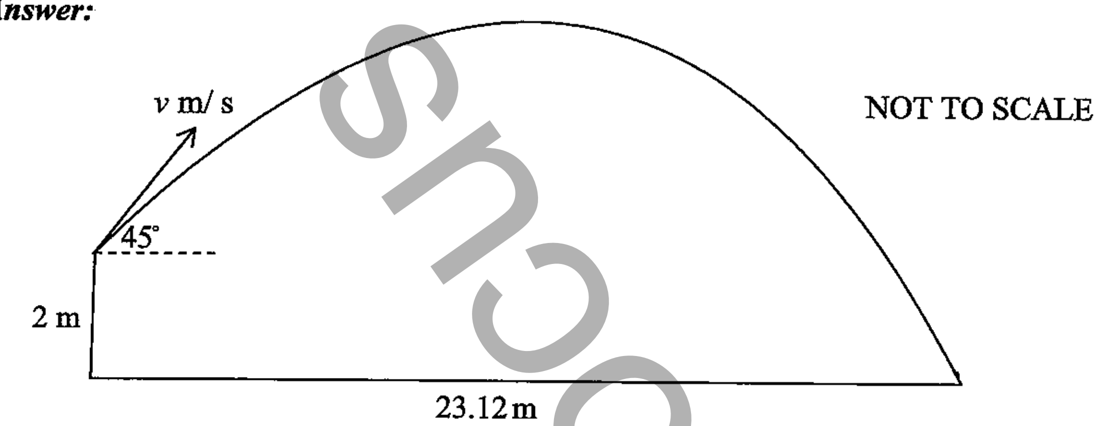
Sample Answer:

For an inverse function domain is $x \geq 2$

(b) (i) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E2-E3

Criteria	Marks
• finds the correct differential equations for the horizontal motion	1
• finds solves the correct differential equations for the vertical motion	1

Sample Answer:



Horizontal

$\ddot{x} = 0$

$\dot{x} = c_1$

at $t = 0$, $\dot{x} = v\cos 45^\circ \Rightarrow c_1 = v\cos 45^\circ$

$\therefore \dot{x} = \dfrac{v}{\sqrt{2}}$

$x = \dfrac{vt}{\sqrt{2}} + c_2$

at $t = 0$, $x = 0 \Rightarrow c_2 = 0$

$x = \dfrac{vt}{\sqrt{2}}$

Vertical

$\ddot{y} = -10$

$\dot{y} = -10t + c_3$

at $t = 0$, $\dot{y} = v\sin 45^\circ \Rightarrow c_3 = v\sin 45^\circ$

$\therefore \dot{y} = -10t + \dfrac{v}{\sqrt{2}}$

$y = -5t^2 + \dfrac{vt}{\sqrt{2}} + c_4$

at $t = 0$, $y = 2 \Rightarrow c_4 = 2$.

$y = -5t^2 + \dfrac{vt}{\sqrt{2}} + 2$

(b) (ii) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• determines the Cartesian equation of motion or other significant progress	1
• substitutes and solves for v	1

Sample Answer:

$$t=\frac{\sqrt{2}x}{v}$$

$$\therefore y=-5\times\frac{2x^2}{v^2}+x+2$$

at world record range $x=23.12$ and $y=0$

$$\therefore 0=-\frac{10\times 23.12^2}{v^2}+25.12$$

$$\frac{5345.344}{v^2}=25.12$$

$v^2=212.7923567$
$v=14.59$ m/s (2 decimal places)

(b) (iii) (2 marks)
Outcomes assessed: HE3
Targeted Performance Bands: E3-E4

Criteria	Marks
• finds the value of t in terms of v	1
• substitutes into y and solves	1

Sample Answer:

maximum height when $\dot{y}=0$

i.e. $t=\dfrac{v}{10\sqrt{2}}$

$$y=-5\times\frac{v^2}{10^2\times 2}+\frac{v^2}{10\times 2}+2$$

$$=\frac{v^2}{20}-\frac{v^2}{40}+2$$

$$=\frac{v^2}{40}+2$$

$$=\frac{212.7923567}{40}+2$$

$= 7.32$ m (2 decimal places)

Question 7 (12 marks)
(a) (i) (2 marks)
Outcomes assessed: H5, PE2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Showing $\frac{\sec^2\theta-\tan^2\theta}{\sec\theta+\tan\theta}$ or other significant progress	1
• completing the proof	1

Sample Answer:

RTP $\sec\theta-\tan\theta=\frac{1}{\sec\theta+\tan\theta}$

LHS

$=\frac{(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}$

$=\frac{\sec^2\theta-\tan^2\theta}{\sec\theta+\tan\theta}$

$=\frac{1}{\sec\theta+\tan\theta}$ =RHS

(a) (ii) (2 marks)
Outcomes assessed: PE2, HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• establishing $\sec\theta+\tan\theta\geq 1$ or other significant progress	1
• justifying the inequality	1

Sample Answer:

From the graphs $\sec\theta\geq 1$ and $\tan\theta\geq 0$ for $0\leq\theta<\frac{\pi}{2}$

$\therefore \sec\theta+\tan\theta\geq 1$

$\therefore 0<\frac{1}{\sec\theta+\tan\theta}\leq 1$

i.e. $0<\sec\theta-\tan\theta\leq 1$ using (i)

(a) (iii) (3 marks)

Outcomes assessed: HE7

Targeted Performance Bands: E3-E4

Criteria	Marks
• progress towards obtaining a quadratic equation e.g. establishing $2\cos\theta = 1 + \sin\theta$	1
• obtaining the correct quadratic in terms of $\sin\theta$ or equivalent progress	1
• correct solution	1

Sample Answer:

$$\sec\theta - \tan\theta = \frac{1}{2}$$

i.e. $\dfrac{1}{\sec\theta + \tan\theta} = \dfrac{1}{2}$

$\therefore \sec\theta + \tan\theta = 2$

$$\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 2$$

i.e. $2\cos\theta = 1 + \sin\theta$

Square and solve the quadratic:

$$4\cos^2\theta = 1 + 2\sin\theta + \sin^2\theta$$
$$4\left(1 - \sin^2\theta\right) = 1 + 2\sin\theta + \sin^2\theta$$
$$5\sin^2\theta + 2\sin\theta - 3 = 0$$
$$(5\sin\theta - 3)(\sin\theta + 1) = 0$$

since $\sin\theta$ is positive in the interval, $0 \le \theta < \dfrac{\pi}{2}$

$$\sin\theta = \frac{3}{5}$$

i.e. $\theta = 0.644$ radians

OR

• progress towards use of auxiliary angle method e.g. establishing $2\sin\theta+\cos\theta=2$	1
• obtaining the correct value for R or α using the auxiliary angle method	1
• correct solution	1

$$\sec\theta-\tan\theta=\frac{1}{2}$$
$$\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}=\frac{1}{2}$$
$$2-2\sin\theta=\cos\theta$$
$$2\sin\theta+\cos\theta=2$$

let $2\sin\theta+\cos\theta\equiv R\sin(\theta+\alpha)$ where R is positive and α is acute

i.e. $2\sin\theta+\cos\theta\equiv R\sin\theta\cos\alpha+R\cos\theta\sin\alpha$

$\Rightarrow R\cos\alpha=2$ and $R\sin\alpha=1$

i.e. $\tan\alpha=\frac{1}{2}$ and $R=\sqrt{5}$

$$\therefore \sqrt{5}\sin\left(\theta+\tan^{-1}\tfrac{1}{2}\right)=2$$
$$\sin\left(\theta+\tan^{-1}\tfrac{1}{2}\right)=\frac{2}{\sqrt{5}}$$
$$\left(\theta+\tan^{-1}\tfrac{1}{2}\right)=\sin^{-1}\tfrac{2}{\sqrt{5}}$$
$$\theta=\sin^{-1}\tfrac{2}{\sqrt{5}}-\tan^{-1}\tfrac{1}{2}$$
$$\theta=0.644 \text{ radians}$$

OR

From simplifying to $2\cos\theta=1+\sin\theta$ i.e. $2\cos\theta-\sin\theta=1$

let $2\cos\theta-\sin\theta\equiv R\cos(\theta+\alpha)$ where R is positive and α is acute

i.e. $2\cos\theta-\sin\theta\equiv R\cos\theta\cos\alpha-R\sin\theta\sin\alpha$

$\Rightarrow R\cos\alpha=2$ and $R\sin\alpha=1$

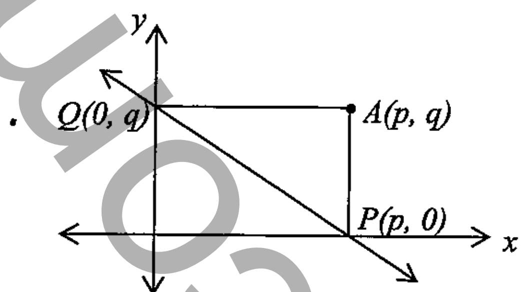
i.e. $\tan\alpha=\frac{1}{2}$ and $R=\sqrt{5}$

$$\therefore \sqrt{5}\cos\left(\theta+\tan^{-1}\tfrac{1}{2}\right)=1$$
$$\cos\left(\theta+\tan^{-1}\tfrac{1}{2}\right)=\frac{1}{\sqrt{5}}$$
$$\left(\theta+\tan^{-1}\tfrac{1}{2}\right)=\cos^{-1}\tfrac{1}{\sqrt{5}}$$
$$\theta=\cos^{-1}\tfrac{1}{\sqrt{5}}-\tan^{-1}\tfrac{1}{2}$$
$$\theta=0.644 \text{ radians}$$

(b) (i) (1 mark)
Outcomes assessed: HE7
Targeted Performance Bands: E2-E3

Criteria	Mark
• finds the correct equation of PQ in any form	1

Sample Answer:



$$y = -\frac{q}{p}x + q$$
$$py = -qx + pq$$
$$\frac{x}{p} + \frac{y}{q} = 1$$

(b) (ii) (3 marks)
Outcomes assessed: HE7
Targeted Performance Bands: E3-E4

Criteria	Marks
• determines the quadratic or other significant progress towards the solution	1
• uses $\Delta = 0$ when PQ is a tangent to the curve	1
• establishes the relationship	1

Sample Answer:

If PQ is a tangent to the parabola then there is one point of intersection.

solve $\frac{x}{p} + \frac{y}{q} = 1$ and $y = \frac{x^2}{4a}$ simultaneously

$$\frac{x}{p} + \frac{x^2}{4aq} = 1$$
$$4aqx + px^2 = 4apq$$
$$px^2 + 4aqx - 4apq = 0$$

for PQ to be a tangent then $\Delta = 0$ in this quadratic (i.e. only one root/solution)

i.e. $16a^2q^2 + 16ap^2q = 0$

$$16aq(aq + p^2) = 0$$

i.e. $aq + p^2 = 0$

(b) (iii) (1 mark)

Outcomes assessed: HE7

Targeted Performance Bands: E3-E4

Criteria	Mark
• finds the locus	1

Sample Answer:

Coordinates of $A \Rightarrow x = p$ and $y = q$

$\therefore$ since $aq + p^2 = 0$ from (ii) then $ay + x^2 = 0$

i.e. $x^2 = -ay$ is the locus of A

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