The Association of Independent Schools of New South Wales

Maths Focus Day

Tara Anglican School - May 22, 2002

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In these notes I summarise three of eight workshops at the AIS Maths Focus Day.

These are:

• Complex Numbers - Morris Needleman - I have added an appendix to this.

• Mechanics - Mick Canty

• Volumes - Ross Dinnell

The Plenary Session was

• Mathematical Induction - Mark Arnold

The other five workshops were

• Projectile Motion - Anne Prescott

• Logarithms - Bill Pender

• Curve Sketching - John Hastings

• The Binomial Theorem and Combinatorial Identities - Peter Brown

• Simple Harmonic Motion - David Sadler

Typeset by $\mathcal{AMS}$-TEX

Complex Numbers - Morris Needleman

Focus Questions.

1. (HSC 1987 Question 8(ii))

a. A polynomial $R(x)$ is given by $R(x) = x^7 - 1$. Let $\rho \neq 1$ be that complex root of $R(x) = 0$ which has the smallest positive argument. Show that:

$\boldsymbol{\alpha}$. $R(x) = (x-1)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)$

$\boldsymbol{\beta}$. $1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6 = 0$

b. Let $\theta = \rho + \rho^2 + \rho^4$ and $\phi = \rho^3 + \rho^5 + \rho^6$.

$\boldsymbol{\alpha}$. Prove that $\theta + \phi = -1$ and $\theta\phi = 2$.

$\boldsymbol{\beta}$. Show that $\theta = \frac{-1+i\sqrt{7}}{2}$ and $\phi = \frac{-1-i\sqrt{7}}{2}$.

c. Given that

$$\begin{aligned} T(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\ &= (x-\rho)(x-\rho^2)(x-\rho^3)(x-\rho^4)(x-\rho^5)(x-\rho^6), \end{aligned}$$

write the polynomial $T(x)$ as a product of two cubics with coefficients involving θ, ϕ and rational numbers.

2. Prove that $e^{i\theta} = \cos\theta + i\sin\theta$

3. Prove that $i^i \subset \mathbb{R}$.

Solutions.

1.

a.

$\boldsymbol{\alpha}$. $$\begin{aligned} R(x) &= x^7 - 1 \\ &= (x-1)\cdot\frac{x^7-1}{x-1} \\ &= (x-1)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \quad \square \end{aligned}$$

$\boldsymbol{\beta}$. $\rho \neq 1 \Rightarrow \rho - 1 \neq 0$ and $R(\rho) = 0 \Rightarrow (\rho-1)(1+\rho+\rho^2+\rho^3+\rho^4+\rho^5+\rho^6) = 0$. Hence $1 + \rho + \rho^2 + \rho^3 + \rho^4 + \rho^5 + \rho^6 = 0$. $\square$

b.

$\boldsymbol{\beta}$. $\theta+\phi = \rho+\rho^2+\rho^4+\rho^3+\rho^5+\rho^6 = (1+\rho+\rho^2+\rho^3+\rho^4+\rho^5+\rho^6)-1 = 0-1 = -1$

and

$$\begin{aligned}\theta\phi &= (\rho+\rho^2+\rho^4)(\rho^3+\rho^5+\rho^6)\\ &= \rho^4+\rho^6+\rho^7+\rho^5+\rho^7+\rho^8+\rho^7+\rho^9+\rho^{10}\\ &= \rho^4+\rho^6+1+\rho^5+1+\rho+1+\rho^2+\rho^3 \qquad (\because \rho^7=1)\\ &= (1+\rho+\rho^2+\rho^3+\rho^4+\rho^5+\rho^6)+2\\ &= 0+2\\ &= 2\end{aligned}$$

using **a.** $\boldsymbol{\beta}$. $\square$

$\boldsymbol{\beta}$**.** Since $\theta+\phi=-1$ and $\theta\phi=2$, then θ and ϕ are the roots of $y^2+y+2=0$.
$\rho=e^{2\pi i/7}$ and so
$\Im(\theta)=\Im(\rho+\rho^2+\rho^4)=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=2\sin\frac{5\pi}{7}\cos\frac{3\pi}{7}+\sin\frac{4\pi}{7}>0$
since $\sin\frac{5\pi}{7}>0, \cos\frac{3\pi}{7}>0$ and $\sin\frac{4\pi}{7}>0$.
Hence $\theta=\frac{-1+i\sqrt{7}}{2}$ and $\phi=\frac{-1-i\sqrt{7}}{2}$ $\square$

c. $$\begin{aligned}T(x) &= 1+x+x^2+x^3+x^4+x^5+x^6\\ &= (x-\rho)(x-\rho^2)(x-\rho^3)(x-\rho^4)(x-\rho^5)(x-\rho^6)\\ &= [(x-\rho)(x-\rho^2)(x-\rho^4)][(x-\rho^3)(x-\rho^5)(x-\rho^6)]\\ &= [x^3-(\rho+\rho^2+\rho^4)x^2+(\rho^3+\rho^5+\rho^6)x-\rho^7]\\ &\quad\cdot[x^3-(\rho^3+\rho^5+\rho^6)x^2+(\rho^8+\rho^9+\rho^{11})x-\rho^{14}]\\ &= [x^3-\theta x^2+\phi x-1][x^3-\phi x^2+(\rho+\rho^2+\rho^4)x-1] \qquad (\because \rho^7=1)\\ &= (x^3-\theta x^2+\phi x-1)(x^3-\phi x^2+\theta x-1) \quad \square\end{aligned}$$

Note. $\theta=\rho^1+\rho^2+\rho^4$ and $\phi=\rho^3+\rho^5+\rho^6$ and 1,2,4 are perfect squares mod 7 whereas 3,5,6 are not. Hence for example, $T(x)$ cannot be written as a product of two cubics with coefficients involving $\tau:=\rho+\rho^3+\rho^5$ and $\sigma:=\rho^2+\rho^4+\rho^6$ and rational numbers. This is related to Galois Theory.

2. $$\begin{aligned}e^{i\theta} &= e^{\int_0^\theta i\,dx} = e^{\int_0^\theta \frac{i(\cos x+i\sin x)\,dx}{\cos x+i\sin x}} = e^{\int_0^\theta \frac{(-sin x+i\cos x)\,dx}{\cos x+i\sin x}} = e^{\int_0^\theta \frac{\frac{d}{dx}(\cos x+i\sin x)\,dx}{\cos x+i\sin x}}\\ &= e^{[\ln(\cos x+i\sin x)]_0^\theta} = e^{\ln(\cos\theta+i\sin\theta)-\ln(\cos 0+i\sin 0)} = e^{\ln(\cos\theta+i\sin\theta)-0}\\ &= \cos\theta+i\sin\theta \quad \square\end{aligned}$$

3. $i^i=\{(e^{\pi i/2-2\pi ki})^i\}_{k\in\mathbb{Z}}=\{e^{(4k-1)\pi/2}\}_{k\in\mathbb{Z}}\subset\mathbb{R}$ $\square$

Following the workshop I spoke to the presenter about a comment he made to the effect that $\mathbb{C}$ contains all other type of numbers. This is not true and so here I present an appendix to his workshop:

Appendix: Standard Notations for Number Systems by Derek Buchanan

Nonnegative integers: $\mathbb{N}$ (**N**atural numbers, or cardinal numbers)
$\mathbb{N} := \{0, 1, 2, \dots\}$.
$\mathbb{N}$ can be constructed purely set theoretically. The *Fundamental Theorem of Arithmetic* states that the natural numbers greater than 1 can be written uniquely as the product $\Pi_{i=1}^{s} p_i^{\alpha_i}$ for *prime* numbers p_i and positive natural numbers s and α_i. The way this works is not yet fully understood and largely depends on the *Riemann Hypothesis* which has not yet been proven.

Nonpositive integers: $\mathbb{N}^-$
$\mathbb{N}^- := \{\dots, -2, -1, 0\}$

Integers: $\mathbb{Z}$ (German: **Z**ahlen=numbers)
$\mathbb{Z} := \mathbb{N} \cup \mathbb{N}^-$

Positive integers: $\mathbb{Z}^+ = \mathbb{P}$
$\mathbb{Z}^+ = \mathbb{P} := \mathbb{N} \backslash \{0\}$.

Negative integers: $\mathbb{Z}^-$
$\mathbb{Z}^- := \mathbb{N}^- \backslash \{0\}$

Rational numbers: $\mathbb{Q}$ (**Q**uotients)
$\mathbb{Q} := \left\{ \frac{p}{q} : p \in \mathbb{Z} \;\&\; q \in \mathbb{Z} \backslash \{0\} \right\}$

Positive rational numbers: $\mathbb{Q}^+$
$\mathbb{Q}^+ := \left\{ \frac{p}{q} : p, q \in \mathbb{Z}^+ \right\}$

Negative rational numbers: $\mathbb{Q}^-$
$\mathbb{Q}^- := \{ \frac{p}{q} : p \in \mathbb{Z}^- \;\&\; q \in \mathbb{Z}^+ \}$

Real numbers: $\mathbb{R}$
$\mathbb{R}$ can be constructed from $\mathbb{Q}$. It is easier but less rigorous to simply say there is a 1-1 correspondence between $\mathbb{R}$ and the real number line. $\mathbb{R}$ is up to isomorphism the only complete ordered field.

Positive real numbers: $\mathbb{R}^+$
$\mathbb{R}^+ := \{r \in \mathbb{R} : r > 0\}$
(where "$r > 0$" means that r occurs to the right of 0 on the real number line).

Negative real numbers: $\mathbb{R}^-$

$\mathbb{R}^- := \{r \in \mathbb{R} : r < 0\}$
(where "$r < 0$" means that r occurs to the left of 0 on the real number line).

All the above number systems are one dimensional.

Complex numbers: $\mathbb{C}$ - two dimensional numbers
$\mathbb{C} := \{a+ib, c+id : a, b, c, d \in \mathbb{R},\ i$ is an *arbitrary symbol* such that $(a+ib)+(c+id) = (a+c)+i(b+d)$ & $(a+ib)(c+id) = (ac-bd)+i(bc+ad)\}$
Letting $a = c = 0,$ & $b = d = 1$ we see from this definition that $i^2 = -1$.

The notations e (for $\sum_{n=0}^{\infty} \frac{1}{n!}$) and i (for $\sqrt{-1}$) were invented by Leonard Euler. These numbers have been very influential in mathematics.

Another notation (William Rowan Hamilton's) for complex numbers is as ordered pairs $(a, b)(= a + ib)$ where $a, b \in \mathbb{R}$ but this is rarely used thesedays.

Quaternion numbers: $\mathbb{H}$ (for **H**amilton who invented them) - four dimensional numbers

Octonion numbers: $\mathbb{O}$ - eight dimensional numbers

$\mathbb{C}$, $\mathbb{H}$, & $\mathbb{O}$ are all based on $\mathbb{R}$ and the units 1 and i, j, k, l, m, n, o which are such that $i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1$. $\mathbb{H}$ and $\mathbb{O}$ are defined similarly to $\mathbb{C}$ together with other conditions on i, j, k, l, m, n, o.

Theorem. $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$.

(This is where I took issue with the presenter of the workshop.)

Occasionally in the literature, we see $\mathbb{N}$ confused with $\mathbb{P}$. We also see the notation $\mathbb{P}$ used for a **p**rojective set. In these notes however, we will use these symbols as they are defined above.

In this notation I am using $\mathbb{BLACKBOARD\ BOLD}$ font, and this is also how we handwrite them, but we often see them typed in **BOLD** font, i.e.:

$$\mathbf{P}, \mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}.$$

An older and less frequently used notation is (in MEDIUM ROMAN font):

J for integers
R for rationals
R* for reals
C for complex numbers

We can generalise complex numbers to n-dimensional *hypercomplex* numbers $\sum_{j=1}^{n} i_j a_j$ where $n \in \mathbb{Z}^+$, $a_j \in \mathbb{R}$, $i_1 = 1$ and $i_2^2 = \cdots = i_n^2 = -1$, as well as several other conditions on the i_j's. However, we actually don't need number systems with dimensions other than 1, 2, 4 or 8. In particular, if $\mathbb{S}$ is a finite dimensional division algebra over $\mathbb{R}$, then the dimension of $\mathbb{S}$ as a vector space over $\mathbb{R}$ is equal to 1, 2, 4 or 8. This was first suspected as early as the early 1800's. There are now several proofs of this and of several equivalent statements in the literature. But it was first proved in the 1950's by Milnor who built upon the work of Bott and independently by Kervaire. The presenter of the workshop could modify his claim to something more along the lines of the Milnor-Kervaire theorem.

Sometimes however, we use the notations $\mathbb{X}^*$ for the *extended* number system of $\mathbb{X}$ where $\mathbb{X}$ is one of the number systems above.
$\mathbb{X}^* := \mathbb{X} \cup \{\infty\}$ or $\mathbb{X} \cup \{-\infty\}$ or $\mathbb{X} \cup \{-\infty, \infty\}$.

The notation $\mathbb{R}^*$ is also sometimes used to represent the *hyperreal* number system, which is the same as $\mathbb{R}$ but with the added axiom of calculus that *infinitessimals* δx exist. **Applied** mathematicians are famous (or infamous, depending on your perspective) for using the idea of infinitessimals as if they were elements of $\mathbb{R}$ and they have had great success in so doing. Pure mathematicians will tell you that they are not elements of $\mathbb{R}$. However, if we use this notion as an **axiom** and in so doing **replace** $\mathbb{R}$ by $\mathbb{R}^* := \mathbb{R} \cup \{\delta x\}$ it is possible to construct a **pure** mathematical calculus system without the use of limits. Calculus questions in $\mathbb{R}$ will still have solutions in $\mathbb{R}$ but with the help of $\mathbb{R}^*$ instead of limits. Applied mathematicians will just tell you "I told you so". But the main difference between pure and applied mathematicians is that applied mathematicians are comfortable with "fudge factors" and pure mathematicians are not. Pure mathematicians will tell you there is no way they will accept the notion of infinitessimals until they have a good way of **understanding why it works**. Applied mathematicians will tell you they will accept an idea **because it works**, whether they understand it or not! The theory of hyperreals has helped to blur the distinction between pure and applied mathematics. In any case, due to Kurt Gödel's undecideablity theorem which states that any logical mathematical system will contain unproveable statements, this may be a vacuous point. But such theorems do not diminish the level of precision present throughout mathematics coupled with the self imposed burden of proof, which is as it always has been, superior in this regard compared with other academic disciplines.

All New South Wales high school students will be familiar with all the abovementioned number systems except $\mathbb{C}$, $\mathbb{H}$, & $\mathbb{O}$. Complex numbers are used in many applications in physics and engineering, and quaternion and octonion numbers are used in high level physics such as quantum mechanical angular momentum and some aspects of the theory of superstrings.

$\mathbb{Z}$ is an *integral domain* (has closure of addition & multiplication, commutivity and associativity of addition and multiplication, distributivity of multiplication over addition, additive and multiplicative identities, additive inverses and a cancellation law) and $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are *fields* (are integral domains and also have multiplicative inverses). Hence these are the most useful of all number systems, and so most of mathematics only uses $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, & $\mathbb{C}$ and nothing else. Nevertheless, many other integral domains and fields are used in pure mathematical research. Note that $\mathbb{H}$ and $\mathbb{O}$ are **not** fields.

There is a 1-1 correspondence between $\mathbb{Z}$ and $\mathbb{Q}$ which is especially interesting since one of these is only an integral domain whereas the other is a field. Anything in 1-1 correspondence with $\mathbb{Z}$ is called *countable*.

There is no 1-1 correspondence between $\mathbb{Z}$ and $\mathbb{R}$. This is often articulated by saying that $\mathbb{R}$ is *uncountable*. There is also no 1-1 correspondence between $\mathbb{R}$ and $\mathbb{C}$.

Theorem. *$\mathbb{R}$ is not countable, $\mathbb{C}$ is not ordered, $\mathbb{H}$ is not commutative, $\mathbb{O}$ is not associative*

If we redefine $\mathbb{Q}$ by saying it is the set of solutions for x to all equations of the form $a_1x + a_0 = 0$ for integers a_0, a_1 with $a_1 \neq 0$ then we can see that we can generalise $\mathbb{Q}$ to $\mathbb{A}$, the set of all *algebraic numbers* which are real solutions to polynomial equations $\sum_{j=0}^{n} a_j x^j = 0$ for integers $a_0, \ldots, a_n$ where $a_n \neq 0$. For example, $\sqrt{2}$, which is irrational, is nevertheless algebraic. $\mathbb{A}$ is countable. So $\mathbb{T} := \mathbb{R} \backslash \mathbb{A}$, the set of all *trancendental numbers*, is uncountable. e and π are transcendental. All polynomial equations with coefficients in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$ have solutions in $\mathbb{C}$ by the Fundamental Theorem of Algebra so sometimes we see a different definition of the set of algebraic numbers than the one given above, allowing it to have complex elements. This is still countable. The trancendental compliment of this set with respect to $\mathbb{C}$ is likewise uncountable. But this definition isn't common because a complex number is of the form $a + ib$ for real a, b so a trancendental complex number can be defined such that inclusively a <u>or</u> $b \in \mathbb{T}$. Similarly an algebraic complex number can be defined such that a <u>and</u> $b \in \mathbb{A}$.

There is also a hypothesis called Georg Cantor's Continuum Hypothesis which says that for any superset $\mathbb{X}$ of $\mathbb{Z}$ which is also a subset of $\mathbb{R}$, i.e., for any set $\mathbb{X}$ such that $\mathbb{Z} \subset \mathbb{X} \subset \mathbb{R}$ there is <u>either</u> a 1-1 correspondence between $\mathbb{Z}$ and $\mathbb{X}$ <u>or</u> there is a 1-1 correspondence between $\mathbb{X}$ and $\mathbb{R}$. This was proved to be undecideable within the confines of Cantorian set theory in 1963 by Paul Cohen (i.e., the truth or otherwise or this mathematical hypothesis not only trancends the mathematics from which it emerged, but is also independent of it). Nevertheless, logical systems can be constructed within which it can be proved **and** other (more exotic) logical systems can be constructed within which it can be disproved!

So I guess we should not frown upon applied mathematicians with too much scorn and derision for their obsessive preponderance on fudge factors.

Mechanics - Mick Canty

Focus Questions.

1. The angular speed of a grindstone is 1800 revolutions per minute. The diameter of the grindstone is 10 cm. Calculate the linear speed of a point on the rim of the grindstone. Give your answer in metres per second.

2. A particle of mass 12 kg rests on a smooth horizontal plane and is attached by a string 1.2 m long to a fixed point on the plane. If the particle describes a horizontal circle at 3.6 ms^{-1} find the tension in the string.

3. A string of length 50 cm can just sustain a mass of 20 kg without breaking. A mass of 4 kg is attached to one end of the string and revolves uniformly in a smooth horizontal table. The other end is being fixed to a point on the table. Find the greatest number of complete revolutions per minute that the body can execute without the string breaking.

4. Two particles of mass $2m$ and m kilograms respectively are connected by a light inextensible string of length l metres. The string passes through a smooth ring and, while the $2m$ kilogram mass hangs in equilibrium a distance x metres below the ring, the mass in kilograms describes a horizontal circle with angular velocity ω rad/sec. Find the inclination of the moving string to the vertical.

5. A water skier of mass 100 kg is being towed by a speed boat, the length of the tow rope being 30 m. Boat and skier are moving with constant speed in concentric circles of radii 40 m and 50 m respectively. The total mass of the boat and its occupant is 300 kg and its speed is 8 m/s while the speed of the skier is 10 m/s. Assuming that the tow rope is horizontal and that the total air and water resistance is proportional to the square of the speed (with the same constant of proportion for both boat and skier), find the tension in the tow rope and the magnitude and direction of the force driving the speed boat. (Fitzpatrick, J.B. (1991), New Senior Mathematics, Four Unit Course for Year 12, Heinemann)

Solutions

1. $0.05 \times 1800 \times 2\pi \div 60\,\text{m/s} = 3\pi\,\text{m/s}.$ □

2. $\frac{12\times3.6^2}{1.2}\,\text{N} = 129.6\,\text{N}.$ □

3. $\left(60 \times \sqrt{\frac{20g}{4\times0.5}} \Big/ 2\pi\right)$ rev/min $= \frac{30\sqrt{10g}}{\pi}$ rev/min □

4. $\cos^{-1}\frac{mg}{2mg} = 60°.$ □

5. Tension $= \left(\frac{100\times10^2}{50}\Big/\frac{30}{50}\right)\text{N} = 333\frac{1}{3}\,\text{N}.$

Angle to tangent $= \tan^{-1}\dfrac{300\times8^2/40}{300\times8^2\times\frac{333\frac{1}{3}}{100\times10^2\big/\frac{40}{50}}+333\frac{1}{3}} = \tan^{-1}\frac{20}{21}$

Force needed $= \dfrac{300\times8^2/40}{20/\sqrt{20^2+21^2}}\,\text{N} = 696\,\text{N}.$ □

Volumes - Ross Dinnell

Focus Questions

1. Find the volume generated by rotating the area bounded by the curve $y^2 = 8x$ in the first quadrant between $x = 0$ and $x = 2$ around the x-axis.

2. Find the volume generated by rotating the area bounded by the curve $y^2 = 8x$ in the first quadrant between $x = 0$ and $x = 2$ around the line $y = 4$.

3. Find the volume generated by rotating the area bounded by the curve $y^2 = 8x$ in the first quadrant between $x = 0$ and $x = 2$ around the line $x = 2$.

4. Find the volume generated by rotating the area bounded by the curve $y^2 = 8x$ in the first quadrant between $x = 0$ and $x = 2$ around the y-axis, using cylindrical shell.

5. The base of a solid is an ellise whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Each plane section perpendicular to the x-axis is a square, one side of which lies on the base of the solid. Find the volume of the solid.

6. Find the volume of a right cone of height h whose base is an ellipse of major axis $2a$ and minor axis $2b$.

Solutions

1. $\lim\limits_{\delta x \to 0} \sum_{x=0}^{2} \pi y^2 \, \delta x = 8\pi \int_0^2 x \, dx = 16\pi \text{ unit}^3 \quad \square$

2. $\lim\limits_{\delta x \to 0} \sum_{x=0}^{2} \pi(4^2 - (4 - y)^2) \, \delta x = \pi \int_0^2 (8\sqrt{8}x^{\frac{1}{2}} - 8x) \, dx = 26\tfrac{2}{3}\pi \text{ unit}^3 \quad \square$

3. $\lim\limits_{\delta y \to 0} \sum_{y=0}^{4} \pi(2 - x)^2 \, \delta y = \pi \int_0^4 (4 - \tfrac{1}{2}y^2 + \tfrac{1}{64}y^4) \, dy = 8\tfrac{8}{15}\pi \text{ unit}^3. \quad \square$

4. $\lim\limits_{\delta x \to 0} \sum_{x=0}^{2} 2\pi xy \, \delta x = \pi \int_0^2 4\sqrt{2}x^{\frac{3}{2}} \, dx = 12\tfrac{4}{5}\pi \text{ unit}^3. \quad \square$

5. $\lim\limits_{\delta x \to 0} \sum_{x=-a}^{a} (2y)^2 \, \delta x = 8b^2 \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = \frac{16}{3}ab^2 \text{ unit}^3. \quad \square$

6. $\lim\limits_{\delta z \to 0} \sum_{z=0}^{h} \frac{\pi ab}{h^2}(h - z)^2 \, \delta z = \frac{\pi ab}{h^2} \int_0^h (h^2 - 2hz + z^2) \, dz = \frac{1}{3}\pi abh. \quad \square$