



SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

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**Trial Higher School Certificate
Examination**

YEAR 12

Mathematics Extension 1

Sample Solutions

Section	Marker
A	RD
B	RB
C	FN
D	AMG

Section A

$$\begin{aligned} \text{Q1 (a)} \quad \frac{3^n}{3^{n+1} - 3^n} &= \frac{3^n}{3^n(3-1)} \\ &= \frac{1}{2} \quad (1) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 0} \frac{\sin 5x}{4x} &= \lim_{5x \rightarrow 0} \frac{\sin 5x}{5x} \times \frac{5}{4} \\ &= \frac{5}{4} \quad (1) \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(3) &= 27 - 27 + 3p - 14 = 1 \\ \therefore 3p &= 15 \\ p &= 5 \quad (2) \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \log_a 2a &= \log_a 2 + \log_a a \\ &= x + 1 \quad (2) \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad P &\equiv \left(\frac{-3 \times 6 + 2 \times -1}{-3 + 2}, \frac{-3 \times -4 + 2 \times 5}{-3 + 2} \right) \\ &\equiv \left(\frac{-20}{-1}, \frac{22}{-1} \right) \\ &\equiv (20, -22) \quad (2) \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad \tan \theta &= \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \\ &= \left| \frac{-\frac{3}{2} - \frac{1}{5}}{1 + -\frac{3}{2} \times \frac{1}{5}} \right| \end{aligned}$$

$$= \left| \frac{-\frac{15}{10} - \frac{2}{10}}{1 - \frac{3}{10}} \right|$$

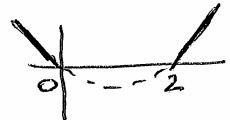
$$= \left| -\frac{17}{7} \right|$$

$$= \frac{17}{7}$$

$$\therefore \theta = 67^\circ 37' \quad (2)$$

$$\begin{aligned} \text{(g)} \quad \frac{2}{x} &\leq 1 \\ 2x &\leq x^2 \\ 0 &\leq x^2 - 2x \\ x(x-2) &\geq 0 \end{aligned}$$

$$x < 0 \text{ or } x \geq 2$$



(2)

12

Q2

$$(a) (i) y = \tan^3(5x+4)$$

$$y' = 3 \tan^2(5x+4) \cdot \sec^2(5x+4) \cdot 5$$

$$= 15 \tan^2(5x+4) \cdot \sec^2(5x+4)$$

$$(ii) y = \ln\left(\frac{2x+3}{3x+4}\right)$$

$$= \ln(2x+3) - \ln(3x+4)$$

$$y' = \frac{1}{2x+3} \times 2 - \frac{1}{3x+4} \times 3$$

$$= \frac{2}{2x+3} - \frac{3}{3x+4}$$

$$(iii) y = \cos(e^{1-5x})$$

$$y' = -\sin(e^{1-5x}) \cdot e^{1-5x} \cdot -5$$

$$= 5 \sin(e^{1-5x}) \cdot e^{1-5x}$$

$$(b) (i) 30 \times 29 \times 28 \times 27 \times 26 \times 25$$

$$= 427\ 518\ 000$$

(ii) Choose the six finalists
and then consider where they
may be placed:

$${}^1C_1 \times {}^{29}C_5 \times 6!$$

$$= 118\ 755 \times 270$$

$$= 85\ 503\ 600$$

$$(c) (i) f\left(\frac{1}{2}\right) = \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

$$(ii) f'(x) = \frac{1}{1+(2x)^2} \times 2$$

$$= \frac{2}{1+4x^2}$$

$$f'\left(\frac{1}{2}\right) = \frac{2}{1+4 \times \frac{1}{4}}$$

$$= 1$$

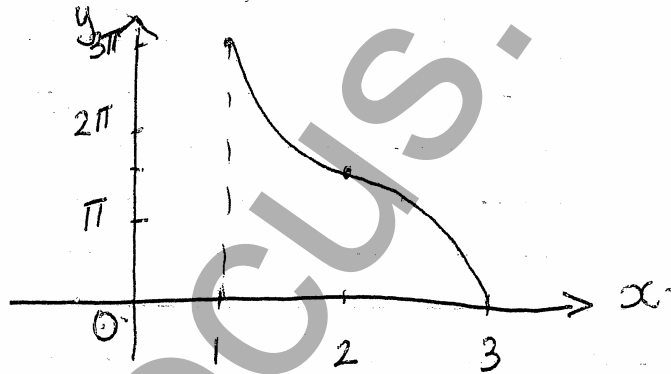
Section B

(3) (a) (i) $y = 3\cos^{-1}(x-2)$

Domain $-1 \leq x-2 \leq 1$
 $\begin{matrix} +2 & +2 & +2 \\ 1 & \leq x & \leq 3. \end{matrix}$

Range $0 \leq \cos^{-1}(x-2) \leq \pi$
 $0 \leq 3\cos^{-1}(x-2) \leq 3\pi$

(ii) $y = 3\cos^{-1}(x-2)$



(b) $\int x \cdot \sqrt{16+x^2} dx$ using $u = 16+x^2$

$u = 16+x^2$
 $\frac{du}{dx} = 2x$
becomes $\int \frac{du}{2} \cdot u^{\frac{1}{2}}$

$du = 2x dx$

$\frac{du}{2} = x dx$

$\frac{1}{2} \int u^{\frac{1}{2}} du$

$= \frac{1}{2} \cdot u^{\frac{3}{2}} \cdot \frac{2}{3} + C$

$= \frac{1}{3} u^{\frac{3}{2}} + C$

$= \frac{1}{3} (16+x^2) \sqrt{16+x^2} + C$

or $\frac{1}{3} [\sqrt{16+x^2}]^3 + C$

(c) general soln to $\sin 2\theta = \sqrt{3} \cos 2\theta$

$$\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \frac{\sqrt{3}}{1}$$

So $2\theta = \frac{\pi}{3} + k\pi$

$$\theta = \frac{\pi}{6} + \frac{k\pi}{2} \text{ where } k = 0, 1, 2, 3, \dots$$

(d) $4x^3 + 6x^2 + C = 0$

$C \neq 0$, roots are $\alpha, \beta, \alpha\beta$.

$$a = 4$$

$$b = 6$$

$$c = 0$$

$$d = C$$

(i) sum of roots $\alpha + \beta + \alpha\beta = -\frac{b}{a} = -\frac{6}{4}$

product $\alpha\beta\alpha\beta = (\alpha\beta)^2 = -\frac{d}{a} = -\frac{C}{4}$

product in twos $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = \frac{C}{a} = \frac{0}{4} = 0$

now since $(\alpha\beta)^2 = -\frac{C}{4}$ and $C \neq 0$

then $\alpha\beta \neq 0$

(ii) from above, since $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = \frac{0}{4} = 0$

then $\alpha\beta(1 + \alpha + \beta) = 0$

So $\alpha\beta = 0$, but it cannot from (i)

So $1 + \alpha + \beta = 0$

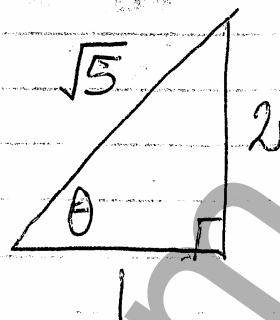
$$\alpha + \beta = -1$$

(iii) since $\alpha + \beta + \alpha\beta = -\frac{6}{4} = -\frac{3}{2}$ from above
and $\alpha + \beta = -1$

$$-1 + \alpha\beta = -\frac{3}{2}$$

$$\alpha\beta = -\frac{3}{2} + 1 = -\frac{1}{2}$$

4 (a) $\tan \theta = 2$, $0 < \theta < \frac{\pi}{2}$
 evaluate $\sin(\theta + \frac{\pi}{4})$



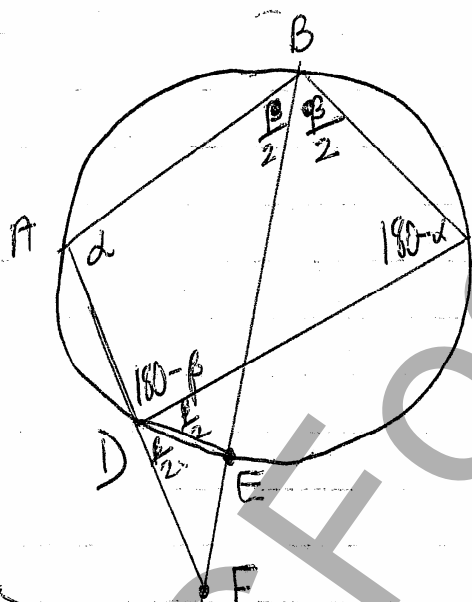
$$\sin(\theta + \frac{\pi}{4}) = \sin \theta \cos \frac{\pi}{4} + \cos \theta \sin \frac{\pi}{4}$$

$$= \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{10}} + \frac{1}{\sqrt{10}} = \frac{3}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

(0.94868...)

(b) (i)



cyclic quad.

$$\alpha + (180 - \alpha) = 180^\circ$$

$$\frac{\beta}{2} + \frac{\beta}{2} + (180 - \beta) = 180^\circ$$

(ii) $\angle CBE = \frac{\beta}{2}$ stands on minor arc CE

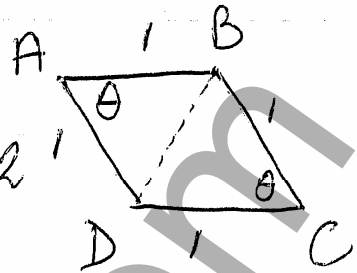
$\angle CDE = \frac{\beta}{2}$ also since it stands on minor arc CE

(iii) now $\angle ADC + \angle CDE + \angle EDF = 180^\circ$ straight line angle
 $(180 - \beta) + \frac{\beta}{2} + \angle EDF = 180^\circ$

$$-\frac{\beta}{2} + \angle EDF = 0 \Rightarrow \angle EDF = \frac{\beta}{2}$$

$\Rightarrow DE$ bisects $\angle CDF$

4 (c) Square ABCD side 1 unit
 $\frac{d\theta}{dt} = -0.1$ radians/sec.



(i) area rhombus $A = \frac{1}{2} \times 1 \times 1 \times \sin \theta \times 2$
 $A = \sin \theta$

Now $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$
 $= \cos \theta \times -0.1$

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = -0.1 \times \frac{\sqrt{3}}{2} = -\frac{1}{10} \times \frac{\sqrt{3}}{2}$
 $= -\frac{\sqrt{3}}{20}$ units²/sec.

Area is decreasing at a rate of $\frac{\sqrt{3}}{20}$ u²/s. (-0.0866...)

(ii) shorter diagonal BD.

$(BD)^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos \theta$
 $= 2 - 2\cos \theta$

$BD = \sqrt{2(1 - \cos \theta)} = \sqrt{2} \cdot (1 - \cos \theta)^{\frac{1}{2}}$

$\frac{dBD}{dt} = \frac{dBD}{d\theta} \times \frac{d\theta}{dt}$

$= \sqrt{2} \times \frac{1}{2} (1 - \cos \theta)^{-\frac{1}{2}} \times \sin \theta \times -0.1$
 $= \frac{\sqrt{2} \times \sin \theta \times -0.1}{2 \cdot \sqrt{1 - \cos \theta}}$

$$\text{At } \theta = \frac{\pi}{3}$$

$$\frac{dBD}{dt} = \frac{\sqrt{2} \times \frac{\sqrt{3}}{2} \times -0.1}{2 \cdot \sqrt{1 - \frac{1}{2}}}$$

$$= \frac{\frac{\sqrt{6}}{2} \times -\frac{1}{10}}{2 \times \sqrt{\frac{1}{2}}}$$

$$= \frac{\frac{\sqrt{6}}{2} \times -\frac{1}{10} \times \frac{\sqrt{2}}{2}}{\frac{2}{\sqrt{2}}}$$

$$= -\frac{\sqrt{12}}{40} = -\frac{2\sqrt{3}}{40} = -\frac{\sqrt{3}}{20} \text{ u/s}$$

shorter diagonal decreasing at $\frac{\sqrt{3}}{20} \text{ u/s}$

Section C

QUESTION 5

(a)

$$(i) \quad \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}$$

$$(ii) \quad \frac{1}{6} + \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \left(\frac{5}{6}\right)^4 \times \frac{1}{6} \dots \dots \dots \text{geometric series}$$

$$S_{\infty} = \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{6}{11}$$

$$(b)(i) \quad y = \frac{x^2}{4a}, y' = \frac{x}{2a} = \frac{2at}{2a} = t = \text{gradient of tangent}$$

$$\text{gradient of normal} = -\frac{1}{t}$$

$$\text{eqn. of normal is } y - at^2 = -\frac{1}{t}(x - 2at)$$

$$yt - at^3 = -x + 2at$$

$$x + ty - 2at - at^3 = 0 \text{ as required.}$$

$$(ii) \quad \text{when } y = 0, x = 2at + at^3 \quad \times (2at + at^3, 0)$$

$$\text{when } x = 0, y = \frac{2at + at^3}{t} = 2a + at^2 \quad \vee (0, 2a + at^2)$$

$$(iii) \quad \text{Midpoint, P is } \left(at + \frac{at^3}{2}, a + \frac{at^2}{2}\right)$$

$$2at = at + \frac{at^3}{2} \quad at^2 = a + \frac{at^2}{2}$$

$$4at = 2at + at^3 \quad 2at^2 = 2a + at^2$$

$$4 = 2 + t^2 \quad 2t^2 = 2 + t^2$$

$$t = \pm\sqrt{2} \quad t = \sqrt{2}, t > 0$$

$$(c)(i) \quad \angle TOP = \pi - \phi$$

$$\tan \angle TOP = \frac{PT}{r} = -\tan \phi, \quad PT = -r \tan \phi$$

$$\text{area } \Delta TOP = \text{area sector TOA (given)}$$

$$\frac{1}{2}r \times PT = \frac{1}{2}r^2\phi$$

$$-r \tan \phi = r\phi$$

$$-\tan \phi = \phi$$

$$\phi + \tan \phi = 0 \text{ as required.}$$

$$(ii) \quad a_1 = a - \frac{f(a)}{f'(a)} = a_1 = 2 - \frac{f(2)}{f'(2)}$$

$$2 - \frac{2 + \tan 2}{1 + \sec^2 2} = 2 - \frac{-0.185}{6.774}$$

$$2.03 \text{ (2d.p.)}$$

QUESTION 6

$$(a)(i) \quad \text{if } x = \alpha \cos(2t + \beta)$$

$$\frac{dx}{dt} = -2\alpha \sin(2t + \beta)$$

$$\frac{d^2x}{dt^2} = -4\alpha \cos(2t + \beta) = -4x \text{ (a possible equation)}$$

$$(ii) \quad v^2 = n^2(\alpha^2 - x^2), \quad n = 2 \text{ and } x = 4 \text{ when } v = 2$$

$$4 = 4(\alpha^2 - 16)$$

$$\alpha = \sqrt{17} \text{ m}$$

$$(ii) \quad \text{Max velocity when displacement} = 0$$

$$v^2 = 4(17 - 0)$$

$$v = 2\sqrt{17} \text{ m/s}$$

$$(b) \quad \text{When } n = 1, 1^3 = \frac{1}{4} \times 1^2 \times 2^2 - P(1) \text{ is true}$$

$$\text{Assume } P(k) \text{ is true } 1^3 + 2^3 \dots + k^3 = \frac{1}{4}k^2(k+1)^2$$

$$\text{if } n = k + 1,$$

$$1^3 + 2^3 \dots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$$

$$\text{LHS} = \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \text{ (using assumption)}$$

$$= (k+1)^2 \left(\frac{1}{4}k^2 + k + 1 \right)$$

$$= (k+1)^2 \frac{1}{4}(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \text{RHS}$$

$$P(k+1) \text{ is true if } P(k) \text{ is true. } P(1) \text{ is true.}$$

$$\therefore, \text{ by Mathematical Induction, } P(n) \text{ is true for any integer } n \geq 1$$

$$(c)(i) \quad 1 - x^2 > 0 \quad -1 < x < 1$$

$$(ii) \quad \text{If } y = f(x), \text{ the inverse function is}$$

$$x = \frac{y}{\sqrt{1-y^2}}$$

$$x^2 = \frac{y^2}{1-y^2}$$

$$x^2 - x^2 y^2 = y^2$$

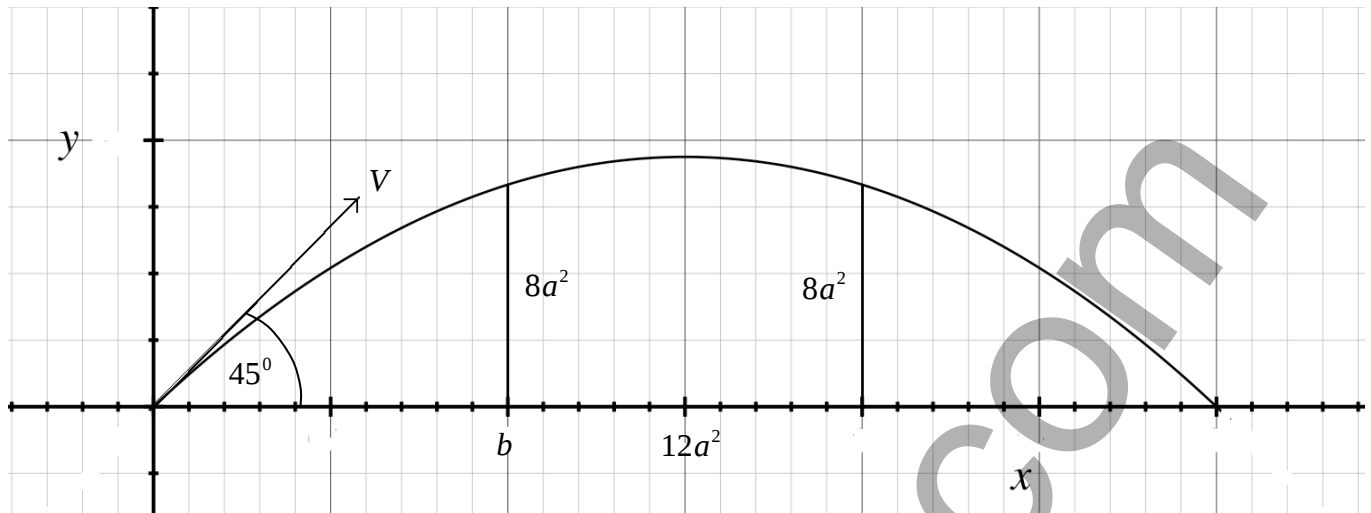
$$y^2(1+x^2) = x^2$$

$$y^2 = \frac{x^2}{1+x^2}$$

$$f^{-1}(x) = \frac{x}{\sqrt{1+x^2}} \text{ (odd function)}$$

Section D

(7) (a)



(i)

$$\ddot{x} = 0$$

Integrate w.r.t. t

$$\dot{x} = K$$

$$\text{When } t = 0, \dot{x} = \frac{V}{\sqrt{2}}$$

$$\therefore K = \frac{V}{\sqrt{2}}$$

$$\therefore \dot{x} = \frac{V}{\sqrt{2}}$$

Integrate w.r.t. t

$$x = \frac{Vt}{\sqrt{2}} + M$$

$$\text{When } t = 0, x = 0$$

$$\therefore M = 0$$

$$\therefore x = \frac{Vt}{\sqrt{2}}$$

$$\ddot{y} = -g$$

Integrate w.r.t. t

$$\dot{y} = -gt + L$$

$$\text{When } t = 0, \dot{y} = \frac{V}{\sqrt{2}}$$

$$\therefore L = \frac{V}{\sqrt{2}}$$

$$\therefore \dot{y} = \frac{V}{\sqrt{2}} - gt$$

Integrate w.r.t. t

$$y = \frac{Vt}{\sqrt{2}} - \frac{1}{2}gt^2 + N$$

$$\text{When } t = 0, y = 0$$

$$\therefore N = 0$$

$$\therefore y = \frac{Vt}{\sqrt{2}} - \frac{1}{2}gt^2$$

(ii) From the equation for x :

$$t = \frac{\sqrt{2}x}{V} \quad \therefore y = \frac{V}{\sqrt{2}} \frac{\sqrt{2}x}{V} - \frac{1}{2}g \left(\frac{\sqrt{2}x}{V} \right)^2$$

$$y = x - \frac{gx^2}{V^2}$$

(iii) The range is achieved when $y = 0$

$$\therefore x - \frac{gx^2}{V^2} = 0$$

$$x \left(1 - \frac{gx}{V^2} \right) = 0$$

$$\therefore 1 - \frac{gx}{V^2} = 0$$

$$x = \frac{V^2}{g} \quad (\text{Range})$$

(iv) (α) By symmetry the second post is b units from point of impact

$$\therefore (x_R =) \frac{V^2}{g} = 2b + 12a^2$$

(β) When $x = b$, $y = 8a^2$, in the equation from (ii):

$$8a^2 = b - \frac{gb^2}{V^2}$$

(v) From (α):

$$2b = \frac{V^2}{g} - 12a^2$$

$$\therefore b = \frac{V^2}{2g} - 6a^2$$

$$\therefore \frac{V^2}{2g} = b + 6a^2$$

$$\therefore V^2 = 2g(b + 6a^2)$$

$$= g(2b + 12a^2)$$

$$\therefore V = \sqrt{g} \sqrt{2b + 12a^2} \quad \text{—————} (*)$$

Hence it remains to prove that $2b = 24a^2$.

$$\text{Now } \frac{g}{V^2} = \frac{1}{2b + 12a^2}$$

$$\begin{aligned}\text{So } 8a^2 &= b - \frac{gb^2}{V^2} \\ &= b - \frac{b^2}{2b + 12a^2} \\ &= \frac{2b^2 + 12a^2b - b^2}{2b + 12a^2}\end{aligned}$$

$$\begin{aligned}\therefore 16a^2b + 96a^4 &= 2b^2 + 12a^2b - b^2 \\ &= b^2 + 12a^2b\end{aligned}$$

$$\therefore b^2 - 4a^2b - 96a^4 = 0$$

$$\begin{aligned}\therefore b &= \frac{4a^2 \pm \sqrt{16a^4 + 4 \times 96a^4}}{2} \\ &= \frac{4a^2 \pm 4\sqrt{a^4 + 24a^4}}{2} \\ &= \frac{4a^2 \pm 4 \times 5a^2}{2} \\ &= 12a^2 \quad (\text{Neg result extraneous})\end{aligned}$$

$\therefore$ In equation (*)

$$\begin{aligned}V &= \sqrt{g} \sqrt{36a^2} \\ &= 6a\sqrt{g} \quad \text{As required.}\end{aligned}$$