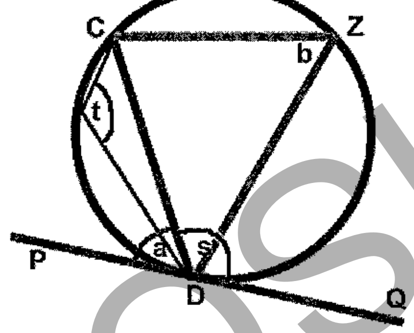
Student Number : ____________________

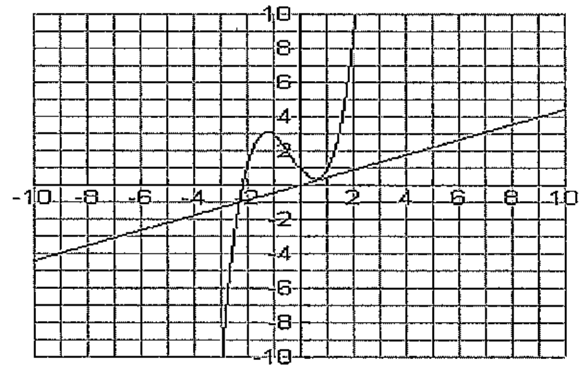
Teacher: ____________________



ST MARY STAR OF THE SEA COLLEGE

HSC MATHEMATICS EXTENSION 1

PORTFOLIO HSC 2007



Newton's Method

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6. A particle is moving so that its velocity, v and displacement, x are related by the equation $v^2 = -6(5 - 6x + x^2)$

a) Find the endpoints of the motion.

b) Show that the motion is SHM and find the centre of motion.

c) What is the maximum speed achieved?

Catholic Secondary 2003 Trial Questions 3b, 4b, 5c, 6c and 7a are recommended for you to complete/

4. Newton's law of cooling states that the rate at which a body loses heat is proportional to the difference between the temperature of the body and room temperature R . This is expressed by the equation: ${}^{dT}/_{dt} = -k(T - R)$

a) Show that $T = R + Ce^{-kt}$, where C is a constant is a solution to this equation.

b) A cup of coffee cools from 90° C to 50°C in 20 minutes in a room with temperature 22°C. Find the temperature of the coffee after an hour.

5. The displacement, x of a particle at time, t is given by $x = 3 + 4\cos\pi t$

a) Show that the particle is undergoing Simple Harmonic Motion and find the period, amplitude and equilibrium position (centre of motion).

b) Find the initial position, velocity and acceleration of the particle.

Topic 1 : Permutations, Combinations and Related Probability

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

1. Counting techniques, including multiplication and addition principles. Definitions and difference between the terms : combination (or selection) and permutation (or arrangement)
2. Formulae for combinations and permutations and use and interpretation of factorial and ${}^nP_r$ and ${}^nC_r$ notation
3. Probability involving permutations and combinations

2. A particle is projected at an angle of α with a speed of 90 m/s from a point on the ground. Using $g = 10 \text{ m/s}^2$

a) Find the value of α if the particle is 270m away from its projection point when it lands.

b) What value of α will make the particle land furthest from its starting point. Justify your answer.

3. Water is poured at a constant rate of $36\text{cm}^3/\text{s}$ into a conical vessel of radius 30cm and height 60cm . After t seconds the height of the water is h cm. Find the rate at which the surface area, S of the top of the water is changing when the depth of the water is 32cm.

Work Sample 7

1. A projectile is fired from a point on the ground at an angle of 60° with a velocity of 80 m/s.

   a) Use $g = 10\text{m/s}^2$ to derive the equations of its motion.

   b) Find the time of its flight and the horizontal distance travelled.

   c) Determine the speed and direction of the projectile 8 seconds after firing.

Work Sample 1

1. A necklace is made up of 20 beads on a continuous elastic band (ie no clasp.) All the beads are different colours. 10 of the beads are spherical and 10 are cubes. How many different necklaces can be created if the spherical beads are separated?

2. A particular car number plate system uses 2 letters of the alphabet and 3 digits. How many different number plates can be made using this system?

3. In a game of LOTTO we select 6 numbers from the numbers 1 to 40. What are our chances of selecting the right 6 numbers and winning LOTTO?

4. From a pack of cards numbered 1,2,3,4,5,6,7,8,9 three cards are selected (without replacement) and placed in the order of selection to form a 3-digit number. For example, if 1,2 and 3 are selected in this order they form the number 123.

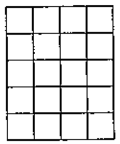
   a) What is the probability that the number formed exceeds 400?

   b) What is the probability that the digits are drawn in descending order? Justify your answer.

5. The Tournament of Mind Competition requires a team of 7 students. The team must include students from at least two year levels with a minimum of two students from year 7 and a maximum of four students from any one year.

At a particular school: five students from year 7, three students from year 8 and two students from year 10 want to enter. Show that 110 different teams can be formed from this group of students.

6. An ant starts at the top left hand corner and walks along the gridlines to the bottom right hand corner of a 5 by 4 grid as shown



a) Explain why the number of arrangements of the letters RRRRDDDDD will give the number of routes the ant can travel.

b) Find the number of routes the ant can travel.

Topic 2: Applications of Calculus

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE4 uses the relationship between functions and their derivatives

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

1. Sketching of curves involving both horizontal and vertical asymptotes
2. Iterative methods for numeric estimation of the roots – halving the interval and Newton's methods.
3. Cases where Newton's method does not give a better approximation.

Topic 7 : Physical Applications of Calculus

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE3 uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion, or exponential growth and decay

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

1. Related rates of change involving the use of the chain rule
2. Exponential growth and decay with consideration of the surrounding situation
3. Velocity and acceleration as functions of displacement
4. Simple Harmonic Motion
5. Projectile Motion

b) i) Prove that $\frac{d}{dx}(x\sin^{-1}x) = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$

ii) Hence evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$

Work Sample 2

1. Sketch $y = \frac{2x^2 - x - 3}{x^2 + x - 2}$ showing asymptotes .(Justify your sketch)

2. Sketch the function $y = (x-2)^2 + 4$. Using a different colour, hence sketch on the same axes :

$y = \frac{1}{(x-2)^2 + 4}$

3. a) i) Find $\int \frac{1}{\sqrt{1-x^2}} dx$

ii) Find $\frac{d}{dx} \tan^{-1}(\frac{x}{\sqrt{1-x^2}})$

iii) Hence show that , for $0 \le x < 1$, $\tan^{-1} \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$

2. a) Show that $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{3}{5}=\frac{\pi}{2}$

b) Evaluate

i) $\int_{1}^{\sqrt{3}}\frac{dx}{\sqrt{3-x^2}}$

ii) $\int_{0}^{\sqrt{3}}\frac{dx}{1+4x^2}$

3. Sketch the functions $y=x^2+2$ and $y=\frac{1}{x}$ on the same graph Hence sketch $y=x^2+2+\frac{1}{x}$

(Use the same graph with different colours- no need to find intercept values or turning points)

4. The function $y=x^2+2+\frac{1}{x}$ which you sketched in question 3 has only one root

a) Show that the root lies between $x = -0.5$ and $x = -0.4$

b) Use one application of the "halving the interval " method to find a smaller interval containing the root.

c) Use one application of Newton's Method with a first approximation of $x = -0.5$ to approximate the root

d) Which gives a better indication of the root in this case : Newton's Method or the Halving the Interval Method ?

5. The actual root of a particular function , $y = f(x)$ is $x = 3$. A first approximation, $x = 2$ is used with Newton's method and results in a second approximation that is undefined. Draw a possible function, $y = f(x)$ for which this occurs. Explain this situation in terms of both the formula and the geometrical conditions.

b) i) Sketch $y = \sin^{-1}2x$. State the domain and range of this function.

ii) Use the Trapezoidal rule with 2 strips to estimate the area bounded by this function the x-axis and the line $x = ½$.

iii) Find the exact value of this area using integration.

Work Sample 6

1. a) Consider the function $f(x) = x - e^{-x}$

i) Show that the function is increasing and that its graph is concave down for all x values.

ii) Given that the x-intercept is near $x = 0.5$, sketch the graph of $y = f(x)$ clearly showing the intercepts (use the given approximation for the x-intercept) and the asymptotes. On the same diagram sketch the graph of the inverse function $y = f^{-1}(x)$. Label each graph, its intercepts and the equations of its asymptotes clearly.

Topic 3 : Binomial Theorem and Related Probability

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

1. Expansion of $(x + y)^n$ for $n = 1,2,3,\ldots$ and $(1 + x)^n$ for $n = 1,2,3,$
2. Use and interpretation of notations : ${}^nC_r$ and $\binom{n}{r}$
3. Use the formula for the expansion of a binomial. For example, find the greatest term, general term, term independent of x.
4. Use identities involving binomial expansions to prove identities for finite series, including those involving differentiation and integration.
5. Proof and use of the related probability formula

Work Sample 3

1. Find the value of the sum :

a) ${}^{8}C_2$

b) $\sum_{n=4}^{7} {}^{n}P_3$

2. Write the expansion of $(2x + 3y)^5$

3. Find the constant term in the expansion of $(2x^4 - {}^{1}/_{x^2})^{12}$

4. In the expansion of $(1 + ax)^9$ the coefficient of x^5 is twice the coefficient of x^6. Find the value of the constant a

5. a) Show that ${}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n = (n+2)2^{n-1}$

b) Find the sum $\frac{{}^nC_0}{1.2} - \frac{{}^nC_1}{2.3} + \frac{{}^nC_2}{3.4} - \ldots + \frac{(-1)^n\,{}^nC_n}{(n+1)(n+2)}$. Write your answer in simplest form.

Topic 6 : Inverse Functions and Inverse Trigonometric Functions

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE4 uses the relationship between functions, inverse functions and their derivatives

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

1. Concept of one–to-one functions and inverse functions and their graphs
2. Inverse trigonometric functions and the graphs of $y = \sin^{-1}x$, $y = \cos^{-1}x$ and $y = \tan^{-1}x$
3. Simple properties of inverse trigonometric functions
4. Derivatives of the inverse trigonometric functions and the corresponding integrations

b) Evaluate $\int_1^{\sqrt{2}} 2\cos^2\frac{x}{2}\,dx$

6. Evaluate $\int_1^{49} \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}}$ using the substitution $u = 1 + \sqrt{x}$

6. A die is biased so that the probability of an odd score is $^3/_8$. In six throws of the die what is the probability of exactly 2 even scores?

Topic 4 : Induction

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE2 uses inductive reasoning in the construction of proofs

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

1. Process of inductive reasoning
2. Application of inductive reasoning to proofs involving inequalities, sums and divisibility. Note: the syllabus requires knowledge of proof by induction of the formula for ${}^nC_r$

4. Use the substitution $u = z^2 + 1$ to find $\int (z^3 \sqrt{z^2+1})dx$.

5. a) Find $\int \sin^2 4x \, dx$

Work Sample 5

1. Evaluate $\lim_{x \to 0} \frac{x \sin\left(\frac{2}{x}\right)}{2}$

2. Use the substitution $x = 4\sin\theta$ to evaluate the area bounded by the semicircle $y = \sqrt{16 - x^2}$, the coordinate axes and the line $x = 2$.

3. Show that $\frac{\sin x}{\cos^3 x} = \tan x \sec^2 x$ and hence find $\int \frac{\sin x dx}{\cos^3 x}$

Work Sample 4

1. Prove the following by Mathematical Induction

   a) $5^n + 2(11)^n$ is divisible by 3 for all positive integers n

Topic 5 : Harder Integration

HE6 determines integrals by reduction to a standard form through a given substitution

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

1. Use of the results $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and the limits as $x \to 0$ of $\sin x$, $\cos x$ and $\tan x$
2. Use of the expansions for $\cos 2x$ to integrate $\sin^2 x$ and $\cos^2 x$
3. Use of trigonometric identities to integrate functions such as $\tan^2 x$, $\cot^2 x$, $\operatorname{cosec}^2 x$
4. Integration by substitution

b) $1 + 4 + 4^2 + 4^3 + \ldots\ldots + 4^{n-1} = \frac{4^n - 1}{3}$ for all positive integral n

2. i) Use Mathematical Induction to show that $\ln(n!) > n$ for all positive integers $n \geq 6$

ii) Hence show that for all positive integers $n \geq 6$

$$\frac{1}{n!} < \frac{1}{e^n}$$

iii) Hence show that

$$\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\ldots < \frac{103}{60}+\frac{1}{e^5(e-1)}$$

3. Explain how the induction process must be modified if proving a statement is true for all

a) positive even integers

b) positive odd integers