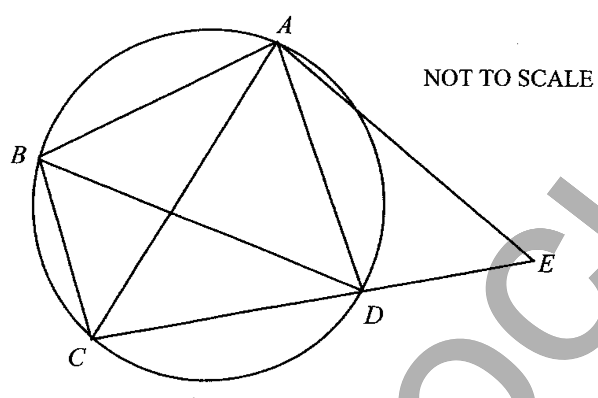
Catholic Schools Trial Examination

2007 Mathematics Extension 1

HSC SUPPORT

07 CT	**1a**	If $y = \sin^2 3x$ find $\frac{dy}{dx}$.	**2**
07 CT	**1b**	A(-3, 1), B(8, -2) and C(4, 16) are the vertices of a triangle. AM is a median of the triangle.	
		(i) Show that M has the coordinates (6, 7).	**1**
		(ii) Hence find the coordinates of the point which divides the interval AM internally in the ratio 2:1.	**2**
07 CT	**1c**	(i) Express $x^3 - 3x^2 + 4$ as a product of three linear factors by first showing that $(x - 2)$ is a factor of this polynomial.	**2**
		(ii) Hence solve the inequality $x^3 - 3x^2 + 4 \geq 0$.	**1**
07 CT	**1d**	 In the diagram, ABCD is a cyclic quadrilateral. E is a point on CD produced.	
		(i) Give a reason why $\angle ADE = \angle ABC$.	**1**
		(ii) If $\Delta ADE \,\|\|\|\, \Delta CBA$, show that AE || BD.	**3**
07 CT	**2a**	If $y = \log_{\frac{1}{a}}\left(\frac{1}{N}\right)$, where $a > 0$ and $N > 0$, show that $y = \log_a N$.	**2**
07 CT	**2b**	The lines $y = mx$ and $y = 2mx$ where $m > 0$, are inclined to each other at an angle θ such that $\tan\theta = \frac{1}{3}$.	
		(i) Show that $2m^2 - 3m + 1 = 0$	**1**
		(ii) Hence find the possible values of m.	**2**
07 CT	**2c**	(i) Show that $\tan 2x + \tan x = \frac{\sin 3x}{\cos 2x \cos x}$.	**2**
		(ii) Hence solve the equation $\tan 2x + \tan x = 0$ for $0 < x < \frac{\pi}{2}$.	**1**
07 CT	**2d**	P($2ap$, ap^2), Q($2aq$, aq^2) and R($2ar$, ar^2), where $p < q < r$, are three points on the parabola $x^2 = 4ay$.	
		(i) Use differentiation to show that the tangent to the parabola at Q has gradient q.	**1**
		(ii) If the chord PR is parallel to the tangent at Q, show that p, q and r are consecutive terms in an arithmetic sequence.	**2**
07	**3a**	Find the number of ways in which the letters of the word EPSILON can be arranged	**2**

CT in straight line so that the three vowels are all next to each other.

07 CT **3b** Use Mathematical induction to show that $5^n > 3^n + 4^n$ for all positive integers $n \geq 3$ **4**

07 CT **3c** Consider the function $f(x) = x + e^{-x}$, $x \geq 0$.

(i) Show that for all values of $x > 0$, the function $f(x)$ is increasing and the curve $y = f(x)$ is concave up. **2**

(ii) Sketch the graph of $y = f(x)$ showing clearly the coordinates of the endpoint and the equation of the asymptote. **2**

(iii) On the same diagram, sketch the graph of the inverse function $y = f^{-1}(x)$. **1**

(iv) Find the domain of the function $g(x) = f(x) + f^{-1}(x)$. **1**

07 CT **4a** (i) Show that the equation $x^3 - 2x - 5 = 0$ has a root α such that $2 < \alpha < 3$. **2**

(ii) Use one application of Newton's method and find an approximation of 2 to find the next approximation for α. **2**

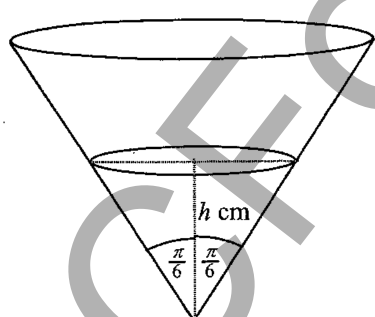
07 CT **4b** (i) Sketch the graph of $y = 2\sin^{-1}2x$, showing clearly the coordinates of the endpoints. **2**

(ii) Find the exact area of the region in the first quadrant bounded by the curve $y = 2\sin^{-1}2x$, the y-axis and the line $y = \pi$. **2**

07 CT **4c** (i) Show that $\dfrac{u^2}{1+u^2} = 1 - \dfrac{1}{1+u^2}$. **1**

(ii) Use the substitution $x = u^2$, $u \geq 0$, to find $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$. **3**

07 CT **5a**



An egg timer in the shape of an inverted right circular cone of semi vertical angle $\frac{\pi}{6}$ contains sand to a depth of h cm. The sand flows out of the apex of the cone at a constant rate of 0.5 cm^3/s.

(i) Show that the volume V cm^3 of sand in the cone is given by $V = \frac{1}{9}\pi h^3$. **1**

(ii) Find the value of h when the depth of sand in the egg timer is decreasing at a rate of 0.05 cm/s, giving your answer correct to 2 decimal places. **3**

07 CT **5b** A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} given by $v = -\frac{1}{8}x^3$, and acceleration a ms^{-2}. The particle is initially 2 metres to the right of O.

(i) Show that $a = \frac{3}{64}x^5$. **1**

(ii) Find an expression for x in terms of t. **2**

(iii) Describe the limiting motion of the particle. **1**

07 CT **5c** A game is played by throwing three fair coins. The game is considered a failure if all three coins show heads or all three coins show tails. Otherwise the game is considered a success.

(i) Show that the probability of success in any play of the game is $\frac{3}{4}$. **2**

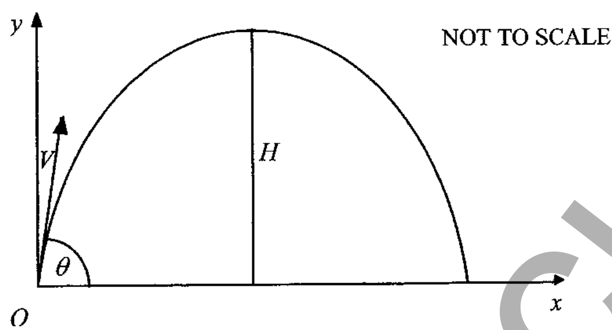
(ii) Find the probability of exactly two successes in four plays of the game. **2**

07 **6a** At time t years the area A km^2 controlled by a colony of animals is given by $A = 300 - 200e^{-kt}$ for some $k > 0$.

(i) Sketch the graph of A as a function of t showing clearly the initial and limiting areas controlled by the colony. **2**

(ii) Find the value of k if the area controlled by the colony is increasing at a rate of 10 km^2 per year when this area is twice its initial value. **2**

07 CT **6b**



A particle is projected from a point O with speed V ms^{-1} at an angle θ above the horizontal, where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. It moves in a vertical plane under gravity where the acceleration due to gravity is g ms^{-2}. At time t seconds its horizontal and vertical displacements from O are x metres and y metres respectively.

(i) Use integration to show that $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$. **2**

(ii) The particle takes T seconds to reach its greatest height H metres. Find expressions for T and H in terms of g, V and θ. **2**

(iii) The path of the particle is inclined at an angle $\frac{\pi}{4}$ above the horizontal at time $\frac{1}{4}T$. Show that $\theta = \tan^{-1}\frac{4}{3}$. **2**

(iv) What fraction of its maximum height has the particle attained at time $\frac{1}{4}T$? **2**

07 CT **7a** A particle is moving in straight line with Simple Harmonic motion. At time t seconds it has displacement x metres from a fixed point O on the line, where $x = (\cos t + \sin t)^2$, velocity v ms^{-1} and acceleration a ms^{-2}.

(i) Show that $a = -4(x - 1)$. **2**

(ii) Find the extreme positions of the particle during its motion. **1**

(iii) Find the time taken by the particle to move from one extreme position of its motion to the other extreme position. **1**

(iv) A second particle moves along a parallel straight line with Simple Harmonic motion so that $x = 1 - \cos t.$, where x metres is its displacement from a fixed **3**

			point level with O. During the first complete oscillation of the particle with the slower average speed, find the number of times the particles pass each other, and state their relative directions of travel on each occasion.	
07 CT	**7b**	(i)	Considering the identity $(1 - t)^n (1 + t)^n = (1 - t^2)^n$, where n is a positive integer, show that for integer values of r, $\sum_{k=0}^{2r}(-1)^k\ {}^nC_k\ {}^nC_{2r-k} = (-1)^r\ {}^nC_r$, provided $0 \le r \le \frac{1}{2}n$.	**2**
		(ii)	Hence show that $\sum_{k=0}^{r}(-1)^k\ {}^nC_k\ {}^nC_{2r-k} = \frac{1}{2}(-1)^r\ {}^nC_r\{1 + {}^nC_r\}$, provided $0 \le r \le \frac{1}{2}n$.	**2**
		(iii)	Hence evaluate $\sum_{k=0}^{10}(-1)^k\ ({}^{20}C_k)^2$.	**1**

A **1a.** $3\sin 6x$ **1b.(ii)** (3, 5) **1c.(i)** $(x - 2)^2(x + 1)$ **(ii)** $x \ge -1$ **1d.(i)** ext $\angle$ equals opp int $\angle$ in cyc. quad. **2b.(ii)** $\frac{1}{2}$, 1 **2c.(ii)** $\frac{\pi}{3}$ **3a.** $5! \times 3! = 720$ **3c.(iv)** $\{x : x \ge 1\}$ **4a.(ii)** 2.1 **4b.(ii)** 1 unit^2 **4c.(ii)** $2(\sqrt{x} - \tan^{-1}\sqrt{x}) + c$ **5a.(ii)** 3.09 cm **5b.(ii)** $x = -\frac{2}{\sqrt{1+t}}$ **(iii)** moving left approaching O and slowing down at a decreasing rate **5c.(ii)** $\frac{27}{128}$ **6a.(ii)** 0.1 **6b.(ii)** $T = \frac{V\sin\theta}{g}$, $H = \frac{V^2\sin^2\theta}{2g}$ **(iv)** $\frac{7}{16}$ **7a.(ii)** O and 2m right of O **(iii)** $\frac{\pi}{2}$ **(iv)** four times: first, second and fourth pass in opposite directions, third time moving in same direction **7b.(iii)** 17 067 482 146