



SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2004

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 3

Mathematics Extension 1

Sample Solutions

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Section A

Question 1

$$\begin{aligned}
 \text{(a)} \quad \int_0^2 \frac{1}{\sqrt{16-x^2}} dx &= \int_0^2 \frac{1}{\sqrt{4^2-x^2}} dx \\
 &= \left[\sin^{-1} \frac{x}{4} \right]_0^2 \\
 &= \sin^{-1} \frac{1}{2} \\
 &= \frac{\pi}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) (i)} \quad \lim_{x \rightarrow \infty} \frac{\sin 3x}{4x} &= \frac{3}{4} \lim_{x \rightarrow \infty} \frac{\sin 3x}{3x} \\
 &= \frac{3}{4} \times 1 \\
 &= \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \lim_{x \rightarrow \infty} \frac{\sin 3x}{\sin 7x} &= \lim_{x \rightarrow \infty} \frac{\sin 3x}{3x} \times \frac{7x}{\sin 7x} \\
 &= \frac{3}{7} \lim_{x \rightarrow \infty} \frac{\sin 3x}{3x} \times \frac{7x}{\sin 7x} \\
 &= \frac{3}{7} \times 1 \\
 &= \frac{3}{7}
 \end{aligned}$$

$$\text{(c)} \quad \int \frac{dx}{x\sqrt{1-(\ln x)^2}}$$

$$\text{Let } u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x} dx$$

$$\begin{aligned}
 \int \frac{dx}{x\sqrt{1-(\ln x)^2}} &= \int \frac{1}{\sqrt{1-(u)^2}} du \\
 &= \sin^{-1} u + C \\
 &= \sin^{-1}(\ln x) + C
 \end{aligned}$$

$$\text{(d)} \quad \log_e(\sin^3 x)$$

$$\text{Let } u = \sin^3 x$$

$$\frac{du}{dx} = 3\sin^2 x \cos x$$

$$\text{Let } y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= \frac{1}{u} \times 3\sin^2 x \cos x \\
 &= \frac{1}{\sin^3 x} \times 3\sin^2 x \cos x \\
 &= \frac{3\cos x}{\sin x} \\
 \therefore \frac{dy}{dx} &= 3\cot x
 \end{aligned}$$

(e) $\frac{d}{dx}(\tan^{-1} x)^2$

Let $u = \tan^{-1} x$

$$\frac{du}{dx} = \frac{1}{1+x^2}$$

Let $y = u^2$

$$\frac{dy}{du} = 2u$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2u \times \frac{1}{1+x^2}\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{2 \tan^{-1} x}{1+x^2}$$

Question 2

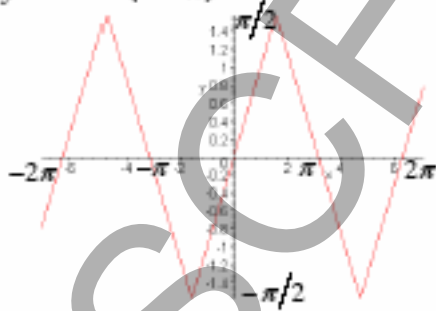
(a) (i) $y = \sin^{-1}(\sin x)$

Domain $\{x : x \in \mathbb{R}\}$

Range $\left\{y : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$

(ii)

$$y = \sin^{-1}(\sin x)$$



(b) $y = \sin^{-1}(\sqrt{x})$

$$\sin y = \sqrt{x}$$

$$\sin^2 y = x$$

$$\therefore x = \sin^2 y$$

$$\frac{dx}{dy} = 2 \sin y \cos y$$

$$= \sin 2y$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sin 2y}$$

(c) (i) $y = x \tan x - \ln(\sec x)$

Now $\frac{d}{dx} x \tan x$

Let $u = x \quad v = \tan x$

$$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \sec^2 x$$

$$\begin{aligned} \therefore \frac{d}{dx} (x \tan x) &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= (x)(\sec^2 x) + (\tan x)(1) \\ &= x \sec^2 x + \tan x \end{aligned}$$

Now $\frac{d}{dx} \ln(\sec x)$

Let $u = \sec x$

$$= (\cos^{-1} x)$$

$$\frac{du}{dx} = -(\cos x)^{-2} (-\sin x)$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \tan x \sec x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u} \times \tan x \sec x \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sec x} \times \tan x \sec x \\ &= \tan x \end{aligned}$$

$$\therefore y = x \tan x - \ln(\sec x)$$

$$\begin{aligned} \frac{dy}{dx} &= x \sec^2 x + \tan x - \tan x \\ &= x \sec^2 x \end{aligned}$$

$$y = \ln u$$

$$\frac{dy}{du} = \frac{1}{u}$$

(c)

$$\begin{aligned} \text{(ii)} \quad \int x \sec^2 x \, dx &= \left[x \tan x - \ln(\sec x) \right]_0^{\frac{\pi}{4}} \\ &= \left\{ \frac{\pi}{4} \tan \frac{\pi}{4} - \ln\left(\sec \frac{\pi}{4}\right) \right\} - \{ 0 \tan 0 - \ln(1) \} \\ &= \left\{ \frac{\pi}{4} (1) - \ln(\sqrt{2}) \right\} - \{ -\ln(1) \} \\ &= \frac{\pi}{4} - \ln \sqrt{2} \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2 \\ &= \frac{\pi - 2 \ln 2}{4} \end{aligned}$$

(d) $y = 10^x$

$$\log_{10} y = \log_{10} 10^x$$

$$\log_{10} y = x \log_{10} 10$$

$$x = \log_{10} y$$

$$x = \frac{\log_e y}{\log_e 10}$$

$$x = \frac{1}{\log_e 10} \times \log_e y$$

$$x \log_e 10 = \log_e y$$

$$\therefore y = e^{x \log_e 10}$$

$$\therefore \frac{dy}{dx} = \log_e 10 \times e^{x \log_e 10}$$

when $x = 1$

$$\frac{dy}{dx} = \log_e 10 \times e^{(1) \log_e 10}$$

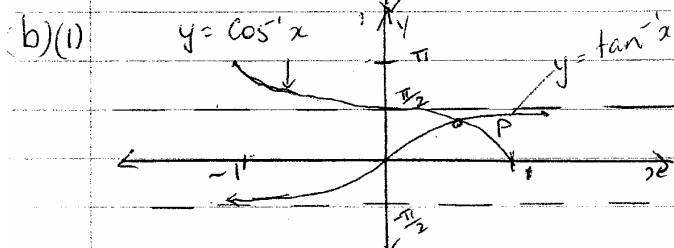
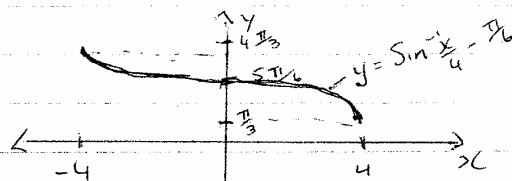
$$= \log_e 10 \times 10$$

$$= 10 \log_e 10$$

QUESTION 3

(a) Inverse $x = 4 \sin(y + \frac{\pi}{6})$ $\frac{\pi}{3} \leq y \leq \frac{4\pi}{3}$ a(i)

(i) $y + \frac{\pi}{6} = \sin^{-1} \frac{x}{4}$
 $y = \sin^{-1} \frac{x}{4} - \frac{\pi}{6}$
 Domain $-4 \leq x \leq 4$



(ii) $\tan^{-1}x = y = \cos^{-1}x$
 $\tan y = x = \cos y$

$$x = \frac{1}{\sqrt{1+x^2}}$$

$$x^2(1+x^2) = 1$$

$$x^4 + x^2 - 1 = 0$$

$$x^2 = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$x^2 = 0.618 \quad (x^2 > 0)$$

$$x = 0.79, \quad y = 0.67 \text{ (2 d.p.)}$$

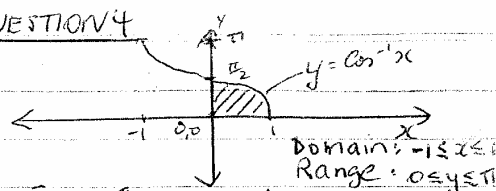
(c) $x = \tan^{-1}(\frac{1}{4})$ $y = \tan^{-1}(\frac{3}{5})$
 $\tan x = \frac{1}{4}$ $\tan y = \frac{3}{5}$
 $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$$\tan(x+y) = \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4} \cdot \frac{3}{5}} = 1$$

$$x + y = \frac{\pi}{4}$$

$$\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{3}{5}) = \frac{\pi}{4}$$

QUESTION 4



$$A = \int_0^{\pi/2} \cos y \, dy$$

$$= [\sin y]_0^{\pi/2}$$

$$= 1 - 0 = 1 \text{ u}^2$$

(b) $y = \log_e 2x - 2 \log_e(x-1)$
 $= \frac{1}{x} - \frac{2}{x-1}$ or $-\frac{x+1}{x(x-1)}$

(c) $\frac{dT}{dt} = k(T-16)$

(i) $\frac{dT}{dt} = \frac{1}{k(T-16)}$

$$t = \frac{1}{k} \log_e(T-16) + C$$

$$k(t-C) = \log_e(T-16)$$

$$T-16 = e^{kt-kC}$$

$$T-16 = Pe^{kt} \quad (P = e^{-kC} = \text{constant})$$

$$T = 16 + Pe^{kt} \text{ as required}$$

(ii) $T = 16 + Pe^{kt}$

$$T=0, t=0 \quad P = -16$$

$$T = 16 - 16e^{kt}$$

$$T=12, \quad 12 = 16 - 16e^{kt}$$

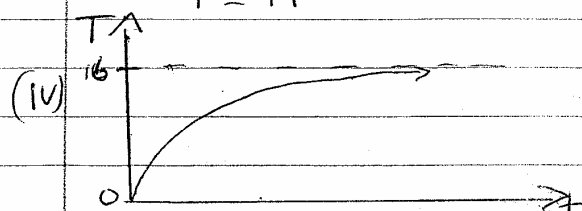
$$-4 = -16e^{10k}$$

$$\frac{1}{4} = e^{10k}, \quad k = \frac{1}{10} \log_e \frac{1}{4}$$

$$k \approx -0.1326$$

(iii) $t=15 \quad T = 16 - 16e^{-0.1326 \times 15}$

$$T = 14^\circ$$



$$\text{as } t \rightarrow \infty, 16e^{kt} \rightarrow 0$$

$$T \rightarrow 16$$

QUESTION 5.

(a). Let $f(x) = \ln x + \sin x$.
 $f'(x) = \frac{1}{x} + \cos x$. (1/2 for diff)

pf $x_1 = 0.5$.
 then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
 $= 0.5 - \frac{\ln 0.5 + \sin 0.5}{\frac{1}{0.5} + \cos 0.5}$ ↓ ① NB
if calculator
is in degree
mode 0.73
(4 MARK)
 $= 0.5 - -0.07427 =$
 $= \boxed{0.57} \text{ (2 D.P.)} \quad \boxed{2}$

(b)(i). $\ddot{x} = 8x(x^2 + 1)$ and when $t=0$, $x=0$, $v=2$.

$\frac{d}{dt}(\frac{1}{2}v^2) = 8x^3 + 8x$

$\frac{1}{2}v^2 = 2x^4 + 4x^2 + c$

now $v=0$ when $x=0$

$\therefore \frac{1}{2}v^2 = 0 + 0 + c$

$c = 2$

$\therefore \frac{1}{2}v^2 = 2x^4 + 4x^2 + 2$

$v^2 = 4x^4 + 8x^2 + 4$

$v^2 = 4(x^4 + 2x^2 + 1)$

$= 4(x^2 + 1)^2$

$v = \pm 2(x^2 + 1)$ (now $v=0$ when $x=0$ $\therefore v \neq -2(x^2 + 1)$)

$\therefore \boxed{v = 2(x^2 + 1)}$

2

must show this.
 ↓

(ii) $\frac{dx}{dt} = 2(x^2 + 1)$

$\frac{dt}{dx} = \frac{1}{2(x^2 + 1)}$

$t = \frac{1}{2} \tan^{-1} x + c$

now $t=0$, when $x=0$.

$\therefore 0 = \frac{1}{2} \tan^{-1} 0 + c$

$c = 0$

$t = \frac{1}{2} \tan^{-1} x$

$\therefore 2t = \tan^{-1} x \Rightarrow \boxed{x = \tan 2t}$

2

(iii)

$$x = \tan 2t$$

$$v = 2 \sec^2 2t$$

↓ □

$$dt \quad t = \frac{\pi}{8}$$

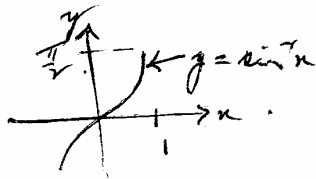
$$v = 2 \times \sec^2 \frac{\pi}{4}$$

$$= 2 \times (\sqrt{2})^2$$

$$= \boxed{4 \text{ m s}^{-1}}$$

□ 2

(c)



$$v = \pi \int_0^1 [\sin^{-1} x]^2 dx$$

$$= \pi \times \frac{1}{3} \left[(\sin^{-1} x)^2 + 4 [\sin^{-1} x] + (\sin^{-1} x)^3 \right]$$

$$= \frac{\pi}{6} \left[0^2 + 4 \times \left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right)^3 \right]$$

$$= \frac{\pi}{6} \left[0 + \frac{\pi^2}{9} + \frac{\pi^2}{4} \right]$$

$$= \frac{\pi}{6} \times \frac{13\pi^2}{36}$$

$$= \frac{13\pi^3}{216}$$

$$= \boxed{1.87 \text{ (2 DP)}}$$

□ 3

QUESTION 6.

$$\begin{aligned} (a) \quad (i) \quad v^2 &= 28 + 24x - 4x^2 \\ &= 4(7 + 6x - x^2) \\ &= 4(7 - x)(1 + x). \end{aligned}$$

Clearly $v^2 \geq 0$

$$\therefore 4(7 - x)(1 + x) \geq 0$$

$$\therefore \boxed{-1 \leq x \leq 7}$$

$\boxed{1}$

$$(ii) \quad \text{Amplitude} = \frac{7 - (-1)}{2} = \boxed{4} \quad \boxed{1}$$

$$(iii) \quad \ddot{x} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \frac{d}{dx} (14 + 12x - 2x^2)$$

$$= 12 - 4x$$

$$= -4(x - 3)$$

$$\left(\begin{array}{l} \text{NB } v^2 = 4 \\ n = 2 \end{array} \right)$$

$$(iv) \quad T = \frac{2\pi}{n} = \frac{2\pi}{2}$$

$$= \boxed{\pi \text{ secs}}$$

$\boxed{1}$

and centre of motion is $x = 3$

$$(v) \quad x = 3 + 4 \cos(2t + \epsilon)$$

if $x = 7$ when $t = 0$.

$$7 = 3 + 4 \cos \epsilon$$

$$4 = 4 \cos \epsilon$$

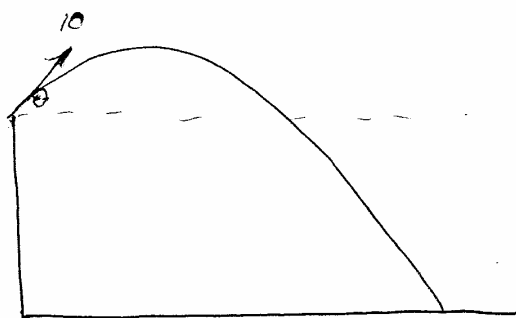
$$\cos \epsilon = 1$$

$$\epsilon = 0$$

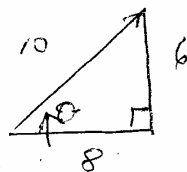
$$\therefore \boxed{x = 3 + 4 \cos 2t}$$

$\boxed{2}$

(b).



$$t=0, x=0, y=8$$



$$\theta = \tan^{-1}\left(\frac{6}{8}\right)$$

(1)

$$\ddot{x} = 0$$

$$\dot{x} = 8$$

$$x = 8t + C_1$$

$$\text{when } t=0, x=0 \therefore C_1=0$$

$$\boxed{x = 8t}$$

$$\ddot{y} = -g$$

$$\dot{y} = -gt + C_2$$

$$\text{Clearly } \dot{y} = 6 \text{ when } t=0$$

$$\therefore C_2 = 6$$

$$\dot{y} = -gt + 6$$

$$\therefore y = -\frac{gt^2}{2} + 6t + C_3$$

$$\text{when } t=0, y=8 \therefore C_3=8$$

$$\therefore \boxed{y = -\frac{1}{2}gt^2 + 6t + 8}$$

3

(ii) If $y=0$.

$$-5t^2 + 6t + 8 = 0 \Rightarrow -(5t^2 - 6t - 8) = 0$$

$$-(5t+4)(t-2) = 0$$

$$t = 2, -\frac{4}{5}$$

$$\therefore \boxed{2 \text{ secs have elapsed.}}$$

1

$$\text{and } \boxed{x = 16}$$

1