Catholic Schools Trial Examinations

2004 Mathematics Extension 1

HSC SUPPORT

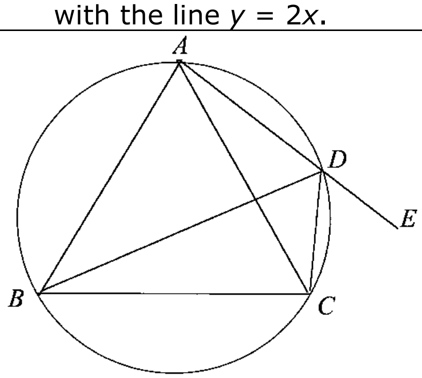
04 CT **1a** Find $\frac{d}{dx}(1 + x^2)\tan^{-1}x$. **2**

04 CT **1b** The polynomial P(x) is given by P(x) = $x^3 + ax + b$ for some real number a. The remainder when P(x) is divided by (x – 1) is equal to the remainder when P(x) is divided by (x – 2). Find the value of a. **2**

04 CT **1c** (i) The line $y = mx$ makes an angle of 45° with the line $y = 2x$. Show that $|m - 2| = |1 + 2m|$. **2**

(ii) Hence find the equations of the lines $y = mx$ which make an angle of 45° with the line $y = 2x$. **2**

04 CT **1d**



ABC is a triangle in which BC = AC. D is a point on the minor arc AC of the circle passing through A, B and C. AD is produced to E.

(i) Copy the diagram.

(ii) Give a reason why ∠CDE = ∠ABC. **1**

(iii) Hence show that DC bisects ∠BDE. **3**

04 CT **2a** A(-5, 6) and B(1, 3) are two points. Find the coordinates of the point P which divides the interval AB externally in the ratio 5:2. **2**

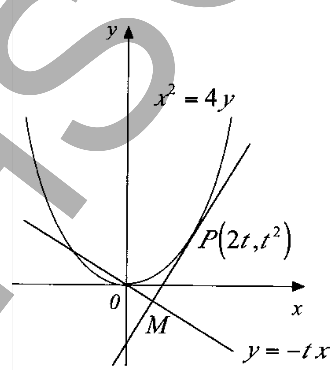
04 CT **2b** The equation $2x^3 + 2x^2 + 4x + 1 = 0$ has roots α, β and γ. Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **2**

04 CT **2c** Consider the geometric series $\sin 2x + \sin 2x \cos 2x + \sin 2x \cos^2 2x + \ldots$ for $0 \leq x \leq \frac{\pi}{2}$.

(i) Show that the limiting sum S of the series exists. **2**

(ii) Show that $S = \cot x$. **2**

04 CT **2d**

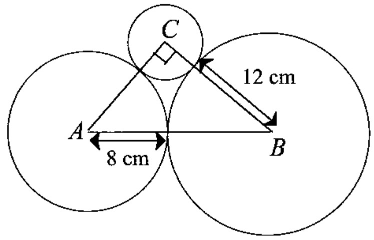


$P(2t, t^2)$ is a point on the parabola $x^2 = 4y$. The tangent to the parabola at P and the line $y = -tx$ intersect at M.

(i) Show that the tangent to the parabola at P has gradient t and equation $tx - y - t^2 = 0$. **2**

(ii) Find the Cartesian equation of the locus of M as t varies. **2**

04 CT 3a



Three circles with centres A, B and C touch externally in pairs with $\angle BCA = 90^o$, as shown in the diagram. The circles with centres A and B have radii 8cm and 12cm respectively.

(i) If the circle with centre C has radius x cm, show that $x^2 + 20x - 96 = 0$. **1**

(ii) Hence, find the radius of the circle with centre C. **2**

04 CT 3b

(i) Show that $\frac{u}{u+1} = 1 - \frac{u}{u+1}$. **1**

(ii) Hence find $\int \frac{1}{1+\sqrt{x}}\, dx$ using the substitution $x = u^2$, where $u \geq 0$. **3**

04 CT 3c

Use Mathematical Induction to show that $5^n > 4^n + 3^n$ for all integers $n \geq 3$. **4**

04 CT 4a

Find the term independent of x in the binomial expansion of $\left(x - \frac{2}{x^2}\right)^{15}$. **3**

04 CT 4b

After t years, $t \geq 0$, the number N of individuals in a population is given by $N = A + Be^{-0.5t}$ for some constants $A > 0$ and $B > 0$. The initial population size is 500 individuals and the limiting population size is 100 individuals.

(i) Find the values of A and B. **2**

(ii) Find the time taken for the population size to fall within 10 of its limiting value, giving the answer correct to the nearest month. **2**

04 CT 4c

Consider the function $f(x) = x - \cos x$.

(i) Show that the equation $f(x) = 0$ has a root α such that $0 < \alpha < 1$. **2**

(ii) Use one application of Newton's Method with an initial approximation of 0.7 to approximate α, giving answer correct to 2 decimal places. **3**

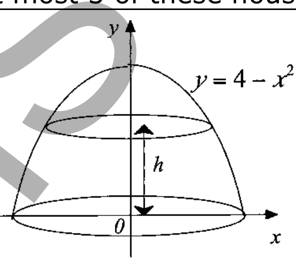
04 CT 5a

In a certain street, 40% of the households have at least 2 cars.
If 4 households are chosen at random, find the probability that

(i) exactly 3 of these households have at least 2 cars. **1**

(ii) at most 3 of these households have at least 2 cars. **2**

04 CT 5b



A mould for a container is made by rotating the part of the curve $y = 4 - x^2$, which lies in the first quadrant through one complete revolution about the y axis. After sealing the base of the container, water is poured through a hole in the top. When the depth of water in the container is h cm, the depth is changing at a rate $\frac{10}{\pi(4-h)}$ cms^{-1}.

(i) Show that when the depth of water is h metres, the surface area S m^2 of the **1** water is given by $S = \pi(4 - h)$.

(ii) Find the rate at which the surface area of the water is changing when the depth of the water is 2cm. **3**

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04 CT 5c Consider the function $f(x) = e^x - x$.

(i) Show that the curve $y = f(x)$ is concave up for all values of x. **1**

(ii) Find the coordinates and nature of the stationary point on the curve $y = f(x)$. **2**

(iii) Hence show that $e^x \geq x + 1$ for all values of x. **2**

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04 CT 6a Consider the function $f(x) = \cos^{-1}(x - 1)$.

(i) Find the domain of the function. **1**

(ii) Sketch the graph of the curve $y = f(x)$ showing clearly the coordinates of the endpoint. **2**

(iii) The region in the first quadrant bounded by the curve $y = f(x)$ and the coordinate axes is rotated through one complete revolution about the y axis. Find the exact value of the volume of revolution. **3**

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04 CT 6b A particle moving in a straight line is performing Simple Harmonic. At time t seconds it has displacement x metres to the right of a fixed point O on the line, where $x = 4\cos^2 t - 2\sin^2 t$.

(i) Show that $x = 1 + 3\cos 2t$. Hence express the acceleration $\ddot{x}$ ms^{-2} of the particle in the form $\ddot{x} = -n^2(x - b)$, where the values of the constant a and b are to be determined. **2**

(ii) Find the set of possible values of x and the period of the motion. **2**

(iii) Find the distance traveled and the time taken (to the nearest tenth of a second) for the particle to first pass through O. **2**

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04 CT 7a A particle is moving in a straight line. At time t seconds it has displacement x metres to the right of a fixed point O on the line and velocity v ms^{-1} given by $v = \sin x \cos x$. The particle starts $\frac{\pi}{4}$ metres to the right of O

(i) Show that $\frac{d}{dx}\ln(\tan x) = \frac{1}{\sin x \cos x}$ **1**

(ii) Hence show that the displacement of the particle is given by $x = \tan^{-1}(e^x)$. **3**

(iii) Find the limiting position of the particle and sketch the graph of x against t. **2**

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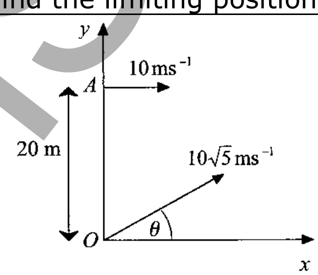
04 CT 7b



OA is a vertical building of height 20 metres. A particle is projected horizontally from A with speed 10 ms^{-1}. At the same instant, a second particle is projected from O with speed $10\sqrt{5}$ ms^{-1} at an angle θ above the horizontal. The two particles travel in the same plane of motion. Take g = 10 ms^{-1} . **2**

(i) Write down expressions for the horizontal and vertical displacements relative to O of each particle after time t seconds. **2**

(ii) Show that if the two particles collide, then they do so after 1 second. **2**

(iii) Show that if the two particles collide, when they do so their paths of motion are perpendicular to each other.

A **1a.** $1 + 2x\tan^{-1}x$ **1b.** $a = -7$ **1c.(ii)** $y = -3x$ and $y = \frac{1}{3}x$ **1d.(ii)** ext $\angle$ of cyclic quad equals opp int $\angle$ **2a.** $P(5, 1)$ **2b.** -4 **2d.(ii)** $y = -2x^2$ **3a.(ii)** 4cm **3b.(ii)** $2\sqrt{x} - 2\ln(1 + \sqrt{x}) + c$ **4a.** -96 096 **4b.(i)** $A = 100$, $B = 400$ **(ii)** 7 yrs 5 months **4c.** 0.74 **5a.(i)** $\frac{96}{625}$ **(ii)** $\frac{609}{625}$ **5b.(ii)** $5\text{cm}^2\text{s}^{-1}$ **5c.(ii)** min (0, 1) **6a.(i)** $0 \le x \le 2$ **(iii)** $\frac{3}{2}$ units^3 **6b.(i)** $\ddot{x} = -2^2(x - 1)$ **(ii)** $-2 \le x \le 4$ and per = π secs **(iii)** 4m **7a.(iii)** $\frac{\pi}{2}$ m to right of O

7b.(i) From A: hor: $x = 10t$ and vert: $y = 20 - 5t^2$ From O: hor: $10\sqrt{5}\cos\theta$ and vert: $y = 10\sqrt{5}\sin\theta - 5t^2$