Sami El Hosri's 4 Unit Mathematics Specimen HSC Paper

QUESTION 1

a) Evaluate $\int_{-1}^{1} \sqrt{(x+2)^2 - 8x}\ dx$.

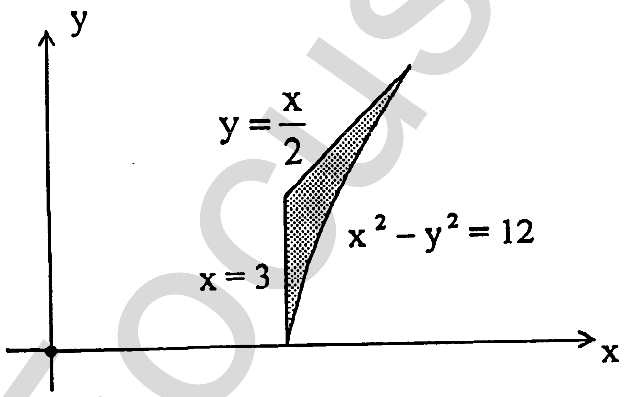
b) Evaluate $\int_{1/16}^{1/9} \frac{\cos(\pi\sqrt{x})}{\sqrt{x}}\ dx$.

c) Find $\int \frac{\tan^2(\frac{x}{2})}{\sin x}\ dx$.

d) Prove that $\cot(\frac{x}{2})\tan x = 1 + \frac{1}{\cos x}$. Hence or otherwise, find $\int \cot(\frac{x}{2})\tan x\ dx$.

QUESTION 2

a)



The diagram shows the shaded region enclosed by the parts of the lines $y = \frac{x}{2}, x = 3$ and the part of the hyperbola $x^2 - y^2 = 12$ the area is rotated about the y axis to form a solid. Find the volume of the solid.

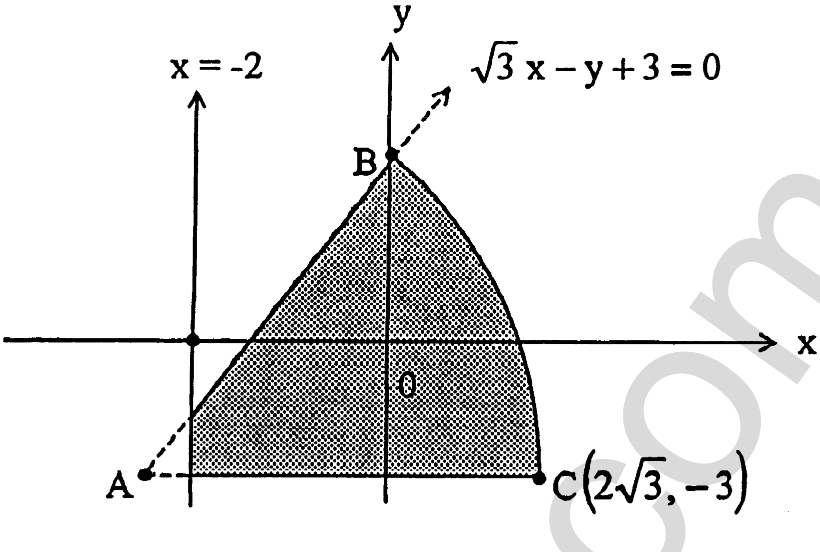
b) Sketch on a separate diagrams the following curves. Indicate clearly any turning points, asymptotes and intercepts with the axes.

i) $y = \frac{x}{\log_e x}$ **ii)** $y = \frac{\log_e x}{x}$ **iii)** $y^2 = \frac{\log_e x}{x}$

iv) $y = \log_e\left(\frac{\log_e x}{x}\right)$ **v)** $y = e^{\frac{x}{\log_e x}}$.

QUESTION 3

a)



NOT TO SCALE

In the Argand diagram above, the shaded region is part of a circle centred at A with radii AC and AB. Find the conditions that should be satisfied by the complex numbers z which are in this region.

b) Considering the complex numbers $\alpha = (m+6) + (2m+1)i, \beta = (2m-1) + (3m+2)i$ and $\phi = (m^2 + 4m + 10) + (m+1)i$ where m is a real number:

i) given that $[\Im(\alpha + 2i)]^2 = 1 + \Re[(\beta + 1 - i)^2]$, what can be said about the complex number ϕ?

ii) given that $|\beta - \alpha|^2 = 2\Re(\phi + 10)$ what can be said about the complex number β?

c) The complex number z satisfies the locus $|z - 2i| = 2\Im(z) - 1$. Show that the locus is a hyperbola and neatly sketch its graph showing its foci, directrices and asymptotes.

QUESTION 4

a) Find the complex square roots of $48 + 14i$, giving your answer in the form $a + bi$, where a and b are real.

b) Consider the polynomial $z^3 - (2 + 7i)z^2 - (23 - 7i)z + 24 = 0$.

i) This equation has one real root α. Find α.

ii) If the other roots are the complex numbers β and ϕ, use expressions of the sum and products ot the roots, or otherwise, to find β and ϕ. (You may use your result of part **a)**)

iii) Show that the points on the Argand diagram α, β and ϕ are the vertices of a right-angled isosceles triangle.

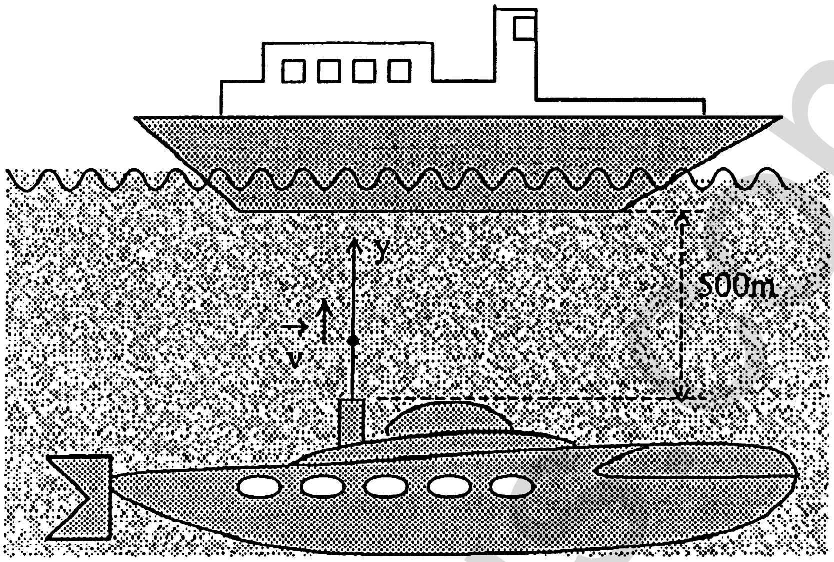
c) If z_1 and z_2 are two complex numbers where $|z_1| = |z_2| = 1$

i) Show that $\overline{z}_1 = \frac{1}{z_1}$ and hence explain why $\overline{z}_2 = \frac{1}{z_2}$.

ii) If ϕ is another complex number such that $\phi = \frac{1+z_1+z_1z_2}{1+z_2+z_1z_2}$ and using your result from **c) i)** show that $\phi \cdot \overline{\phi} = 1$ and hence show that $|\phi| = 1$.

QUESTION 5

a) A stationary submarine fires a missile of mass 40 kg with a speed of 500 m/s towards a ship at rest 500 m directly above it, as shown in the diagram below.



The missile is subject to a downward gravitational force of 400 Newtons and water resistance of $\frac{3v^2}{100}$ Newtons in the opposite direction to the velocity v m/s.

i) Using $\ddot{y} = \frac{v\ dv}{dy}$, show that, while the missile is rising, $\frac{3v^2}{10000} = 79e^{\frac{-3y}{2000}} - 4$.

ii) Show that the velocity of the missile at the time of impact with the ship is approximately 333 m/s.

iii) Hence, using $\ddot{y} = \frac{dy}{dt}$, show that the time taken by the missile to hit the ship is $\frac{20\sqrt{3}}{3} \tan^{-1}\left[\frac{334\sqrt{3}}{5395}\right]$ seconds.

b) i) Express $\alpha = \frac{-1+i}{3-i\sqrt{3}}$ in modulus/argument form. Hence evaluate $\cos\frac{11\pi}{12}$ in surd form.

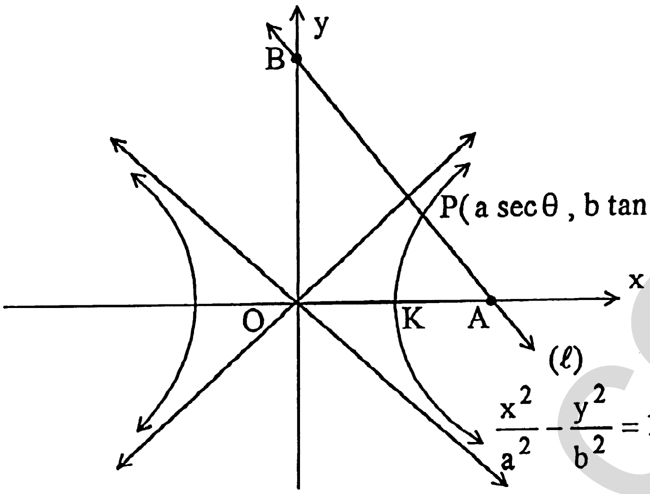
ii) Find the smallest positive integer K such that α^K is imaginary and evaluate K.

QUESTION 6

a) Solve $a^x = e^{2x-1}$ in terms of a where $a > 0, a \neq \frac{1}{e^2}$.

QUESTION 6 (continued)

b)



The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > b$ cuts the positive x axis at the point K. The line (ℓ) which is the normal to the hyperbola at the point $P(a\sec\theta, b\tan\theta)$ cuts the x axis at A and the y axis at B, as shown on the diagram.

i) Show that the equation of the line (ℓ) is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$.

ii) Find the midpoint M of AB.

iii) Find the point G such that G cuts the interval OM in the ratio $2:1$.

iv) Show that the locus of G is a hyperbola and find the point L at which this locus intersects with the positive x axis.

v) Show that for the rotio $\frac{OL}{OK} < 1$ is valid only if the condition $1 < e < \sqrt{3}$ must be satisfied.

QUESTION 7

a) Let $I_n = \int_0^1 \frac{x^n}{\sqrt{9-x^2}}\, dx$, where n is a positive integer.

i) Find the value of I_0.

ii) Using integration, show that $nI_n = 9(n-1)I_{n-2} - 2\sqrt{2}$, for $n \geq 2$.

iii) Hence evaluate $L = \int_0^1 \frac{x^4}{\sqrt{9-x^2}}\, dx$

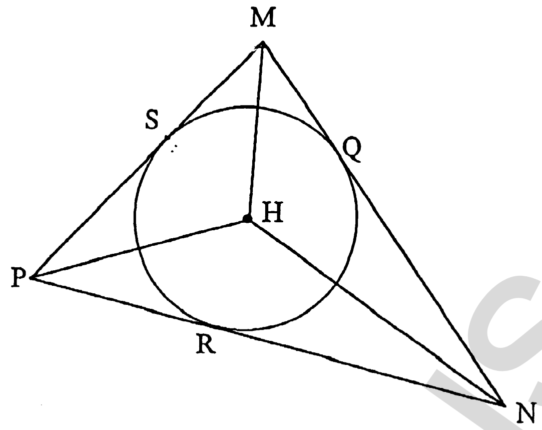
QUESTION 7 (continued)

b) PM, PN and MN three tangents to the incircle, of centre H, at the points S, R and Q respectively as shown in the diagram below.

i) Show that HP bisects $\angle MPN$

ii) Show that HM bisects $\angle PMN$

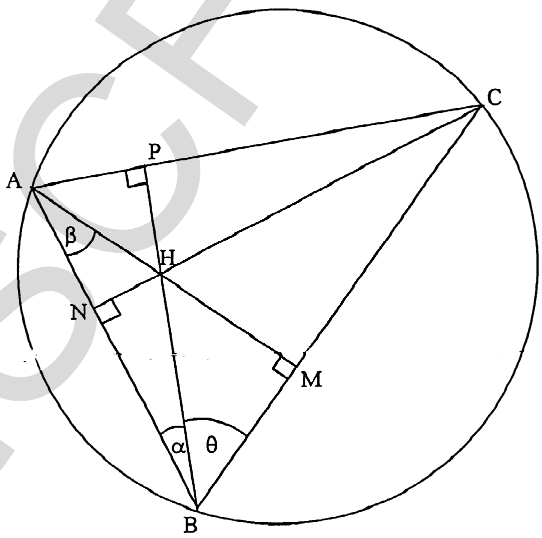
iii) Hence explain a method to draw the incircle of a triangle.



c) ABC is a triangle inscribed inside a circle. AM, BP and CN are the altitudes to the sides BC, AC and AB respectively, where H is their point of intersection. Let $\angle ABP = \alpha, \angle BAM = \beta, \angle CBP = \theta$.

i) Show that $\angle NCA = \alpha, \angle BCN = \beta$ and $\angle MAC = \theta$

ii) Use parts **b)** and **c) i)** to show that H is the centre of the incircle to $\triangle MNP$.



NOT TO SCALE

QUESTION 8

a) i) Show that $(m+n+p)^2 \geq 3(mn+mp+np)$ where m, n and p are positive numbers.

ii) Hence, or otherwise, show that $m^2n^2 + m^2p^2 + n^2p^2 \geq mnp(m+n+p)$

b) A group of 6 boys and 6 girls went to the theatre with two teachers. In how many ways could they be seated in a row if:

i) The boys sat apart from the girls, and the teachers sat apart from all the students?

ii) The teachers sat on each end of the row, and the boys and girls sat in alternate positions on the remaining seats?

iii) No two boys are seated together?

c) i) Prove that $\tan^{-1}(x+2) - \tan^{-1}(x+1) = \cot^{-1}(x^2+3x+3)$ where $x \geq 0$.

ii) Simplify the sum $K_n := \cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \cdots + \cot^{-1}(n^2+3n+3)$ where n is a nonnegative integer.

iii) Show that $\lim\limits_{n \to \infty} K_n = \frac{\pi}{4}$.