CATHOLIC SECONDARY SCHOOLS ASSOCIATION

2007 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 1

Question 1

a. Outcomes assessed : H5, PE5

Marking Guidelines

Criteria	Marks
• applies the chain rule, writing one factor of the derivative as $2\sin 3x$	1
• obtains the second factor $3\cos 3x$ (even if final simplification is not carried out)	1

Answer

$$y = \sin^2 3x \qquad \therefore \frac{dy}{dx} = 2\sin 3x \,.\, 3\cos 3x = 3(2\sin 3x \cos 3x) = 3\sin 6x$$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • identifies M as the midpoint of BC and obtains its coordinates	1
ii • finds the x coordinate of the division point	1
• finds the y coordinate of the division point	1

Answer

i. M is the midpoint of BC. Hence M has coordinates $\left(\frac{8+4}{2}, \frac{-2+16}{2}\right) = (6, 7)$

ii.

$$\begin{array}{ccc} A & & M \\ (-3, 1) & & (6, 7) \\ 2 & : & 1 \end{array}$$

$$\left(\frac{2\times 6 + 1\times(-3)}{2+1}, \frac{2\times 7 + 1\times 1}{2+1}\right)$$

Required point has coordinates $(3, 5)$

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c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • shows that $(x-2)$ is a factor	1
• completes the factorisation	1
ii • solves the inequality	1

Answer

i. Let $P(x)=x^3-3x^2+4$. Then $P(2)=8-12+4=0$ $\therefore$ $(x-2)$ is a factor of $P(x)$

Then by inspection (or by division) $x^3-3x^2+4=(x-2)(x^2-x-2)=(x-2)^2(x+1)$

ii. Since $(x-2)^2\geq 0$ for all real x, $x^3-3x^2+4\geq 0$ whenever $x+1\geq 0$. $\therefore x\geq -1$

d. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • quotes an appropriate geometrical property as a reason	1
ii • uses the similarity to identify a pair of equal angles	1
• identifies equal alternate or corresponding angles using an appropriate circle property	1
• quotes an appropriate test for parallel lines to support the required deduction	1

Answer

i. In cyclic quadrilateral *ABCD*, the exterior angle *ADE* is equal to the interior opposite angle *ABC*.

ii. If $\Delta ADE \,|||\, \Delta CBA$, then $\angle AED=\angle CAB$ (*corresponding $\angle$'s in similar triangles are equal*)

But $\angle CAB=\angle CDB$ (*$\angle$'s subtended at the circumference by the same arc BC are equal*)

$\therefore \angle AED=\angle BDC$ (*both equal to $\angle CAB$*)

$\therefore AE \parallel BD$ (*equal corresponding angles on transversal CDE*)

Question 2

a. Outcomes assessed : H3

Marking Guidelines

Criteria	Marks
• converts the logarithm statement to an equivalent index statement	1
• after taking reciprocals, converts the index statement to an equivalent logarithm statement	1

Answer

The statement $y=\log_{\frac{1}{a}}\left(\frac{1}{N}\right)$ is equivalent to $\left(\frac{1}{a}\right)^y=\frac{1}{N}$.

Taking reciprocals, $a^y=N$ $\therefore y=\log_a N$

b. Outcomes assessed : P4, H5

Marking Guidelines

Criteria	Marks
i • uses the formula for the angle between two lines to obtain result	1
ii • factors quadratic expression	1
• writes down both possible values of m	1

Answer

i. θ is the acute angle between lines with gradients m and $2m$, where $m>0$.

$\therefore \left|\dfrac{2m-m}{1+2m^2}\right| = \tan\theta \Rightarrow \dfrac{m}{1+2m^2} = \frac{1}{3}$. Hence $2m^2-3m+1=0$.

ii. $(2m-1)(m-1)=0 \quad \therefore m=\frac{1}{2},\ 1$

c. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • writes expressions for tan in terms of sin and cos	1
• takes a common denominator and recognises the expression for sine of an angle sum	1
ii • solves $\sin 3x = 0$ in the required domain	1

Answer

i.

$$\tan 2x + \tan x = \frac{\sin 2x}{\cos 2x} + \frac{\sin x}{\cos x} = \frac{\sin 2x \cos x + \cos 2x \sin x}{\cos 2x \cos x} = \frac{\sin 3x}{\cos 2x \cos x}$$

ii. $\sin 3x = 0,\quad 0<3x<\frac{3\pi}{2} \quad \Rightarrow\ 3x=\pi$

Hence $\tan 2x + \tan x = 0,\quad 0<x<\frac{\pi}{2}$

has solution $x=\frac{\pi}{3}$

d. Outcomes assessed : PE3, PE4, H5

Marking Guidelines

Criteria	Marks
i • uses direct or parametric differentiation to establish result	1
ii • finds simplified expression for gradient of PR	1
• equates gradients of parallel lines	1
• rearranges relation to show $p,\ q,\ r$ in arithmetic progression	1

Answer

i. $y=\frac{1}{4a}x^2$

$\therefore \dfrac{dy}{dx} = \frac{1}{2a}x = q$ at Q

Hence tangent at Q has gradient q.

ii. $gradient\ PR = \dfrac{a(r^2 - p^2)}{2a(r-p)} = \dfrac{r+p}{2}$

If PR is parallel to tangent at Q

$$\frac{r+p}{2} = q$$
$$r + p = 2q$$
$$r - q = q - p$$

$\therefore\ p,\ q,\ r$ are in arithmetic progression

Question 3

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• groups vowels and arranges 5 objects • multiplies by number of arrangements of the 3 vowels	1 1

Answer

(EIO), P, S, L, N arranged in 5! ways, then E, I, O in 3! ways. Hence $5! \times 3! = 720$ arrangements.

b. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• defines a sequence of statements and establishes the truth of the first $5^3 > 3^3 + 4^3$ • writes an inequality for 5^{k+1} in terms of $3^k + 4^k$, conditional on the truth of $S(k)$ • works with this inequality to show that $5^{k+1} > 3^{k+1} + 4^{k+1}$ • deduces the required result from the previous steps	1 1 1 1

Answer

Define the sequence of statements $S(n)$: $5^n > 3^n + 4^n$, $n = 3, 4, 5, \ldots$

Consider $S(3)$: $5^3 = 125$, $3^3 + 4^3 = 91 \Rightarrow 5^3 > 3^3 + 4^3$ $\therefore S(3)$ is true.

If $S(k)$ is true : $5^k > 3^k + 4^k$ **

Consider $S(k+1)$:
$$\begin{aligned} 5^{k+1} &= 5\,.\,5^k \\ &> 5\left(3^k + 4^k\right) \quad \textit{if } S(k) \textit{ is true, using } ** \\ &= 5\,.\,3^k + 5\,.\,4^k \\ &> 3\,.\,3^k + 4\,.\,4^k \\ &= 3^{k+1} + 4^{k+1} \end{aligned}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(3)$ is true, hence $S(4)$ is true, and then $S(5)$ is true and so on. Hence by Mathematical induction, $5^n > 3^n + 4^n$ for all positive integers $n \geq 3$.

c. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
i • uses the first derivative to show the function is increasing	1
• uses the second derivative to show the curve is concave up	1
ii • sketches curve with correct shape showing the endpoint $(0,1)$	1
• shows the oblique asymptote with equation $y=x$	1
iii • sketches the inverse as the reflection in the line $y=x$	1
iv • writes the domain of the function g	1

Answer

i. $f(x)=x+e^{-x}\ ,\ x\ge 0$

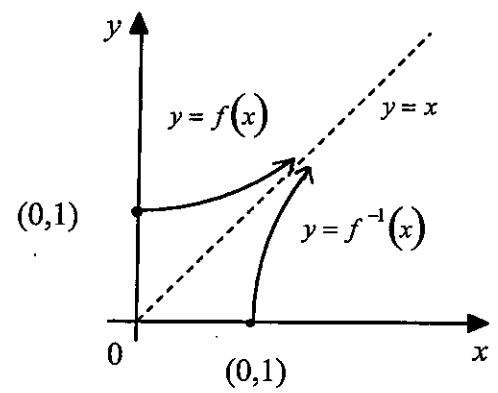
$f'(x)=1-e^{-x}>0$ for $x>0$

Hence function is increasing for $x>0$.

$f''(x)=e^{-x}>0$ for $x>0$

Hence the curve is concave up for $x>0$.

ii., iii.



iv. Domain of g is the intersection of the domains of f and f^{-1}.

$\therefore$ g has domain $\{x : x\ge 1\}$.

Question 4

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • shows that expression is negative for $x=2$ and positive for $x=3$.	1
• uses the continuity of the function to deduce equation has a root between 2 and 3	1
ii • calculates the value of the derivative at $x=2$	1
• applies Newton's method once to calculate the next approximation	1

Answer

i. Let $f(x)=x^3-2x-5$

Then $f(2)=8-4-5=-1<0$

and $f(3)=27-6-5=16>0$

But f is continuous. Hence there exists some real number α, $2<\alpha<3$, such that $f(\alpha)=0$.

ii. $f'(x)=3x^2-2 \Rightarrow f'(2)=10$

Using Newton's method, next approximation for the root α is

$$2-\frac{f(2)}{f'(2)}=2-\frac{-1}{10}=2\cdot 1$$

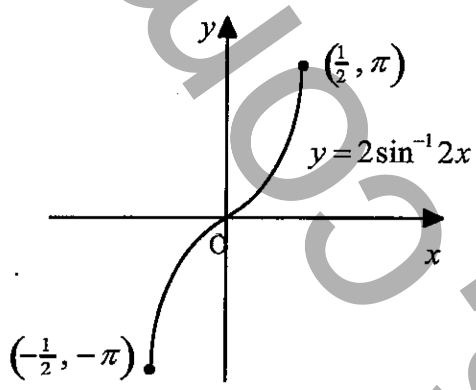
b. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i • sketches curve with correct shape and position	1
• shows the coordinates of the endpoints	1
ii • writes an expression for the area as the definite integral of a function of y	1
• finds the primitive then evaluates the definite integral	1

Answer

i.



ii. Area is A square units where $A = \int_0^{\pi} x\, dy$.

$$A = \int_0^{\pi} \tfrac{1}{2}\sin\left(\tfrac{1}{2}y\right)\ dy$$
$$= -\left[\cos\left(\tfrac{1}{2}y\right)\right]_0^{\pi}$$
$$= -\left(\cos\tfrac{\pi}{2} - \cos 0\right)$$
$$= 1$$

Hence area is 1 square unit.

c. Outcomes assessed : P4, HE6

Marking Guidelines

Criteria	Marks
i • shows the required result	1
ii • writes du in terms of dx and writes integrand as a function of u	1
• finds primitive as a function of u	1
• substitutes for u in terms of x	1

Answer

i.

$$\frac{u^2}{1+u^2} = \frac{\left(1+u^2\right)-1}{1+u^2}$$
$$= 1 - \frac{1}{1+u^2}$$

ii. $x = u^2 \qquad u > 0$

$dx = 2u\ du$

$$\int \frac{\sqrt{x}}{1+x}\ dx = \int \frac{u}{1+u^2}\ 2u\ du$$
$$= 2\int \frac{u^2}{1+u^2}\ du$$
$$= 2\int \left(1 - \frac{1}{1+u^2}\right)\ du$$
$$= 2\left(u - \tan^{-1} u\right) + c$$
$$= 2\left(\sqrt{x} - \tan^{-1}\sqrt{x}\right) + c$$

Question 5

a. Outcomes assessed : P4, HE5, HE7

Marking Guidelines

Criteria	Marks
i • expresses the radius of the cone of sand in terms of h then uses $V=\frac{1}{3}\pi r^2 h$	1
ii • expresses $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$	1
• substitutes appropriate negative values for both of these derivatives	1
• calculates depth to required accuracy	1

Answer

i. The cone of sand has radius $h\tan\frac{\pi}{6}$

$$\therefore V=\tfrac{1}{3}\pi\left(\tfrac{1}{\sqrt{3}}h\right)^2 h$$
$$=\tfrac{1}{9}\pi h^3$$

ii.
$$\frac{dV}{dt}=\tfrac{1}{9}\pi . 3h^2\frac{dh}{dt}$$
$$-0\cdot 5=\tfrac{1}{3}\pi h^2(-0\cdot 05)$$
$$\frac{3\times 0\cdot 5}{0\cdot 05\times\pi}=h^2$$
$$h=\sqrt{\tfrac{30}{\pi}}$$

Depth is $3\cdot 09$ cm (to 2 dec. pl.)

b. Outcomes assessed : HE5

Marking Guidelines

Criteria	Marks
i • writes a in terms of either $\frac{dv}{dx}$ or $\frac{dv^2}{dx}$ to obtain required result	1
ii • integrates $\frac{dt}{dx}$ and evaluates the constant to find t in terms of x	1
• rearranges to find x in terms of t, choosing the appropriate square root	1
iii • describes the limiting position, speed and acceleration as $t\to\infty$	1

Answer

i. $a=v\dfrac{dv}{dx}=-\tfrac{1}{8}x^3\cdot\left(-\tfrac{3}{8}x^2\right)$

$\therefore a=\tfrac{3}{64}x^5$

ii.
$$\frac{dx}{dt}=-\tfrac{1}{8}x^3$$
$$\frac{dt}{dx}=-8x^{-3}$$
$$t=4x^{-2}+c$$

$$\left.\begin{array}{l}t=0\\x=2\end{array}\right]\Rightarrow\quad c=-1$$
$$t=\frac{4}{x^2}-1$$
$$t+1=\frac{4}{x^2}$$

$\therefore x^2=\dfrac{4}{t+1}$ and $x>0$ for $t>0$

$$\therefore x=\frac{2}{\sqrt{1+t}}$$

iii. As $t\to\infty$, the particle is moving left, approaching O and slowing down at an ever decreasing rate with speed approaching 0.

c. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes an expression for the probability of success in terms of outcomes	1
• shows how this probability is calculated	1
ii • recognises this as a binomial probability and writes corresponding numerical expression	1
• calculates this probability	1

Answer

i. $P(success) = 1 - \{P(H,H,H) + P(T,T,T)\}$

$= 1 - \left\{\left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^3\right\}$

$= \tfrac{3}{4}$

ii. $P(exactly\ 2\ successes) = {}^4C_2 \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right)^2$

$= \tfrac{27}{128}$

Question 6

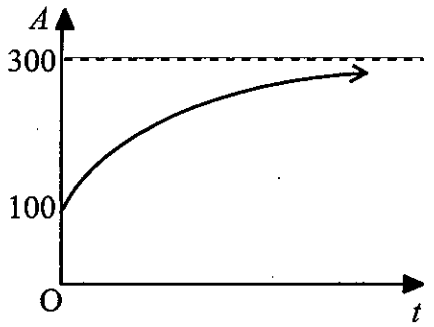
a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • sketches curve with correct shape and vertical intercept 100	1
• shows horizontal asymptote $A = 300$	1
ii • expresses $\dfrac{dA}{dt}$ in terms of A and k	1
• writes and solves an equation for k	1

Answer

i.



ii. $A = 300 - 200e^{-kt}$

$\dfrac{dA}{dt} = k\left(200e^{-kt}\right)$

$= k(300 - A)$

$10 = k(300 - 200)$

$\therefore k = 0 \cdot 1$

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • uses integration to show result for x	1
• uses integration to show result for y	1
ii • finds expression for T	1
• find expression for H	1
iii • finds $\dot{x}$ and $\dot{y}$ when $t=\frac{1}{4}T$	1
• deduces equality of horizontal and vertical components of velocity at this time to find θ	1
iv • finds y when $t=\frac{1}{4}T$	1
• expresses this y value as a fraction of H	1

Answer

i. $\ddot{x}=0$

$\dot{x}$ is constant $\quad \therefore \dot{x}=V\cos\theta$

$x=Vt\cos\theta+c_1$

$\left.\begin{matrix} t=0 \\ x=0 \end{matrix}\right] \Rightarrow c_1=0 \qquad \therefore x=Vt\cos\theta$

$\ddot{y}=-g$

$\dot{y}=-gt+c_2$

$\left.\begin{matrix} t=0 \\ \dot{y}=V\sin\theta \end{matrix}\right] \Rightarrow c_2=V\sin\theta \qquad \therefore \dot{y}=V\sin\theta-gt$

$y=Vt\sin\theta-\frac{1}{2}gt^2+c_3$

$\left.\begin{matrix} t=0 \\ y=0 \end{matrix}\right] \Rightarrow c_3=0 \qquad \therefore y=Vt\sin\theta-\frac{1}{2}gt^2$

ii. At greatest height, $\dot{y}=0,\ \ y=H$

$\therefore gT=V\sin\theta$

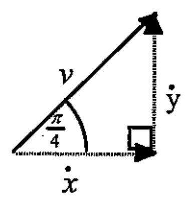
$T=\dfrac{V\sin\theta}{g}$

$$H=V\left(\frac{V\sin\theta}{g}\right)\sin\theta-\tfrac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2=\frac{V^2\sin^2\theta}{2g}$$

iii. When $t=\frac{1}{4}T$,

$\dot{y}=\frac{3}{4}V\sin\theta$

$\dot{x}=V\cos\theta$



$\therefore \quad \dot{y}=\dot{x}$

$\frac{3}{4}\sin\theta=\cos\theta$

$\tan\theta=\frac{4}{3}$

$\therefore \quad \theta=\tan^{-1}\frac{4}{3}$

iv. When $t=\frac{1}{4}T,\quad y=V\left(\dfrac{V\sin\theta}{4g}\right)\sin\theta-\frac{1}{2}g\left(\dfrac{V\sin\theta}{4g}\right)^2=\dfrac{7V^2\sin^2\theta}{32g}=\frac{7}{16}H$

Hence particle has attained $\frac{7}{16}$ of its maximum height.

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • uses trigonometric identities to simplify expression for x	1
• finds second derivative then rearranges to obtain required result	1
ii • states both extreme positions	1
iii • states time taken	1
iv • realises that the second particle with period 2π seconds has the slower average speed	1
• graphs x as a function of t for both particles on the same axes and counts the intersections (or solves simultaneous trigonometric equations) for $0 \le t \le 2\pi$	1
• investigates the signs of the gradients of the graphs at the intersection points (or the values of v at the solutions of the corresponding trigonometric equation) and relates these to the relative directions of travel	1

Answer

i. $x = (\cos t + \sin t)^2$

$= \cos^2 t + \sin^2 t + 2\sin t \cos t$

$= 1 + \sin 2t$

$v = 2\cos 2t$

$a = -4\sin 2t$

$= -4(x-1)$

ii. $-1 \le \sin 2t \le 1$

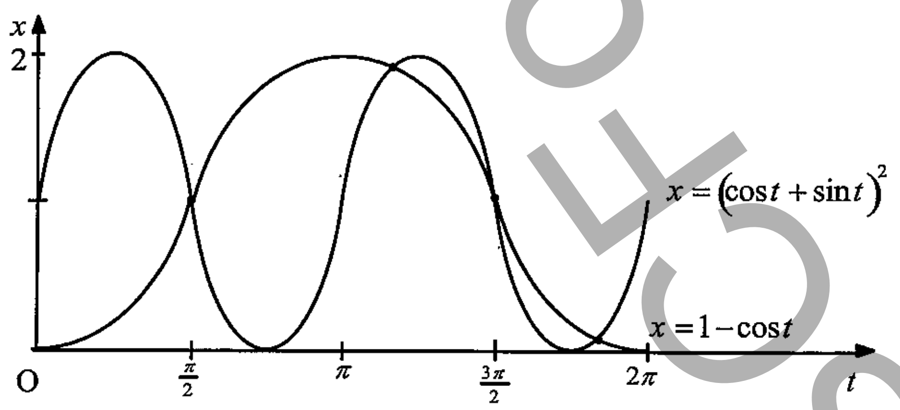
$\therefore 0 \le x \le 2$, where x attains both extremes.

Extreme positions are O and the point 2 m to right of O.

iii. Period of motion is $\frac{2\pi}{2} = \pi$ seconds.

Hence time taken is $\frac{\pi}{2}$ seconds.

iv. The second particle moves between extreme positions where $x = 0$ and $x = 2$, starting at $x = 0$ and taking 2π seconds for each complete oscillation, and hence has the slower average speed.



The 4 intersection points correspond to 4 times when the particles pass each other while the second particle completes its first oscillation.

The gradient of the curve gives the velocity of the corresponding particle, and hence the sign of the gradient gives the direction of the particle.

Hence on the first, second and fourth occasions that they pass each other they are travelling in opposite directions, but on the third occasion they are travelling in the same direction.

b. Outcomes assessed : PE6, HE3

Marking Guidelines

	Criteria	Marks
i	• applies Binomial expansions to $(1-t)^n, (1+t)^n$ and $(1-t^2)^n$	1
	• equates coefficients of t^{2r} on both sides of the identity to obtain required result	1
ii	• realises that in the summation in (i), the terms given by $k=r-j$ and $k=r+j$ are the same for $j=1,2,3,\dots,r$.	1
	• uses this fact with appropriate rearrangement and regrouping to obtain required result	1
iii	• substitutes appropriate values for n and r then uses (ii) to calculate given sum.	1

Answer

i. $(1-t)^n(1+t)^n \equiv (1-t^2)^n$ ** Using the binomial expansion for each factor on the LHS,

$$(1-t)^n = 1 - {}^nC_1 t + {}^nC_2 t^2 - {}^nC_3 t^3 + \dots + (-1)^k\, {}^nC_k t^k + \dots + (-1)^n\, {}^nC_n t^n$$

$$(1+t)^n = 1 + {}^nC_1 t + {}^nC_2 t^2 + {}^nC_3 t^3 + \dots + {}^nC_k t^k + \dots + {}^nC_n t^n$$

The term in t^{2r} on the LHS of ** is formed by adding the products of terms $(-1)^k\, {}^nC_k t^k$ and ${}^nC_{2r-k} t^{2r-k}$ taken from the first and second expansions respectively in all possible ways. Provided $2r \le n$, k can take all the values $0, 1, 2, \dots, 2r$ when forming this product, giving the coefficient of t^{2r} as $\sum_{k=0}^{2r} (-1)^k\, {}^nC_k\, {}^nC_{2r-k}$.

The binomial expansion of the RHS of ** gives the coefficient of t^{2r} as $(-1)^r\, {}^nC_r$.

Hence, provided $0 \le r \le \frac{1}{2}n$, equating coefficients of t^{2r} on both sides of the identity ** gives

$$\sum_{k=0}^{2r} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} = (-1)^r\, {}^nC_r.$$

ii. For $j = 1, 2, 3, \dots, r$,

$$\begin{aligned}(-1)^{r-j}\, {}^nC_{r-j}\, {}^nC_{2r-(r-j)} &= (-1)^r(-1)^{-j}\, {}^nC_{r-j}\, {}^nC_{r+j}\\ &= (-1)^r(-1)^j\, {}^nC_{r-j}\, {}^nC_{r+j}\\ &= (-1)^{r+j}\, {}^nC_{r+j}\, {}^nC_{r-j}\\ &= (-1)^{r+j}\, {}^nC_{r+j}\, {}^nC_{2r-(r+j)}\end{aligned}$$

Hence the terms in the sum obtained by putting $k = r - j$ and $k = r + j$ are equal.

$$\begin{aligned}\therefore \sum_{k=0}^{2r} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} &= \sum_{k=0}^{r-1} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} + (-1)^r\, {}^nC_r\, {}^nC_r + \sum_{k=r+1}^{2r} (-1)^k\, {}^nC_k\, {}^nC_{2r-k}\\ &= 2\sum_{k=0}^{r-1} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} + (-1)^r\, {}^nC_r\, {}^nC_r\\ &= 2\sum_{k=0}^{r} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} - (-1)^r\, {}^nC_r\, {}^nC_r\end{aligned}$$

But $\sum_{k=0}^{2r}(-1)^k \, {}^nC_k \, {}^nC_{2r-k} = (-1)^r \, {}^nC_r$ for $0 \le r \le \frac{1}{2}n$.

$$\therefore \ (-1)^r \, {}^nC_r = 2\sum_{k=0}^{r}(-1)^k \, {}^nC_k \, {}^nC_{2r-k} - (-1)^r \, {}^nC_r \, {}^nC_r$$

$$(-1)^r \, {}^nC_r + (-1)^r \, {}^nC_r \, {}^nC_r = 2\sum_{k=0}^{r}(-1)^k \, {}^nC_k \, {}^nC_{2r-k}$$

$$\therefore \sum_{k=0}^{r}(-1)^k \, {}^nC_k \, {}^nC_{2r-k} = \tfrac{1}{2}(-1)^r \, {}^nC_r \left\{1 + {}^nC_r\right\} \quad \text{for } 0 \le r \le \tfrac{1}{2}n.$$

iii. Substituting $n = 20$ and $r = 10$ gives $\sum_{k=0}^{10}(-1)^k \, {}^{20}C_k \, {}^{20}C_{20-k} = \frac{1}{2}(-1)^{10} \, {}^{20}C_{10}\left\{1 + {}^{20}C_{10}\right\}$ since $10 \le \frac{1}{2} \times 20$.

But ${}^{20}C_k = {}^{20}C_{20-k}$ *for* $k = 1, 2, \ldots, 20$.

Hence

$$\sum_{k=0}^{10}(-1)^k \left({}^{20}C_k\right)^2 = \tfrac{1}{2}(-1)^{10} \, {}^{20}C_{10}\left\{1 + {}^{20}C_{10}\right\}$$

$$= \tfrac{1}{2} \times 184756 \times 184757$$

$$= 17067482146$$