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NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 56/67

by James Coroneos*

1. Find the equation of the tangent and normal at the point $P(4,2)$ on the rectangular hyperbola $xy = 8$.
   (i) The tangent meets the asymptotes in M, N. Find the length of MN.
   (ii) The normal intersects the curve again in Q. Find the coordinates of Q.

2. **(i)** Show that the line $x + y + 1 = 0$ is a tangent to the rectangular hyperbola $xy = \frac{1}{4}$ and to the parabola $y^2 = 4x$. Find the points of contact.
   (ii) Prove that the line $y = mx + b$ will touch the rectangular hyperbola $xy = c^2$ if $b^2 = -4mc^2$. Hence find the equations of the tangents from $(-8c, c)$ to the curve.

3. P is any point $(ct, c/t)$ on the hyperbola $xy = c^2$, whose centre is O.
   (i) Perpendiculars PM, PN are drawn to the asymptotes. Prove that $PM.PN$ is constant.
   (ii) Prove that the tangent at P has equation $x + t^2y = 2ct$. Show that
   (a) OP and the tangent at P are equally inclined to the asymptotes;
   (b) if the tangent at P meets the asymptotes in A, B and Q is the fourth vertex of the rectangle $OAQB$, then Q lies on the hyperbola $xy = 4c^2$.
   (c) P is the midpoint of AB and the area of the triangle OAB is independent of the position of P on the curve.

4. A, B are the points $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ on the rectangular hyperbola $xy = c^2$. Show that the chord AB has equation $x + t_1t_2y = c(t_1 + t_2)$

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*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{AMS}$-TEX.

(i) The chord AB meets the asymptotes in M, N. Prove that AB, MN have the same midpoint, and hence show that $AM = BN$.

(ii) If the chord AB is fixed in direction with slope m, show that $t_1 t_2$ is constant and deduce that the midpoints of all chords parallel to AB lie on a diameter of the hyperbola.

5. **(i)** Find the equation of the tangent at $P(ct, c/t)$ on the hyperbola $xy = c^2$. NP is the ordinate of P and the tangent at P meets the y-axis in M. The line through M parallel to the x-axis meets the hyperbola in Q. Show that NQ is the tangent at Q to the hyperbola.

(ii) P_1P_2 is a variable chord of the rectangular hyperbola $xy = c^2$ which subtends a right angle at the point P_3 on the curve. (The parameters of P_1, P_2, P_3 are T_1, T_2, T_3). Show that P_1P_2 is parallel to the normal at P_3.

6. The tangents at $P_1(ct_1, c/t_1)$ and $P_2(ct_2, c/t_2)$ to the rectangular hyperbola $xy = c^2$ whose centre is C, intersects at T. Show that the coordinates of T are $\{2ct_1t_2/(t_1 + t_2), 2c/(t_1 + t_2)\}$.

(i) Prove that C, T and the midpoint M of P_1P_2 are collinear.

(ii) If $t_1t_2 = k$, show that the locus of T is a diameter of the hyperbola.

(iii) $AMBT$ is a rectangle with its sides parallel to the asymptotes. Show that A, B lie on the hyperbola.

7. P and Q are variable points on the rectangular hyperbola $xy = c^2$. The tangent at Q passes through the foot of the ordinate of P. If P, Q have parameters t_1, t_2 show that $t_1 = 2t_2$. Hence prove hat the locus of

(i) the midpoint of PQ is the rectangular hyperbola $xy = 9c^2/8$;

(ii) T, the point of intersection of the tangents at P, Q is a hyperbola with the same asymptotes as the given hyperbola, and state its equation.

8. Show that the normal at $P(ct, c/t)$ to the rectangular hyperbola $xy = c^2$ has equation $t^2x - y = c(t^4 - 1)/t$.

(i) The normal at P meets the y-axis in A, whilst the tangent at P meets the x-axis in B. Q is the fourth vertex of the rectangle $APBQ$. Find the coordinates of Q in terms of c, t. {*Hint*: *Use midpoints.*}

(ii) The normal at P meets the x-axis in A and the tangent at P meets the y-axis in B. Show the midpoint of AB lies on the curve whose equation is $2xyc^2 = c^4 - y^4$.

9. Find the equation of the normal at the point $P(cp, c/p)$ on the rectangular hyperbola $xy = c^2$ and prove that it cuts the hyperbola again at the point Q whose parameter q is $-1/p^3$.

(i) If R is the opposite end of the diameter of the hyperbola through P, prove that PR is perpendicular to RQ.

(ii) If PQ is also a normal at Q, show that $q = -p$. Hence prove that for the rectangular hyperbola $xy = c^2$, there is only one chord which is normal at both ends, and find its equation.

10. The tangent at a variable point P on the hyperbola $xy = c^2$ meets the asymptotes at M, N. The line OP (O is the origin) meets the circle on MN as diameter in Q. Show that P is the midpoint of MN and prove the locus of Q is the rectangular hyperbola $xy = 4c^2$.

11. PQ is a variable chord of the rectangular hyperbola $xy = c^2$, whose centre is O. If PQ meets an asymptote in R and M is the midpoint of the chord PQ, prove that $OM = MR$. If the chord PQ passes through the fixed point (a, b) show that the locus of the midpoint of PQ has equation $(x - \frac{1}{2}a)(y - \frac{1}{2}b) = \frac{1}{4}ab$.

12. **(i)** P, Q are fixed points on the rectangular hyperbola $xy = c^2$ and R is a variable point. If PR, QR meet the same asymptote in H, K prove that HK is of constant length.

(ii) T is a variable point $(ct, c/t)$ on the hyperbola $xy = c^2$. The perpendicular from the centre O to the tangent at T meets it in N. Find the coordinates of N and prove that N lies on the curve $(x^2 + y^2)^2 = 4c^2xy$.

13. Find the equation of the tangent to the reactangular hyperbola $xy = c^2$ at the point $(ct, c/t)$. The tangent at P meets the x, y axes at L, M respectively. O is the origin and POQ a diameter. The line MQ meets the x-axis at T. Prove that the area of the

(a) triangle MOL is $2c^2$ **(b)** triangle QOT is $c^2/3$.

14. PP' is a diameter of the rectangular hyperbola $xy = c^2$. The tangent at P meets the lines through P' parallel to the asymptotes in Q, Q'. Prove that P is the midpoint of PP' and show that the locus of Q has equation $xy + 3c^2 = 0$.

15. Prove that the equation of the line joining the points $P(cp, c/p)$ and $Q(cq, c/q)$ on the hyperbola $xy = c^2$ is $x + pqy = c(p + q)$. Determine also the equation of the tangent at P. The tangents at P, Q meet one asymptote in L, M and the other asymptote in L', M'. Prove that PQ bisects both LM and $L'M'$.

16. Prove that the normal at $P(ct, c/t)$ on the rectangular hyperbola $xy = c^2$ has equation $t^3x - ty = c(t^4 - 1)$. This normal meets the rectangular hyperbola $x^2 - y^2 = a^2$ at Q and R. Prove that P is the midpoint of QR.

17. Verify that the point $(ct, c/t)$ having parameter t, lies on the rectangular hyperbola $xy = c^2$. Let P, Q, R be three points on this rectangular hyperbola with parameters t_1, t_2, t_3 respectively and let H be the point of intersection of the altitudes of the triangle PQR (H is the orthocentre of $\triangle PQR$).

(i) Find the equations of QR and PH and thus determine the coordinates of H.

(ii) Hence prove the theorem: 'if a rectangular hyperbola circumscribes a triangle, it also passes through the orthocentre of the triangle.'

(iii) Further, if the parameter of H is t_4, prove that $t_1t_2t_3t_4 = -1$.

18. P_1, P_2, P_3, P_4 are points with parameters t_1, t_2, t_3, t_4 on the rectangular hyperbola $x = ct, y = ct^{-1}$. Prove that if P_1P_2 and P_3P_4 are at right-angles then $t_1t_2t_3t_4 = -1$. Deduce that the orthocentre of a triangle inscribed in a rectangular hyperbola lies on the hyperbola.

19. The circle $x^2+y^2+2gx+2fy+k = 0$ intersects the rectangular hyperbola $x = ct, y = c/t$ in four points with parameters t_1, t_2, t_3, t_4. Show that $t_1t_2t_3t_4 = 1$. Prove that the tangents at P_1 and P_2 meet on the diameter of the hyperbola perpendicular to P_3P_4.

20. PQ is a diameter of the rectangular hyperbola $xy = c^2$, and P has coordinates $(cp, c/p)$. The circle centre P, with radius PQ, cuts the hyperbola in the points A, B, C, D. If A, B, C have parameters t_1, t_2, t_3 show that $t_1+t_2+t_3 = 3p$, and prove that the centroid (mean centre) of the triangle ABC is P. Deduce that the triangle ABC is equilateral.

21. Find the equation of the tangent at $P(cp, c/p)$ on the hyperbola $xy = c^2$ and prove the point of intersection $T(X, Y)$ of the tangents at $P, Q(cq, c/q)$ are given by $p + q = 2c/Y$ and $pq = X/Y$. If the chord PQ is of fixed length k, prove $k^2p^2q^2 = c^2\{(pq)^2 - 4pq\}(1 + p^2q^2)$. Deduce the locus of T is $4c^2(c^2 - xy)(x^2 + y^2) = k^2x^2y^2$. Also show that the locus of M, the midpoint of PQ, has equation $4(xy - c^2)(x^2 + y^2) = k^2xy$.

22. Find the equations of the tangent and normal to the rectangular hyperbola $xy = c^2$ at the point $P(ct, c/t)$. If the tangent at P meets the axes of x and y at X and Y respectively, and the normal at P meets the lines $y = x$, $y = -x$ at L and M respectively, prove that $LYMX$ is a rhombus. (Assume that $t^2 \neq 1$)

23. Find the coordinates of the point of intersection of the tangents drawn to the rectangular hyperbola $xy = c^2$ at the points $(cp, c/p)$ and $(cq, c/q)$. A variable chord of the hyperbola is such that its mid-point lies on a fixed straight line parallel to the axis of y. Show that the point of intersection of the two tangents to the hyperbola at the extremities of the chord lies on a fixed straight line parallel to the axis of x.

24. A circle is drawn with diameter PQ, where P and Q are two points on opposite branches of the hyperbola $xy = c^2$. Prove that the circle cuts the hyperbola again at two real points which are the ends of a diameter of the hyperbola.

25. $P_i,\ i = 1, 2, 3, 4$ is the point (x_i, y_i) and K is the point (h, k). $P_1, \ldots, P_4$ are the feet of the four normals from K to the rectangular hyperbola $xy = c^2$. Prove that

(i) $\sum_{i=1}^{4} x_i = h$ **(ii)** $\sum_{i=1}^{4} y_i = k$ **(iii)** $\sum_{i=1}^{4} x_i^2 = h^2$ **(iv)** $\sum_{i=1}^{4} y_i^2 = k^2$

Show that the locus of a point K which moves so that $\sum KP_i^2$ is constant is a circle.