

## 3/4 UNIT MATHEMATICS FORM VI

Time allowed: 2 hours (plus 5 minutes reading)

Exam date: 8th August, 2000

## Instructions:

- All questions may be attempted.
- All questions are of equal value.
- Part marks are shown in boxes in the left margin.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

## Collection:

- Each question will be collected separately.
- Start each question in a new 8-leaf answer booklet.
- If you use a second booklet for a question, place it inside the first. Don't staple.
- Write your candidate number on each answer booklet.

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QUESTION ONE (Start a new answer booklet)

Marks

**1** (a) Differentiate  $\tan^{-1}(\pi x)$  with respect to  $x$ .

**2** (b) Find:

(i)  $\int \frac{dx}{4+x^2},$

(ii)  $\int \frac{dx}{\sqrt{4-x^2}}.$

**3** (c) Evaluate  $\int_0^{\frac{\pi}{4}} \tan^2 x \, dx.$

**2** (d) Find the value of  $\tan \alpha$  if  $\alpha$  is the acute angle between the lines  $y = \frac{1}{2}x$  and  $y = -\frac{1}{\sqrt{3}}x + 1.$

**2** (e) Prove that  $\frac{\sin 2\theta}{\sin \theta} - \frac{\cos 2\theta}{\cos \theta} = \sec \theta.$

**2** (f) Solve the inequation  $\frac{x-2}{x} \geq 1.$

QUESTION TWO (Start a new answer booklet)

Marks

**3** (a) Use the substitution  $u = e^x$  to evaluate  $\int_0^{\ln \sqrt{3}} \frac{e^x}{1+e^{2x}} \, dx.$

**3** (b) Find the term independent of  $x$  in the expansion of  $\left(x^2 + \frac{2}{x}\right)^6.$

**3** (c) The volume  $V$  of a spherical balloon is expanding at the rate of  $10 \text{ mm}^3/\text{s}$ . Find the rate of increase of its radius  $r$  when the surface area  $S$  is  $1000 \text{ mm}^2$ . (Note: You may use the formulae  $V = \frac{4}{3}\pi r^3$  and  $S = 4\pi r^2$ ).

**3** (d) (i) Sketch, without the use of calculus, the polynomial  $P(x) = (2x-1)^2(x+1)^3$ , showing the  $x$ - and  $y$ -intercepts.

(ii) Hence solve the inequation  $P(x) \geq 0.$

**QUESTION THREE** (Start a new answer booklet)

Marks

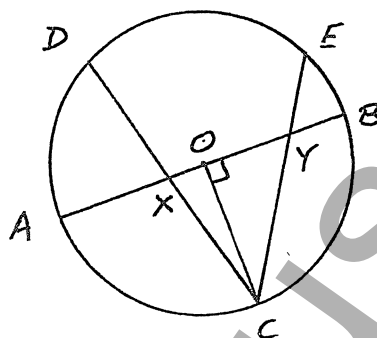
- 4 (a) The velocity  $v$  of a particle moving along the  $x$ -axis satisfies the equation

$$v^2 = 5 + 14x - 3x^2.$$

Show that the particle is moving in simple harmonic motion, and find the centre, amplitude and period of the motion.

- 4 (b) (i) Express  $\cos 2x - \sin 2x$  in the form  $R \cos(2x + \alpha)$ , where  $\alpha$  is acute and  $R > 0$ .  
 (ii) Hence solve the equation  $\cos 2x - \sin 2x = 1$ , for  $0 \leq x \leq \pi$ .

- 4 (c)



In the diagram above,  $AB$  is the diameter of a circle with centre  $O$ . The radius  $OC$  is drawn perpendicular to  $AB$ . The chords  $CD$  and  $CE$  intersect the diameter in the points  $X$  and  $Y$  respectively.

Copy the diagram into your examination booklet.

- (i) Prove that  $\angle CBA = \angle CAB = 45^\circ$ .  
 (ii) Give a reason why  $\angle DCA = \angle DBA$  and  $\angle CBD = \angle CED$ .  
 (iii) Prove that  $\angle CBD = \angle CXB$ . (Hint: Let  $\angle DCA = \alpha$ ).  
 (iv) Prove that  $XYED$  is a cyclic quadrilateral.

**QUESTION FOUR** (Start a new answer booklet)

Marks

- 4 (a) Prove by mathematical induction that for all positive integers  $n$ ,  

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1.$$

- 4 (b) The variable point  $P$  has coordinates  $P(a \cos 2\theta, a \sin \theta)$ .

(i) Show that  $P$  lies on the curve  $y^2 = -\frac{a}{2}(x - a)$ .

(ii) Sketch the locus of  $P$  as  $\theta$  varies, taking account of any restrictions on  $x$  and  $y$ .

- 4 (c) The point  $P(x, y)$  divides the interval joining the points  $A(-1, 3)$  and  $B(2, 8)$  internally in the ratio  $k : 1$ .

(i) Find the coordinates of  $P$  in terms of  $k$ .

(ii) Hence find the ratio in which the line  $5x + 2y - 10 = 0$  divides the interval  $AB$ .

**QUESTION FIVE** (Start a new answer booklet)

Marks

- 4 (a) When the polynomial  $P(x)$  is divided by  $(x + 1)(x - 2)$ , the result can be written as

$$P(x) = (x + 1)(x - 2)Q(x) + R(x), \text{ where } R(x) = ax + b.$$

(i) Given that  $P(-1) = 3$ , find the value of  $R(-1)$ .

(ii) Given also that the remainder is  $-2$  when  $P(x)$  is divided by  $x - 2$ , find the values of  $a$  and  $b$ .

- 4 (b) Let the expansion of  $(2 + 3x)^{12}$  be written in the form  $\sum_{r=0}^{12} t_r x^r$ .

(i) Write down expressions for  $t_r$  and  $t_{r+1}$ , and show that  $\frac{t_{r+1}}{t_r} = \frac{36 - 3r}{2r + 2}$ .

(ii) Hence find the greatest coefficient in the expansion of  $(2 + 3x)^{12}$ . You need not simplify your answer.

- 4 (c) (i) Show that the coefficient of  $x^n$  in the expansion of  $(1 + x)^n(1 + x)^n$  is given by

$$\sum_{r=0}^n ({}^nC_r)^2.$$

(ii) Hence, by equating the coefficients of  $x^n$  on both sides of the identity

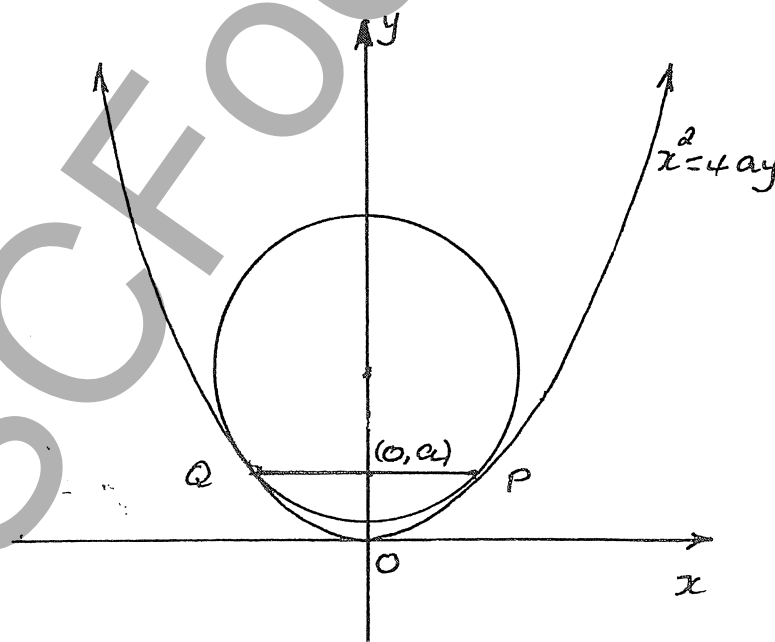
$$(1 + x)^n(1 + x)^n = (1 + x)^{2n},$$

prove that  $\sum_{r=0}^n ({}^nC_r)^2 = \frac{(2n)!}{(n!)^2}.$

**QUESTION SIX** (Start a new answer booklet)

Marks

- 5** (a) (i) Show that the function  $f(x) = x^3 - 3x + 1$  has stationary points at  $x = 1$  and at  $x = -1$ .
- (ii) Show that  $x^3 - 3x + 1 = 0$  has a root  $\alpha$  between  $x = 0$  and  $x = 0.5$ .
- (iii) Taking  $x = 0.1$  as a first approximation, use one application of Newton's method to find a closer approximation to  $\alpha$ . Give your answer correct to three decimal places.
- (iv) Explain, with the aid of a neat sketch of the curve, why  $x = 1.1$  would not be a suitable first approximation to  $\alpha$ .
- 7** (b) (i) Show that the equation of the chord  $PQ$  joining the points  $P(2ap, ap^2)$  and  $Q(2aq, aq^2)$  on the parabola  $x^2 = 4ay$  is  $y = \frac{1}{2}(p+q)x - apq$ .
- (ii) Suppose now that  $PQ$  is the *latus rectum* of the parabola, that is, the chord parallel to the directrix passing through the focus.
- ( $\alpha$ ) Show that  $p + q = 0$  and  $pq = -1$ , and find the coordinates of  $P$  and  $Q$ .
- ( $\beta$ ) Use calculus to find the equations of the normals at  $P$  and  $Q$ , and show that they intersect at  $N(0, 3a)$ .
- ( $\gamma$ )



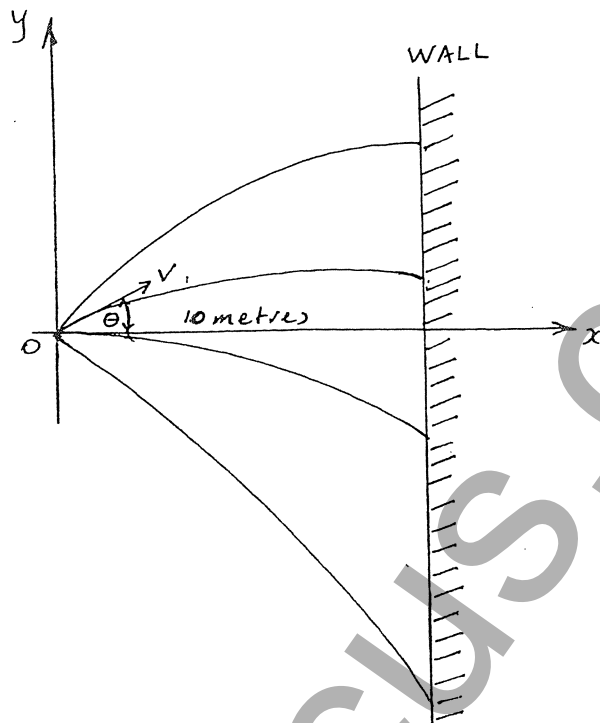
The diagram above shows the circle that touches the parabola  $x^2 = 4ay$  at the endpoints of the latus rectum. Use ( $\beta$ ) above to prove that the equation of the circle is  $x^2 + y^2 - 6ay + a^2 = 0$ .

**QUESTION SEVEN** (Start a new answer booklet)

Marks

**3** (a) Find the general solution of the equation  $\cos\left(2\pi\left(1 - \frac{1}{3}x\right)\right) = -\frac{1}{2}$ .

**9** (b)



In the diagram above, a large number of projectiles are fired simultaneously from  $O$ , each with the same velocity  $V$  but various angles of elevation  $\theta$ , at a wall distant 10 metres from  $O$ . The projectiles are fired so that their trajectories all lie in the same vertical plane perpendicular to the wall.

You may assume that the equations for the coordinates of a projectile at time  $t$  are

$$x = Vt \cos \theta \text{ and } y = -\frac{1}{2}gt^2 + Vt \sin \theta.$$

- (i) Use the identity  $\cos^2 \theta + \sin^2 \theta = 1$  to eliminate  $\theta$  from these two equations, and hence prove that the relationship between height  $y$  and time  $t$  is

$$4y^2 + 4gt^2y + k = 0, \text{ where } k = g^2t^4 + 4x^2 - 4V^2t^2.$$

- (ii) Show that the first impact on the wall occurs at time  $t = \frac{10}{V}$ , and that this projectile was fired horizontally. Also find where this projectile hits the wall.

- (iii) Show that for  $t > \frac{10}{V}$ , there are two impacts at time  $t$ , and that the distance between these impacts is

$$2\sqrt{V^2t^2 - 100}.$$

- (iv) Given that  $V = 10$  m/s, what are the initial angles of elevation of the two projectiles that strike the wall simultaneously  $20\sqrt{3}$  metres apart.

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QUESTION ONE

$$(a) \frac{d}{dx}(\tan^{-1} \pi x) = \frac{\pi}{1 + \pi^2 x^2}. \quad \boxed{\checkmark}$$

$$(b) \quad (i) \int \frac{dx}{4 + x^2} = \frac{1}{2} \tan^{-1} \frac{x}{2} + C. \quad \boxed{\checkmark}$$

$$(ii) \int \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1} \frac{x}{2} + C. \quad \boxed{\checkmark}$$

$$\begin{aligned} (c) \int_0^{\frac{\pi}{4}} \tan^2 x \, dx &= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \, dx, \quad \boxed{\checkmark} \\ &= \left[ \tan x - x \right]_0^{\frac{\pi}{4}} \quad \boxed{\checkmark} \\ &= 1 - \frac{\pi}{4} \\ &= \frac{4 - \pi}{4}. \quad \boxed{\checkmark} \end{aligned}$$

$$(d) \text{ Gradients are } \frac{1}{2} \text{ and } -\frac{1}{\sqrt{3}}.$$

$$\begin{aligned} \tan \alpha &= \left| \frac{\frac{1}{2} + \frac{1}{\sqrt{3}}}{1 - \frac{1}{2\sqrt{3}}} \right| \\ &= \frac{\sqrt{3} + 2}{2\sqrt{3} - 1}. \quad \boxed{\checkmark\checkmark} \end{aligned}$$

$$\begin{aligned} (e) \, LHS &= \frac{\sin 2\theta \cos \theta - \cos 2\theta \sin \theta}{\sin \theta \cos \theta} \quad \boxed{\checkmark} \\ &= \frac{\sin \theta}{\sin \theta \cos \theta} \\ &= \sec \theta \quad \boxed{\checkmark} \\ &= RHS. \end{aligned}$$

$$(f) \quad \frac{x-2}{x} \geq 1$$

$$(x-2)x \geq x^2, \quad x \neq 0 \quad \boxed{\checkmark}$$

$$x^2 - 2x \geq x^2$$

$$-2x \geq 0$$

$$\text{so} \quad x \leq 0 \quad \boxed{\checkmark}$$

QUESTION TWO

(a) Let  $u = e^x$

$$du = e^x dx.$$

When  $x = 0$

$$u = 1.$$

When  $x = \ln\sqrt{3}$

$$u = \sqrt{3}.$$

$$\begin{aligned} \int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}} dx &= \int_1^{\sqrt{3}} \frac{du}{1+u^2} \quad \checkmark \\ &= \left[ \tan^{-1} u \right]_1^{\sqrt{3}} \quad \checkmark \\ &= \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{\pi}{12}. \quad \checkmark \end{aligned}$$

(b) General term  $= {}^6C_r (x^2)^{6-r} (2x^{-1})^r$

$$= {}^6C_r \times 2^r \times x^{12-3r} \quad \checkmark$$

$$12 - 3r = 0$$

$$r = 4. \quad \checkmark$$

Required term  $= {}^6C_4 \times 2^4$

$$= 240. \quad \checkmark$$

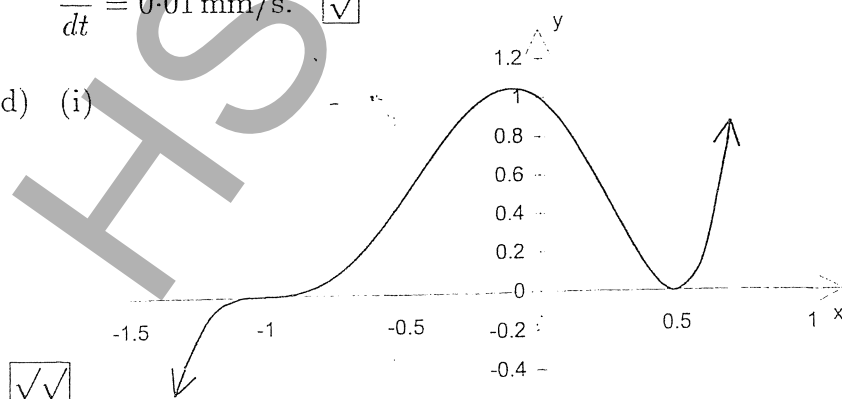
(c)  $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$$10 = 4\pi r^2 \times \frac{dr}{dt} \quad \checkmark$$

$$10 = 1000 \times \frac{dr}{dt} \quad \checkmark$$

$$\frac{dr}{dt} = 0.01 \text{ mm/s.} \quad \checkmark$$

(d) (i)



(ii)  $x \geq -1. \quad \checkmark$



QUESTION THREE

(a)  $v^2 = 5 + 14x - 3x^2$

$$\frac{d}{dx}(\frac{1}{2}v^2) = \frac{1}{2}(14 - 6x)$$

$$= 7 - 3x$$

$$= -3(x - \frac{7}{3}), \quad \boxed{\checkmark}$$

which is of the required form,

The centre of motion is  $x = \frac{7}{3}$ .  $\boxed{\checkmark}$

Let  $v^2 = 0$

$$3x^2 - 14x - 5 = 0$$

$$(3x + 1)(x - 5) = 0$$

$$x = -\frac{1}{3} \text{ or } 5.$$

$$\begin{aligned} \text{Amplitude} &= 5 - \frac{7}{3} \\ &= \frac{8}{3}. \quad \boxed{\checkmark} \end{aligned}$$

$$n = \sqrt{3}$$

$$\text{Period} = \frac{2\pi}{\sqrt{3}}. \quad \boxed{\checkmark}$$

(b) (i) Let  $\cos 2x - \sin 2x = R \cos(2x + \alpha)$   
$$= R \cos 2x \cos \alpha - R \sin 2x \sin \alpha$$

So  $R \cos \alpha = 1$

and  $R \sin \alpha = 1.$

So  $R = \sqrt{2}$

and  $\alpha = \frac{\pi}{4}.$

So  $\cos 2x - \sin 2x = \sqrt{2} \cos(2x + \frac{\pi}{4}). \quad \boxed{\checkmark}$

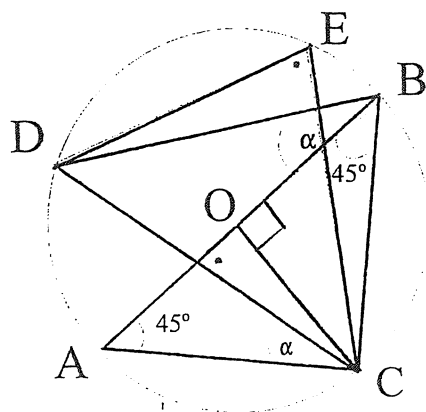
(ii) Now  $\sqrt{2} \cos(2x + \frac{\pi}{4}) = 1$

$$\cos(2x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}, \text{ for } \frac{\pi}{4} \leq 2x + \frac{\pi}{4} \leq \frac{9\pi}{4} \quad \boxed{\checkmark}$$

$$2x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \text{ or } \frac{9\pi}{4}$$

$$x = 0, \frac{3\pi}{4} \text{ or } \pi. \quad \boxed{\checkmark\checkmark}$$

(c)



(i)  $\angle CBA = \angle CAB = 45^\circ$  (angle at centre is twice angle at circumference). ☒

(ii)  $\angle DCA = \angle DBA$  and  $\angle CBD = \angle CED$ ,  
(angles at the circumference standing on the same arc). ☒

(iii) Let  $\angle DCA = \angle DBA = \alpha$ .

$\angle CBD = \alpha + 45^\circ$  (from (i) and (ii)).

But  $\angle CXB = \alpha + 45^\circ$  (exterior angle of  $\triangle AXC$  equals sum of interior opposite angles)

so  $\angle CBD = \angle CXB$ . ☒

(iv)  $\angle DEY = \alpha + 45^\circ$  from (ii) above,

$\angle CXB = \alpha + 45^\circ$  from (iii) above,

so  $\angle DEY = \angle CXB$ ,

so  $XYED$  is a cyclic quadrilateral as the exterior angle equals interior opposite angle. ☒

#### QUESTION FOUR

(a) Prove that for positive integers  $n$ ,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1.$$

##### STEP 1.

When  $n = 1$ :

$$\text{LHS} = 1$$

$$\text{RHS} = 2! - 1$$

$$= 1.$$

So proposition is true for  $n = 1$ . ☒

##### STEP 2.

Assume the proposition true for some positive integer  $k$  so that,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! = (k+1)! - 1. \quad \text{☒$$

We are required to prove the proposition true for  $k+1$ . That is,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + k \times k! + (k+1) \times (k+1)! = (k+2)! - 1.$$

$$\begin{aligned}
 \text{Now LHS} &= 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! + (k+1) \times (k+1)!. \\
 &= (k+1)! - 1 + (k+1) \times (k+1)!, \text{ from the assumption } \boxed{\checkmark} \\
 &= (k+1)! \times (1+k+1) - 1 \\
 &= (k+2)(k+1)! - 1 \\
 &= (k+2)! - 1 \\
 &= \text{RHS. } \boxed{\checkmark}
 \end{aligned}$$

It follows from steps one and two above by mathematical induction that for all positive integers  $n$ ,

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1.$$

(b) (i)  $y^2 = -\frac{a}{2}(x-a).$

$$\begin{aligned}
 \text{LHS} &= y^2 \\
 &= a^2 \sin^2 \theta \\
 \text{RHS} &= -\frac{a}{2}(a \cos 2\theta - 1) \\
 &= -\frac{a^2}{2}(1 - 2\sin^2 \theta - 1) \boxed{\checkmark} \\
 &= a^2 \sin^2 \theta \\
 &= \text{LHS. } \boxed{\checkmark}
 \end{aligned}$$

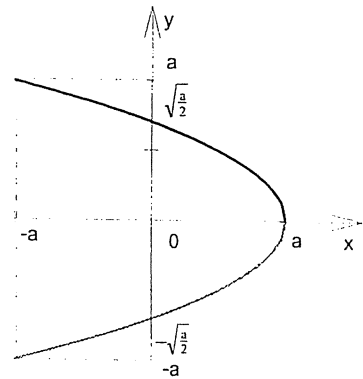
So  $P$  lies on the curve.

(ii)  $x = a \cos 2x$

so  $-a \leq x \leq a,$

and  $y = a \sin \theta$

so  $-a \leq y \leq a. \boxed{\checkmark}$



$$y^2 = -\frac{a}{2}(x-a). \boxed{\checkmark}$$

(c) (i) Given points are  $A(-1, 3)$  and  $(2, 8).$

$$\begin{aligned}
 \text{For } P \quad x &= \frac{k \times 2 + 1 \times -1}{k+1} \\
 &= \frac{2k-1}{k+1}
 \end{aligned}$$

$$\begin{aligned}
 \text{and } y &= \frac{k \times 8 + 1 \times 3}{k+1} \\
 &= \frac{8k+3}{k+1}.
 \end{aligned}$$

$P$  is the point  $\left(\frac{2k-1}{k+1}, \frac{8k+3}{k+1}\right). \boxed{\checkmark\checkmark}$

(ii) Let  $P$  lie on  $5x + 2y - 10 = 0,$

$$\text{so } 5 \times \left( \frac{2k-1}{k+1} \right) + 2 \times \left( \frac{8k+3}{k+1} \right) - 10 = 0 \quad \checkmark$$

$$10k - 5 + 16k + 6 - 10k - 10 = 0$$

$$16k - 9 = 0$$

$$k = \frac{9}{16}. \quad \checkmark$$

The line  $5x + 2y - 10 = 0$  divides the interval in the ratio 9 : 16.

### QUESTION FIVE

(a) (i)  $P(x) = (x+1)(x-2)Q(x) + R(x)$

$$P(-1) = R(-1)$$

$$\text{so } R(-1) = 3. \quad \checkmark$$

(ii)  $P(2) = 0 + R(2)$

$$\text{so } R(2) = -2. \quad \checkmark$$

$$\text{Now } -a + b = 3$$

$$\text{and } 2a + b = -2$$

$$\text{so } 3a = -5$$

$$a = -\frac{5}{3} \quad \checkmark$$

$$\text{and } b + \frac{5}{3} = 3$$

$$b = \frac{4}{3}. \quad \checkmark$$

(b) (i)  $t_r = {}^{12}C_r \times 2^{12-r} \times 3^3 \quad \checkmark$

$$t_{r+1} = {}^{12}C_{r+1} \times 2^{11-r} \times 3^{r+1}$$

$$\frac{t_{r+1}}{t_r} = \frac{{}^{12}C_{r+1} \times 2^{11-r} \times 3^{r+1}}{{}^{12}C_r \times 2^{12-r} \times 3^3}$$

$$= \frac{12!}{(r+1)!(11-r)!} \times \frac{r!(12-r)!}{12!} \times \frac{3}{2} \quad \checkmark$$

$$= \frac{12-r}{r+1} \times \frac{3}{2}$$

$$= \frac{36-3r}{2r+2} \quad \checkmark$$

(ii) For increasing coefficients,

$$\frac{t_{r+1}}{t_r} > 1$$

$$\frac{36-3r}{2r+2} > 1$$

$$36-3r > 2r+2$$

$$r < 6\frac{4}{5}.$$

Since  $t_r < t_{r+1}$ ,  $t_1 < t_2 < t_3 < \dots < t_7$ .

So the greatest coefficient is  $t_7 = {}^{12}C_7 \times 2^5 \times 3^7. \quad \checkmark$

$$(c) \quad (i) \quad (1+x)^n \times (1+x)^n = \left( \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \right)^2.$$

Term in  $x^n$

$$= \binom{n}{0} \binom{n}{n} x^n + \binom{n}{1} \binom{n}{n-1} x^n + \cdots + \binom{n}{n} \binom{n}{0} x^n$$

$$= \left( \binom{n}{0} \binom{n}{n} + \binom{n}{1} \binom{n}{n-1} + \cdots + \binom{n}{n} \binom{n}{0} \right) x^n$$

Coefficient of term in  $x^n$

$$= \binom{n}{0} \binom{n}{n} + \binom{n}{1} \binom{n}{n-1} + \cdots + \binom{n}{n} \binom{n}{0} \quad \boxed{\checkmark}$$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2, \text{ since } \binom{n}{r} = \binom{n}{n-r}$$

$$= \sum_{r=0}^n \binom{n}{r}^2. \quad \boxed{\checkmark}$$

(ii) For coefficient of term in  $x^n$  in the expansion of  $(1+x)^{2n}$ ,

$$\text{coefficient} = \binom{2n}{n}$$

$$= \frac{(2n)!}{(n!)^2} \quad \boxed{\checkmark}$$

$$\text{so } \sum_{r=0}^n \binom{n}{r}^2 = \frac{(2n)!}{(n!)^2} \quad \boxed{\checkmark}$$

### QUESTION SIX

$$(a) \quad (i) \quad f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3.$$

$$\text{Let } f'(x) = 0$$

$$3x^2 - 3 = 0$$

$$3(x-1)(x+1) = 0$$

$$x = 1 \text{ or } -1. \quad \boxed{\checkmark}$$

$$(ii) \quad f(0) = 1$$

$$> 0$$

$$\text{and } f(0.5) = -0.375$$

$$< 0,$$

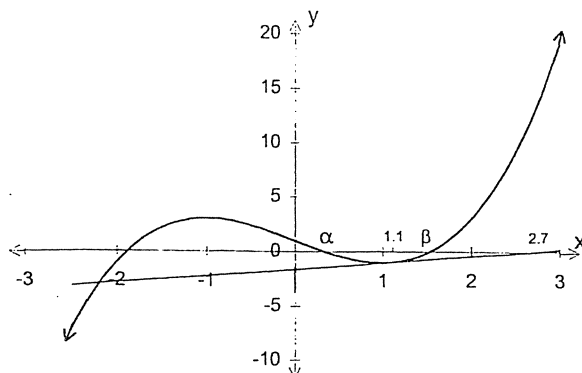
$$\text{so } \alpha \text{ is a root between } x = 0 \text{ and } x = 0.5. \quad \boxed{\checkmark}$$

$$(iii) \quad x_2 = 0.1 - \frac{f(0.1)}{f'(0.1)}$$

$$= 0.1 + \frac{0.701}{2.97} \quad \boxed{\checkmark}$$

$$= 0.336 \text{ (correct to three decimal places).} \quad \boxed{\checkmark}$$

(iv)



The tangent at  $x = 1.1$  crosses the  $x$ -axis to the right of the root  $\beta$  shown, where  $\alpha < \beta$ . Further applications will approximate to the root at  $\beta$ . ☒

$$\begin{aligned} \text{(b) (i) Gradient} &= \frac{ap^2 - aq^2}{2ap - 2aq} \\ &= \frac{p + q}{2} \end{aligned}$$

Equation of the chord is,

$$y - ap^2 = \frac{p + q}{2}(x - 2ap)$$

$$y = \frac{p + q}{2}x - apq. \quad \boxed{\checkmark}$$

(ii) ( $\alpha$ )  $S(0, a)$  lies on the chord,

$$\text{so } a = 0 - apq$$

$$pq = -1.$$

$PQ$  is parallel to  $x$ -axis so gradient is zero.

$$\frac{p + q}{2} = 0$$

$$\text{so } p + q = 0 \quad \boxed{\checkmark \text{ for both results}}$$

$$\text{and so } p = 1 \text{ and } q = -1.$$

It follows that the coordinates are  $P(2a, a)$  and  $Q(-2a, a)$ . ☒

$$(\beta) \quad y = \frac{1}{4a}x^2$$

$$\frac{dy}{dx} = \frac{x}{2a}$$

At  $P$ ,  $\frac{dy}{dx} = 1$ , so gradient of normal is  $-1$ .

Equation of normal:

$$y - a = -1(x - 2a)$$

$$y = -x + 3a \dots (1)$$

$$\text{At } Q, \frac{dy}{dx} = -1, \text{ so gradient of normal is } 1. \quad \boxed{\checkmark \text{ for either gradient}}$$

Equation of normal:

$$y - a = 1(x + 2a)$$

$$y = x + 3a \dots (2) \quad \boxed{\checkmark \text{ for normals}}$$

For  $N$ , (1) + (2):

$$2y = 6a$$

so  $y = 3a$

and  $3a = x + 3a$

so  $x = 0$ .

$N$  is the point  $(0, 3a)$ .

( $\gamma$ ) Since the tangents to the parabola at  $P$  and  $Q$  are also tangents to the circle, the normals at these points are radii of the circle. These intersect at the centre of the circle.

The centre of the circle is  $N(0, 3a)$ .

$$\begin{aligned} \text{radius} &= \sqrt{(2a - 0)^2 + (3a - a)^2} \\ &= \sqrt{4a^2 + 4a^2} \\ &= 2a\sqrt{2}. \quad \boxed{\checkmark} \end{aligned}$$

The equation of the circle is:

$$(x - 0)^2 + (y - 3a)^2 = (2a\sqrt{2})^2$$

$$x^2 + y^2 - 6ay + 9a^2 = 8a^2$$

$$x^2 + y^2 - 6ay + a^2 = 0. \quad \boxed{\checkmark}$$

### QUESTION SEVEN

(a)  $\cos\left(2\pi\left(1 - \frac{1}{3}x\right)\right) = -\frac{1}{2}$

$$2\pi\left(1 - \frac{1}{3}x\right) = 2n\pi + \frac{2\pi}{3} \text{ or } 2n\pi - \frac{2\pi}{3}, \text{ where } n \text{ is an integer.} \quad \boxed{\checkmark}$$

$$1 - \frac{1}{3}x = n + \frac{1}{3} \text{ or } n - \frac{1}{3}$$

$$-\frac{1}{3}x = n - \frac{2}{3} \text{ or } n - \frac{4}{3}$$

$$x = 2 - 3n \text{ or } 4 - 3n \quad \boxed{\checkmark\checkmark}$$

(Note: Check variations.)

(b) (i)

$$V \cos \theta = \frac{x}{t}$$

$$V \sin \theta = \frac{y}{t} + \frac{1}{2}gt$$

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t} + \frac{1}{2}gt\right)^2 = V^2$$

$$x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = V^2t^2$$

$$4x^2 + (2y + gt^2)^2 = 4V^2t^2$$

$$\text{so } 4y^2 + 4ygt^2 + (4x^2 + g^2t^4 - 4V^2t^2) = 0$$

$$4y^2 + 4gt^2y + k = 0 \quad \boxed{\checkmark}$$

(ii) Now  $t = \frac{x}{V \cos \theta}$  which, for fixed  $x$  and  $V$ , is a minimum when  $\cos \theta$  is a maximum.

So minimum  $t = \frac{x}{V}$  when  $\theta = 0$  ☒

$$= \frac{10}{V} \cdot \quad \text{☒$$

So  $y = -\frac{1}{2}g \times \left(\frac{10}{V}\right)^2$   
 $= -\frac{50g}{V^2} \cdot \quad \text{☒$

- (iii) For  $t > \frac{10}{V}$  there is a solution for  $\theta$ , which means that the projectile hits the wall and hence there will be a solution for  $y$ .

Now for the quadratic in  $y$  :

$$\Delta = (4gt^2)^2 - 16(g^2t^4 + 4 \times 10^2 - 4V^2t^2)$$

$$= 64(V^2t^2 - 100).$$

So  $\Delta > 64\left(V^2 \times \frac{10^2}{V^2} - 100\right)$  since  $t > \frac{10}{V}$   
 $> 0. \quad \text{☒$

Hence there are two real and distinct roots and so there are two impacts at time  $t$ . ☒

Now distance between impacts equals difference between roots.

$$\text{Difference} = \frac{-b + \sqrt{\Delta}}{2a} - \frac{-b - \sqrt{\Delta}}{2a}$$

$$= \frac{2\sqrt{\Delta}}{2a}$$

$$= \frac{\sqrt{\Delta}}{a}$$

So Distance  $= \frac{8\sqrt{V^2t^2 - 100}}{4}$   
 $= 2\sqrt{V^2t^2 - 100}. \quad \text{☒$



$$\begin{aligned} \text{(iv)} \quad \text{Distance} &= 2\sqrt{V^2t^2 - 100} \\ &= 2\sqrt{100t^2 - 100} \\ &= 20\sqrt{t^2 - 1} \end{aligned}$$

$$\text{so} \quad 20\sqrt{t^2 - 1} = 20\sqrt{3}$$

$$\sqrt{t^2 - 1} = \sqrt{3}$$

$$t^2 - 1 = 3$$

$$t^2 = 4$$

$$t = 2. \quad \boxed{\checkmark}$$

Now

$$\cos \theta = \frac{x}{Vt}$$

$$= \frac{10}{10 \times 2}$$

$$= \frac{1}{2}$$

and so

$$\theta = 60^\circ \text{ and } -60^\circ \text{ are the angles of elevation.} \quad \boxed{\checkmark}$$

GJ