SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2006

YEAR 12
TRIAL HIGHER SCHOOL
CERTIFICATE

Mathematics Extension 1

General Instructions

- Working time – 2 Hours
- Reading time – 5 Minutes
- Write using black or blue pen
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work.
- Hand in your answer booklets in 4 sections. Section A (Questions 1 and 2), Section B (Questions 3 and 4), Section C (Questions 5 and 6) and Section D (Question 7).

Total Marks – 84

- Attempt Questions 1 – 7.
- All QUESTIONS are of equal value.

Examiner: *R. Boros*

Section A – Start a new booklet **Marks**

Question 1. (12 marks)

a)

(i) Evaluate $\int_0^1 \frac{x}{x^2+1}\,dx$ leaving your answer in exact form. 2

(ii) Evaluate $\int_{-2}^{2\sqrt{3}} \frac{1}{x^2+4}\,dx$ leaving your answer in exact form. 2

b) Find the gradient of the tangent to the curve $y = \tan^{-1}(\sin x)$ at $x = 0$. 2

c) Solve for x, $\frac{1}{x+1} < 3$. 2

d) Give the general solution of the equation, $\cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. 2

e) If $f(x) = 8x^3$, then find the inverse function $f^{-1}(x)$. 2

Question 2. (12 marks)

a) Find the co-ordinates of the point P that divides the interval $A(-4,-6)$ and $B(6,-1)$ externally in the ratio 3:1. 2

b)

(i) Sketch the graph of $y = |2x-4|$. 2

(ii) Using your graph, or otherwise, solve the inequation $|2x-4| > x$. 2

c) Use the substitution $u = 1 + x$ to evaluate, $\int_{-1}^{3} x\sqrt{1+x}\,.dx$. 2

d) Solve for n, $2 \times {}^nC_4 = 5 \times {}^nC_2$. 2

e) What is the least distance between the circle $x^2 + y^2 + 2x + 4y = 1$ and the line $3x + 4y = 6$? (Leave your answer in exact form.) 2

End of Section A

Section B – Start a new booklet **Marks**

Question 3. (12 marks)

a) If the roots of the equation, $x^4 - 2x^3 - 5x + 1 = 0$, are t_1, t_2, t_3, t_4,

find $\sum_{1}^{4} (t_i t_j t_k)^{-1}$, such that $i \neq j \neq k$. 2

b) State the domain and range of the function $y = 2\sin^{-1}\left(\frac{x}{3}\right)$.

Hence sketch the curve. 3

c) A bowl of water heated to $100^\circ C$ is placed in a coolroom where the temperature is maintained at $-5^\circ C$. After t minutes, the temperature $T^\circ C$ of the water is changing so that $\frac{dT}{dt} = -k(T+5)$.

(i) Prove that $T = Ae^{-kt} - 5$ satisfies this equation and find the value of A. 1

(ii) After 20 minutes, the temperature of the water has fallen to $40^\circ C$. How long, to the nearest minute, will the water need to be in the coolroom before ice begins to form, (i.e. the temperature falls to $0^\circ C$). 2

d) (i) Show that the equation $\ln x + x^2 - 4x = 0$ has a root lying between $x = 3$ and $x = 4$. 2

(ii) By taking $x = 4$ as a first approximation, use one application of Newton's Method to obtain another approximation for the root, to 2 decimal places. Is this newer approximation a better one? Explain. 2

Question 4. (12 marks) **Marks**

a) The points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$. It is given that the chord PQ has equation $y = \left(\frac{p+q}{2}\right)x - apq$.

(i) Show that the gradient of the tangent at P is p. 1

(ii) Prove that if PQ passes through the focus, then the tangent at P is parallel to the normal at Q. 2

b) A committee of five is to be formed from 4 Liberal senators, 3 Labor senators and 2 Democrat senators.

(i) How many different committees can be formed that have 3 Liberals, 1 Labor and 1 Democrat? 1

(ii) If the committee is to be chosen at random, what is the probability that there will be a Liberal majority in the committee? 2

c) (i) Express $7\cos\theta - \sin\theta$ in the form $R\cos\left(\theta + \alpha\right)$, where $R > 0$ and $0^\circ \le \alpha \le 90^\circ$. 2

(ii) Hence solve $7\cos\theta - \sin\theta = 5$ for $0^\circ \le \theta \le 360^\circ$, giving your answer to the nearest degree. 2

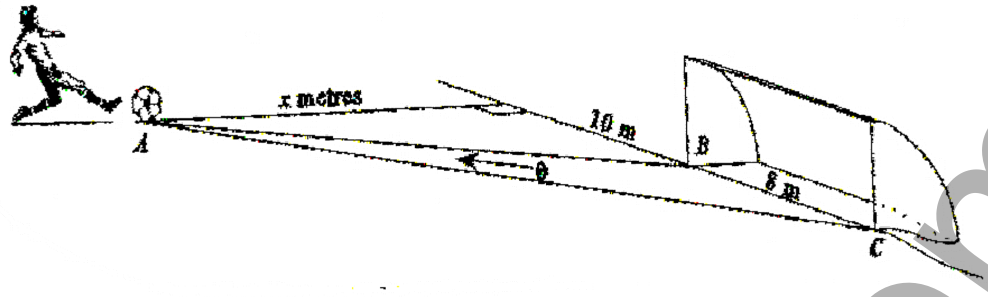
d) Find the values of the constants a and b if $x^2 - 2x - 3$ is a factor of the polynomial $P\left(x\right) = x^3 - 3x^2 + ax + b$. 2

End of Section B

Section C – Start a new booklet **Marks**

Question 5. (12 marks)

a)



A soccer player A is x metres from a goal line of a soccer field. He takes a shot at the goal BC, with the ball not leaving the ground.

(i) Show that the angle θ within which he must shoot is given by $\theta = \tan^{-1}\left(\frac{8x}{180+x^2}\right)$ when he is 10 metres to one side of the near goal post and 18 metres to the same side of the far post. 2

(ii) Find the value of x which makes this angle a maximum. (Leave your answer in exact form). 2

b) A particle moves in a straight line such that its velocity V m/s is given by $V = 2\sqrt{2x-1}$ when it is x metres from the origin. If $x = \frac{1}{2}$ when $t = 0$ find:

(i) the acceleration. 1

(ii) an expression for x in terms of t. 2

c) Find the volume of the solid obtained by rotating $y = \sin^{-1} x$ about the y-axis between $y = -\frac{\pi}{4}$ and $y = \frac{\pi}{4}$. Answer in exact form. 3

d) The perimeter of a circle is increasing at 3 cm/s. Leaving your answer in terms of π, find the rate at which the area is increasing when the perimeter is 1m. 2

Question 6. (12 marks) **Marks**

a)

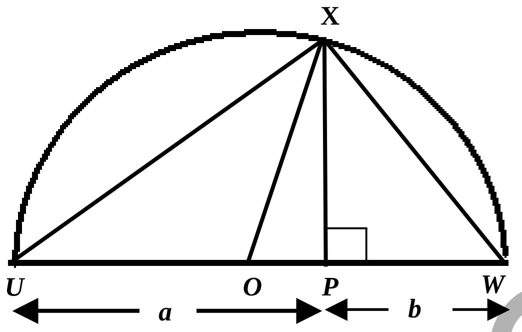
Consider the following three expressions involving n, where n is a positive integer: $5^n+3,\ 7^n+5,\ 5^n+7$

(i) By substituting values of n, show that 7^n+5 is the only one of these expressions which could be divisible by 6 for all positive integers n. 1

(ii) Use mathematical induction to show that the expression 7^n+5 is in fact divisible by 6 for all positive integers n. 2

b)

Not to scale



In the diagram UXW is a semi-circle with O as a midpoint of diameter UW. The point P lies on UW and XP is perpendicular to UW. The length of $UP=a$ units and $PW=b$ units are shown.

(i) Explain why $OX=\dfrac{a+b}{2}$. 1

(ii) Show that $\triangle UXP \,|||\, \triangle XWP$. 1

(iii) Deduce that $XP=\sqrt{ab}$. 1

(iv) By using the diagram show that $\dfrac{a+b}{2}\ge\sqrt{ab}$. 1

c)

The displacement x metres of a particle from the origin is given by $x=5\cos\left(3t-\dfrac{\pi}{6}\right)$, where t is the time lapsed in seconds.

(i) Show that $\ddot{x}=-9x$. 1

(ii) Find the period of the motion 1

Marks

d) Suppose that $\left(5+2x\right)^{12} = \sum_{k=0}^{12} a_k x^k$.

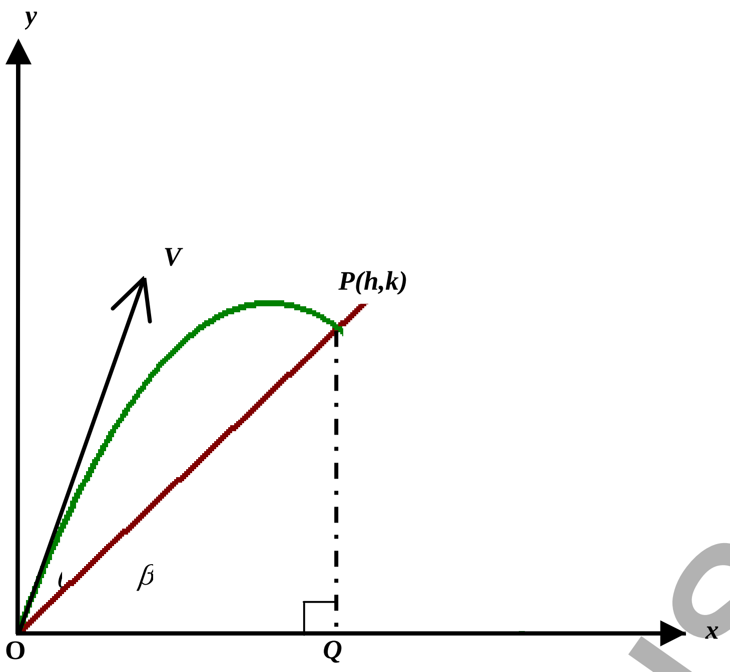
(i) Use the binomial theorem to write the expression for a_k . 1

(ii) Show that $\dfrac{a_{k+1}}{a_k} = \dfrac{24-2k}{5k+5}$ 2

End of Section C

Section D – Start a new booklet **Marks**

Question 7. (12 marks)



A projectile is fired from the origin with a velocity V and an angle of elevation θ, where $\theta \neq 90^\circ$. You may assume that $x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$, where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing, and g is the acceleration due to gravity.

(i) Show that the Cartesian equation of the flight of the projectile is: **1**

$$y = x\tan\theta - \frac{g}{2V^2\cos^2\theta}x^2$$

(ii) Suppose the projectile is fired up a plane inclined at β to the horizontal so that $0^\circ \leq \beta \leq \theta$. If the projectile strikes the plane at $P(h,k)$, show that: **2**

$$h = \frac{(\tan\theta - \tan\beta)2V^2\cos^2\theta}{g}$$

(iii) Hence, show that the range OP of the projectile can be given by **4**

$$OP = \frac{2V^2\sin(\theta-\beta)\cos\theta}{g\cos^2\beta}$$

Marks

(iv) Given the fact that $2\sin(x-\beta)\cos x = \sin(2x-\beta) - \sin\beta$. Show that the maximum value of the range of OP is given by:

$$\frac{V^2}{g(1+\sin\beta)}$$ 4

(v) If the angle of inclination of the plane is 14°, at what angle to the horizontal should the projectile be fired in order to attain maximum range? 1

End of Examination

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$