(10)

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics Extension 1

Sample Solutions

Section	Marker
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C	Mr Bigelow

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

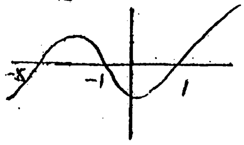
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Section A

1 a) $(x^2-1)(x+5)>0$



$x>1$

AND

$-5<x<-1$ (2 marks)

b) $y=\ln\sqrt{x+1}$

$=\frac{1}{2}\ln(x+1)$

$y'=\frac{1}{2(x+1)}$ (2 marks)

c) $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$

$=\frac{1}{2}\sec 2x\Big]_0^{\pi/6}$

$=\frac{1}{2}\sec\frac{\pi}{3}-\frac{1}{2}\sec 0$

$=\frac{1}{2}\times 2-\frac{1}{2}\times 1$

$=\frac{1}{2}$ (2 marks)

d) $\int_0^{\sqrt{3}} \frac{dx}{9+x^2}=\frac{1}{3}\tan^{-1}\frac{x}{3}\Big]_0^{\sqrt{3}}$

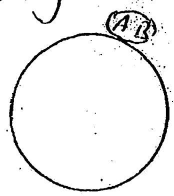
$=\left[\frac{1}{3}\tan^{-1}\frac{\sqrt{3}}{3}\right]$

$=\frac{1}{3}\tan^{-1}\frac{1}{\sqrt{3}}$

$=\frac{1}{3}\times\frac{\pi}{6}$

$=\frac{\pi}{18}$ (2 marks)

e) Total number of arrangements $=7!$



If A and B are together

Then $2\times 6!$

Hence not together

$=7!-2\times 6!$

$=6!(7-2)$

$=5\times 6!$

$=3600$ (2 marks)

f) $LHS=\frac{1-\cos\theta}{\sin\theta}+\frac{\sin\theta}{1+\cos\theta}$

$=\frac{1-\cos^2\theta+\sin^2\theta}{\sin\theta(1+\cos\theta)}$

$=\frac{2\sin^2\theta}{\sin\theta(1+\cos\theta)}$

$=\frac{2\sin\theta}{1+\cos\theta}$

Let $t=\tan\frac{\theta}{2}$

$=\frac{2\times\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$

$=\frac{\frac{4t}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}}=\frac{4t}{2}=2t$

$=2\tan\frac{\theta}{2}=RHS$ (2 marks)

QUESTION TWO

a) $y = \sin^{-1} 2x$

Let $u = 2x$

Then $y = \sin^{-1} u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{\sqrt{1-u^2}} \times 2$

$= \frac{2}{\sqrt{1-4x^2}}$ (2 marks)

b) $y = 3\sin^{-1}\sqrt{1-x^2}$

Consider $\sqrt{1-x^2}$

$-1 \le x \le 1$ ~~Range~~ Domain

Then

$y = 3\sin^{-1} 0$ to $3\sin^{-1} 1$

ie $0 \le y \le \frac{3\pi}{2}$ Range (2 marks)

c) $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\cos\alpha = \sqrt{3}$

$R\sin\alpha = 1$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$R^2(\cos^2\alpha + \sin^2\alpha) = 3+1$

$R = 2$

c) continued

$2\cos\left(x+\frac{\pi}{6}\right) = 1$ (2 marks)

$\cos\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$

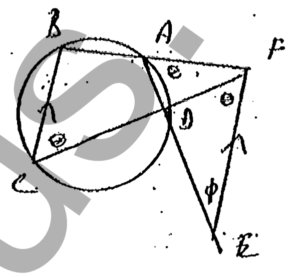
$x+\frac{\pi}{6} = \pm\frac{\pi}{3}$

$x = 2k\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$

$x = 2k\pi + \frac{\pi}{6}$ (2 marks)

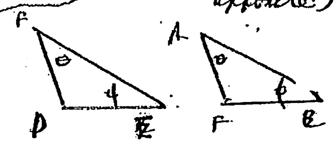
or $2k\pi - \frac{\pi}{2}$

d)



$F\hat{A}E = F\hat{B}C$ (angle in alternate segment)

$B\hat{C}F = C\hat{F}E$ (alternate opposite)



Hence $\Delta DEF \parallel\!| \Delta FEA$ (2 marks)

$\frac{EF}{EA} = \frac{ED}{EF}$

$EF^2 = EA \times ED$

(2 marks)

QUESTION THREE

a) Prove $2^{3n}-1$ is divisible by 7 for $n \geq 1$ (integers)

Let $n=1$ Then $2^3-1=7$

ie True for $n=1$

Assume

$2^{3k}-1 = 7K$ where K is an integer

Try to prove

$2^{3k+3}-1 = 7N$ where N is an integer

$LHS = 2^3 . 2^{3k} - 1$

$= 8(7K+1)-1$ from assumption.

$= 56K+7$

$= 7(8K+1)$

$= 7N$

True for $n=1$

$n=1+1=2$

$n=2+1=3$

$\vdots$

All integers $n \geq 1$ (3 marks)

b) i) $y = 1 + 2\cos x - 2\cos^2 x$

$y' = -2\sin x + 4\cos x \sin x$

$= 2\sin x\,(2\cos x - 1)$ (1 mark)

ii) $y'=0$ when $\sin x = 0$

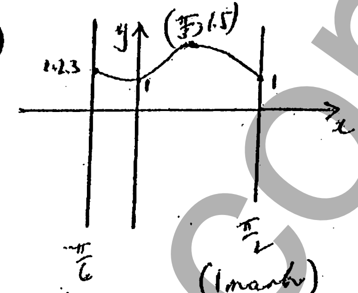
$\cos x = 1/2$

ie $x = 0, \frac{\pi}{3}$

When $x=0$, $y=1$

$x=\frac{\pi}{3}$, $y=\frac{3}{2}$ } 2 marks

iii)



(1 mark)

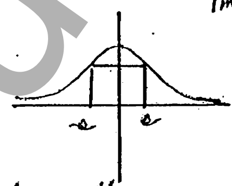
Max of 1.5 at $x=\frac{\pi}{3}$

Minimum of 1 at

$x=0$ or $x=\frac{\pi}{2}$

1 mark

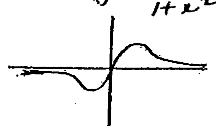
c)



Area $= lb$

$= 2a \times \frac{1}{1+a^2} = \frac{2a}{1+a^2}$ (1 mark)

Consider $y = \frac{2x}{1+x^2}$



$y' = \frac{(x^2+1)2 - 2.2x}{(1+x^2)^2}$

$= \frac{2-2x^2}{(1+x^2)^2}$

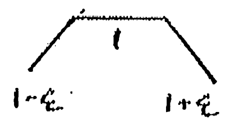
$y'=0$ when $x = \pm 1$

THREE

c ii) continued.

When $x = 1+\varepsilon$ $\quad y' < 0$

$x = 1-\varepsilon$ $\quad y' > 0$



Hence $x = 1$ produces maximum

$$\text{Area} = \frac{2}{1+1} = 2 \text{ square units}$$ (3 marks)

or

$$y'' = \frac{(1+x^2)^2(-4x) - (2-2x^2)\,4x(1+x^2)}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)\left[1+x^2+(2-2x^2)\right]}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)(3-x^2)}{(1+x^2)^4}$$

When $x = 1$ $\quad y'' = \dfrac{-4\times 2\times 2}{2^4}$

$y'' < 0$ Hence maximum.

QUESTION 4

i) $x = -2t, \quad t = -\frac{x}{2}$

$y = \frac{1}{4}x^2$

$y' = \frac{x}{2} = -t$

eqn of tangent $y - t^2 = -t(x+2t)$

$y - t^2 + tx + 2t^2 = 0$

$tx + y + t^2 = 0$

ii) $tx + y + t^2 = 0$

at A, $y = 0$

$tx + t^2 = 0$

$t(x+t) = 0, \quad x = -t$

$A(-t, 0) \quad T(-2t, t^2)$

Midpoint M $\frac{-t-2t}{2}, \left(\frac{0+t^2}{2}\right)$

$M = \left(-\frac{3t}{2}, \frac{t^2}{2}\right)$

$x = -\frac{3t}{2}, \quad t = -\frac{2x}{3}$

$y = \frac{t^2}{2}$

$= \frac{1}{2}\left(-\frac{2x}{3}\right)^2$

$y = \frac{2x^2}{9}$

Locus of M $x^2 = \frac{9}{2}y$

$4x^3 - 12x^2 + 11x - 3 = 0$

roots $\alpha - d, \alpha, \alpha + d$ (arith series)

Sum of roots $= 3\alpha = -\frac{b}{a} = 3$

$\alpha = 3$

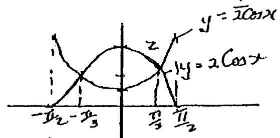
product $1(1-d) + 1(1+d) + (1-d)(1+d) = \frac{c}{a}$

$3 - d^2 = \frac{11}{4}$

$d^2 = \frac{1}{4}$

$d = \pm\frac{1}{2}$

roots $\frac{1}{2}, 1, 1\frac{1}{2}$.



$2\cos x = \frac{1}{2\cos x}$

$4\cos^2 x = 1$

$\cos x = \frac{1}{2}$ or $\cos x = -\frac{1}{2}$

$x = -\frac{\pi}{3}, \frac{\pi}{3}$ or no soln in domain

$V = \pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(4\cos^2 x - \frac{1}{4}\sec^2 x\right) dx$

$2\cos^2 x = \cos 2x + 1$

$V = 2\pi \int_0^{\frac{\pi}{3}} \left(2\cos 2x + 2 - \frac{1}{4}\sec^2 x\right) dx$

$= 2\pi \left[\sin 2x + 2x - \frac{1}{4}\tan x\right]_0^{\frac{\pi}{3}}$

$= 2\pi\left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{4}\right) - 0$

$V = \left(\frac{4\pi^2}{3} + \frac{\sqrt{3}}{2}\right) u^3$

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5 (a) Find $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$\frac{dr}{dt} = -5$ cm/s $\quad V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$r = 10$ cm

$\frac{dV}{dt} = -5 \times 4 \times \pi \times 100$

$= -2000\pi$ cm^3/s

(b) $x = 2\cos\left(t + \frac{\pi}{6}\right)$

$\dot{x} = -2\sin\left(t + \frac{\pi}{6}\right)$

$\ddot{x} = -2\cos\left(t + \frac{\pi}{6}\right)$

$\ddot{x} = -1^2 x$, in the form $-n^2 x$, $n = 1$

∴ motion is SHM

(ii) Period $= \frac{2\pi}{n} = 2\pi$

(iii) $x = 2\cos\left(t + \frac{\pi}{6}\right) = 0$

$t + \frac{\pi}{6} = \frac{\pi}{2} + 2n\pi$

$t = \frac{\pi}{3}$ sec (1st osc.)

(iv) $2\cos\left(t + \frac{\pi}{6}\right) = 1$

$t + \frac{\pi}{6} = \pm\frac{\pi}{3} + 2n\pi$

$t = \frac{\pi}{6}$ (1st osc.)

$\dot{x} = -2\sin\frac{\pi}{3}$

$v = -2 \times \frac{\sqrt{3}}{2}$

$v = -\sqrt{3}$ cm/s

QUESTION 5(c)

$$\int \sqrt{16-x^2}\,dx$$

$x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$

$dx = 4\cos\theta\,d\theta$

$$= \int \sqrt{16-16\sin^2\theta}\;4\cos\theta\,d\theta$$

$$\int \sqrt{16\cos^2\theta}\,.\,4\cos\theta\,d\theta$$

$$\int 4\cos\theta\,.\,4\cos\theta\,d\theta$$

$$16\int \cos^2\theta\,d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

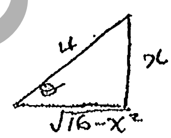
$2\cos^2\theta = \cos 2\theta + 1$

$$8\int (\cos 2\theta + 1)\,d\theta$$

$$8\left(\tfrac{1}{2}\sin 2\theta + \theta\right)$$

$$4\sin 2\theta + 8\theta + c$$

$$4\,.\,2\sin\theta\cos\theta + 8\theta$$



$$4\times 2\,.\,\frac{x}{4}\,\frac{\sqrt{16-x^2}}{4} + 8\sin^{-1}\frac{x}{4}$$

$\theta = \sin^{-1}\frac{x}{4}$

$$= \frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\frac{x}{4} + c$$

QUESTION 6.

(a) If $y' = \frac{3x}{4+x^2}$

$\boxed{y = \frac{3}{2}\ln(4+x^2)+c.}$ ✓

(b). $P(x) = 8x^3 - 12x^2 + 6x + 13$

$P'(x) = 24x^2 - 24x + 6.$

$= 6(2x-1)^2$

(i) $P(x)$ is increasing when $P'(x) > 0$

ie $6(2x-1)^2 > 0$

$\therefore$ all Reals, except $x = \frac{1}{2}.$ ✓

(ii) Since $P(x) \to -\infty$ as $x \to -\infty$, $P(0) = 13$. and $P(x)$ is increasing for all $x \neq \frac{1}{2}$. it follows that there must be a root x_1 where $x_1 < 0$. ✓

(iii) $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$

if $a_1 = -1$ then $a_2 = -1 - \frac{-8-12-6+13}{24+24+6}$

$= -1 - \frac{-13}{54}$

$= -\frac{41}{54}$

$= \boxed{-0.76}$ (2.D.P). ✓✓

18

(c) (i) $T = S + Ae^{-kt}$ — (A)

$\therefore \frac{dT}{dt} = -kAe^{-kt}$

$= -k(T-S)$ from (A)

(ii) when $t=0$, $T=1390$ and $S=30$ (constant)

$\therefore 1390 = 30 + Ae^{0}$

$\therefore A = 1360$

$\therefore T = 30 + 1360e^{-kt}$

when $t=10$, $T = 1060$

$\therefore 1060 = 30 + 1360e^{-10k}$

$\frac{1030}{1360} = e^{-10k}$

$-10k = \ln \frac{103}{136}$

$k \doteqdot 0.0278$

$\therefore T = 30 + 1360e^{-0.0278t}$

Set $T = 110$

$110 = 30 + 1360e^{-0.0278t}$

$\frac{80}{1360} = e^{-0.0278t}$

$\therefore \ln \frac{8}{136} = -0.0278t$

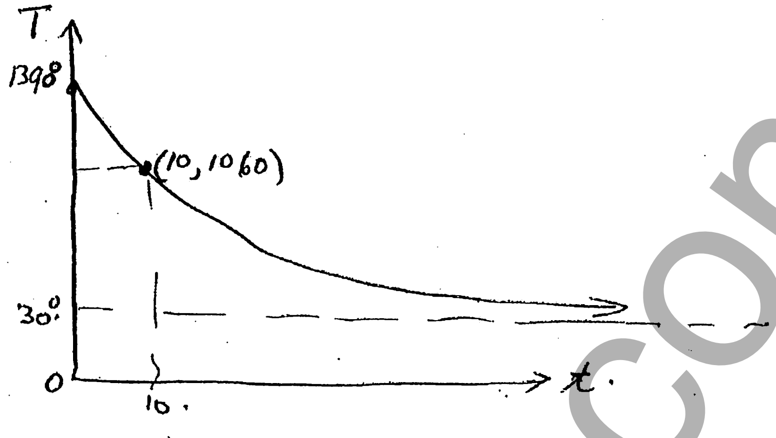
$t = \frac{\ln \frac{1}{17}}{-0.0278}$

$\doteqdot 102$ mins

$\therefore$ it takes 92 mins longer.

19

(iii)



QUESTION 7.

(a) now

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{r}x^r + \cdots$$
$$\cdots + \binom{n}{n}x^n \quad (A)$$

(i) Differentiate both sides of (A) above.

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + r\binom{n}{r}x^{r-1} + \cdots$$
$$\cdots + n\binom{n}{n}x^{n-1}.$$

let $x = 1$

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + r\binom{n}{r} + \cdots + n\binom{n}{n}$$

ie. $\boxed{\sum_{r=1}^{n} r\binom{n}{r} = n\,2^{n-1}}$ ✓✓ (NB this is equivalent to $\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$.)

(ii) R.T.P. $\sum_{r=0}^{n} (r+1)\binom{n}{r} = 2^{n-1}(n+2)$

$$LHS = \sum_{r=0}^{n} r\binom{n}{r} + \sum_{r=0}^{n}\binom{n}{r}$$

$$= n2^{n-1} + 2^n$$ (If we let $x=1$ in (A) $2^n = \sum_{r=0}^{n}\binom{n}{r}$)

$$= \boxed{2^{n-1}(n+2)}$$ ✓✓

$= RHS.$

(b) (i) $x = Vt \Rightarrow t = \frac{x}{V}$.

$\therefore y = -\frac{1}{2}gt^2 + h$. becomes.

$y = -\frac{1}{2}g\left(\frac{x}{V}\right)^2 + h$.

$\boxed{y = h - \frac{1}{2}g\frac{x^2}{V^2}}$ ✓

(ii) $x = Vt\cos\alpha \Rightarrow t = \frac{x}{V\cos\alpha}$ $\therefore y = -\frac{1}{2}gt^2 + Vt\sin\alpha + h$.

becomes $y = -\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 + V\frac{x}{V\cos\alpha}\sin\alpha + h$.

ie $\boxed{y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V^2} + h}$ ✓

(iii) Substitute $(d, 0)$ in (i) $0 = h - \frac{gd^2}{2V^2}$

$\therefore \boxed{h = \frac{gd^2}{2V^2}}$ ✓

(iv) Substitute $(d, 0)$ in (ii)

$0 = d\tan\alpha - \frac{gd^2}{2V^2}\sec^2\alpha + h$.

$0 = d\tan\alpha - h(1 + \tan^2\alpha) + h$ $\left(h = \frac{gd^2}{2V^2}\right)$

$\therefore h\tan^2\alpha - d\tan\alpha = 0$

$\tan\alpha(h\tan\alpha - d) = 0$

$\therefore \tan\alpha = 0$ or $\tan\alpha = \frac{d}{h}$

Clearly $\tan\alpha \neq 0$ $\therefore \boxed{\tan\alpha = \frac{d}{h}}$ ✓✓

(v). Substitute $(2d, 0)$ into (ii).

$$2d\tan\alpha - \frac{g\cdot 4d^2}{2v^2}\sec^2\alpha + h = 0.$$

$$2d\tan\alpha - 4h\sec^2\alpha + h = 0.$$

$$2d\tan\alpha - 4h(1+\tan^2\alpha) + h = 0$$

$$2d\tan\alpha - 4h - 4h\tan^2\alpha + h = 0$$

$$4h\tan^2\alpha - 2d\tan\alpha + 3h = 0$$

For $\tan\alpha$ to be real $4d^2 - 4\times 4h\times 3h \geq 0.$

ie $4d^2 - 48h^2 \geq 0.$

$$4d^2 \geq 48h^2$$

$$d^2 \geq 12h^2$$

$$\boxed{d \geq 2h\sqrt{3}}$$ ✓✓✓