NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 55/67

by James Coroneos*

1. For the hyperbola $H: \ x^2/16 - y^2/9 = 1$, find the coordinates of the foci S, S', P is any point $(4\sec\theta, 3\tan\theta)$ on H.

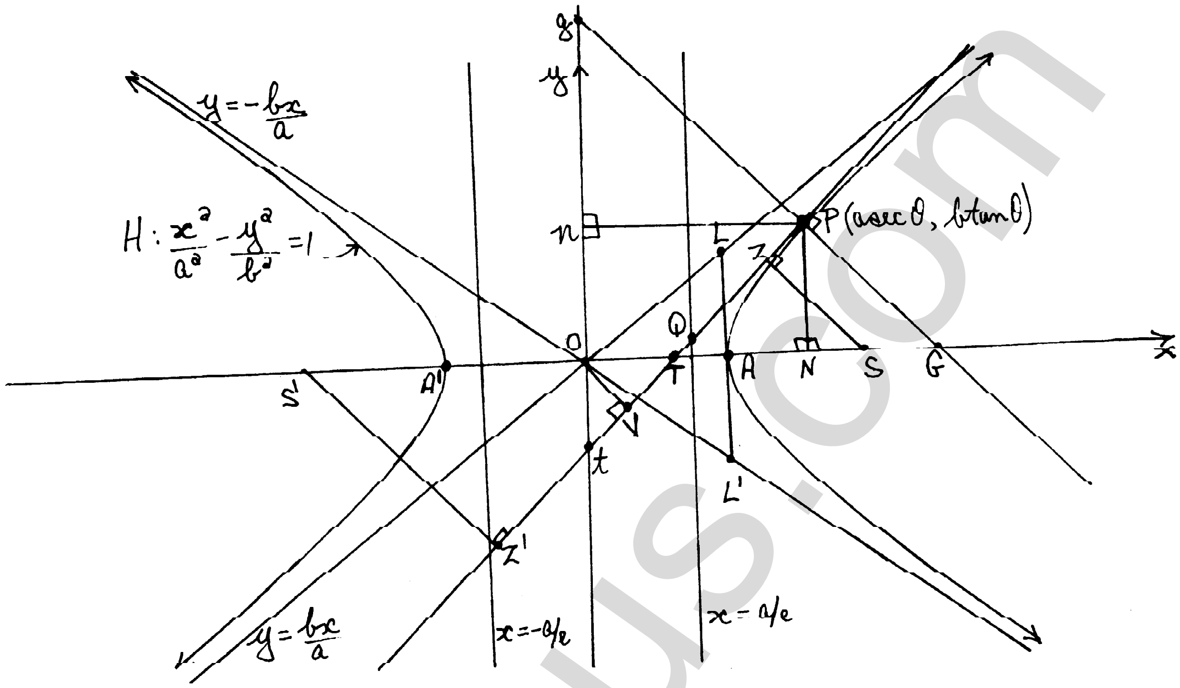
- **(i)** Show that $PS = 5\sec\theta - 4$ and $PS' = 5\sec\theta + 4$; and hence prove the difference of the focal distances to P is independent of the position of P on H.
- **(ii)** Prove the tangent at P has equation $x\sec\theta/4 - y\tan\theta/3 = 1$. If this tangent meets the transverse axis in T, show that $S'P : PS = S'T : TS$. Deduce that the tangent at P bisects the angle $S'PS$.
- **(iii)** Show the normal at P has equation $4x/\sec\theta + 3y/\tan\theta = 25$. If this normal meets the x-axis in G, prove that $S'P : PS = S'G : SG$.

2. For the hyperbola $H: \ 3x^2 - y^2 = 3$ find the coordinates of the foci S, S'; the equations of the directrices and of the asymptotes. P is any point (x_1, y_1) on H.

- **(i)** Find the equation of the line through S perpendicular to the asymptote with positive gradient. If these lines intersect in Z, show that Z lies on the directrix corresponding to S and also on the circle centre O (the origin) with radius equal to the semi-transverse axis.
- **(ii)** Show the tngent at P has equation $xx_1 - yy_1/3 = 1$. If the tangent meets the x, y axes in T, R respectively, prove that $|OT : OR| = |y_1 : 3x_1|$
- **(iii)** If p_1, p_2 are the perpendiculars from P to the asymptotes, show that p_1p_2 is constant.

*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

3.



In the diagram, P is the point $(a\sec\theta, b\tan\theta)$ on the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$, which has centre O and foci S, S'. PT, PG are the tangent and normal at P meeting the axes at T, t and G, g. PN, Pn are perpendicular to the axes; A, A' are the vertices of H. The tangent at A meets the asymptotes at L, L'. $SZ, OV, S'Z'$ are perpendiculars from S, O, S' to the tangent PT. PT cuts the directrix $x = a/e$ in Q. Prove the following results:

(i) The equation of the tangent PT is $bx\sec\theta - ay\tan\theta = ab$ and of the normal PG is $ax\tan\theta + by\sec\theta = (a^2+b^2)\sec\theta\tan\theta$.

(ii) $ON.OT = a^2$ **(iii)** $On.Ot = -b^2$ **(iv)** $PN^2 : A'N.NA = -b^2 : a^2$

(v) $OT : Ot = -a\sin\theta : b$ **(vi)** $OG = e^2.ON$ **(vii)** $Og = a^2e^2.On/b^2$

(viii) $SG = e(e.ON - a)$ **(ix)** $S'G = e(e.ON + a)$ **(x)** $OS = OL = OL'$

(xi) $b^2\sec^2\theta + a^2\tan^2\theta = a^2(e^2\sec^2\theta - 1)$ **(xii)** $SZ.S'Z' = b^2$

(xiii) $PG.OV = b^2$ **(xiv)** Tg and tG are at right-angles **(xv)** $P\hat{S}Q = 90^\circ$

(xvi) $SP = a(e\sec\theta - 1)$ and $S'P = a(e\sec\theta + 1)$. Hence show that $S'P - SP$ is constant and equal to the length of the transverse axis.

(xvii) $SG = e.SP$ and $S'G = e.S'P$. Hence show that $S'P : SP = S'G : SG$

{In the above diagram, if P is (x_1, y_1) show that the tangent and normal at P have equations $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$, $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$ respectively. Hence prove the results **(ii)**-**(iv)**, **(vi)**-**(x)**, **(xii)**-**(xv)**.}

4. Write down the equations of the asymptotes and of the directrices of the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$. Show that

(i) The angle between the asymptotes is $2\tan^{-1}\sqrt{e^2-1}$, where e is the eccentricity of H;

(ii) The asymptotes meet the directrices on the auxiliary circle;

(iii) The perpendicular from a focus to an asymptote meets it on the corresponding directrix;

(iv) The product of the lengths of the perpendiculars from any point $(a\sec\theta, b\tan\theta)$ of H to its asymptotes is constant and equal to b^2/e^2.

5. Show that the length of a latus rectum of $H:\ x^2/a^2 - y^2/b^2 = 1$ is $2b^2/a$ units.

(i) If the latus rectum through $S(ae, 0)$ meets H in the first quadrant at G, show that the tangent at G has equation $ex - y = a$. Prove this tangent intersects the transverse axis where a directrix does and intersects the conjugate axis at a point on the auxiliary circle.

(ii) Also prove that the tangents to H at the ends of the latus rectum through S meet at the foot of the direcrix corresponding to S.

6. P is the point (x_1, y_1) on the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$, with foci S, S'. If e is the eccentricity of H, show that

(i) $SP = ex_1 - a$ and $S'P = ex_1 + a$;

(ii) If the tangent at P meets the transverse axis in T, then $S'P : PS = S'T : TS$ and hence deduce that $SP, S'P$ are equally inclined to the tangent at P;

(iii) If the normal at P meets the conjugate axis in R, show that $SR^2 : SP.S'P$ is constant. $\{$*Hint*: *Note* $y_1^2/b^2 = (x_1^2/a^2 - 1)\}$

7. Find the equation of the tangent and normal at the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$, whose centre is O.

(i) If the tangent meets the x, y axes in T, R and $OTQR$ is a rectangle, show that Q lies on the curve $a^2/x^2 - b^2/y^2 = 1$.

(ii) If the normal meets the x, y axes in L, M and $OLZM$ is a rectangle, show that Z lies on the curve $x^2/b^2 - y^2/a^2 = (a^2+b^2)^2/a^2b^2$. Find the eccentricity of this hyperbola in terms of e, the eccentricity of H.

8. P, Q are the points $\theta, \pi/2 - \theta$ on the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$.

(i) Show that the midpoint of PQ lies on the curve $x^2/a^2 - y^2/b^2 = y/b$.

(ii) Prove that the tangent at P has equation $bx - ay\sin\theta = ab\cos\theta$ and find the equation of the tangent at Q in similar form. Hence prove that the tangents at P, Q meet on the line $y = b$.

9. Write down the equations of the asymptotes of the hyperbola $H:\ x^2/9 - y^2/16 = 1$.

(i) Prove that the part of the tangent at $P(3\sec\theta, 4\tan\theta)$ on H which is intercepted between the asymptotes is bisected at P.
(ii) Show that this tangent forms with the asymptotes a triangle of constant area.
(iii) Prove that the tangent intercepted between the asymptotes subtends a constant angle at a focus.

10. Find the equation of the tangent at $P(x_1, y_1)$ on $H:\ x^2/3 - y^2/2 = 1$.
(i) If h, k are the intercepts made by this tangent on the coordinate axes, show that $3/h^2 - 2/k^2 = 1$.
(ii) If the length of the perpendicular from the origin O to this tangent is p and $OP = r$, show that $1/p^2 = (r^2 - 1)/6$.

11. The tangent at $P(a\sec\theta, b\tan\theta)$ on the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$ meets a directrix in Q and S is the corresponding focus. O is the centre of H.
(i) Prove that SQ is perpendicular to SP, and deduce that tangents at the ends of a focal chord meet on the directrix.
(ii) Show the perpendicular from S to the tangent at P intersects the line OP on the directrix.
(iii) If the point P is on H such that the line joining PS is parallel to an asymptote, find the coordinates of the point where the tangent at P to H meets this asymptote.

12. A, A' are the vertices of the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$, whose centre is O. The tangent at P meets the tangents at A, A' in L, L' respectively. Prove that
(i) $A'P$ is parallel to OL; **(ii)** $AL.A'L = -b^2$;
(iii) the circle on diameter LL' passes through the foci of H;
(iv) the midpoint of AP lies on the hyperbola $(x - \frac{1}{2}a)^2/a^2 - y^2/b^2 = 1/4$. State the centre of this hyperbola, and show the axes are half the lengths of the axes of H.

13. The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$, whose centre is O; the foci are S, S'. Show that $SP = a(e\sec\theta - 1)$ and $S'P = a(e\sec\theta + 1)$
(i) Perpendiculars are drawn from S, S' to meet the tangent at P in M, M' respectively. Prove that $\sin A\hat{P}M = \sin S'\hat{P}M'$, and hence deduce that the tangent at P bisects the angle $S'PS$.
(ii) The tangent meets the transverse axis at T and PN is the ordinate of P. If the vertex A of H is the midpoint of TS, prove that $ON = \frac{1}{2}S'P$.

14. From the point R in which the tangent at any point P of the hyperbola $H: \; x^2/a^2 - y^2/b^2 = 1$ meets an asymptote, perpendiculars RM, RN are drawn to the axes. Prove that MN passes through P. The tangent and normal at P meet the transverse axis in T, G respectively. Show the midpoint of TG cannot be at a focus.

15. Show that the equation of the tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is $x\sec\theta/a - y\tan\theta/b = 1$. Find also the equation of the normal at P. The ordinate at P meets an asymptote at Q. The tangent at P meets the same asymptote at R. The normal at P meets the x-axis at G. Prove that the angle RQG is a right angle.

16. Show that the equation of the normal to the hyperbola $H: \; x^2/a^2 - y^2/b^2 = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is $ax\tan\theta + by\sec\theta = (a^2 + b^2)\sec\theta\tan\theta$.

- **(i)** This normal meets the x-axis at Q and the y-axis at R. Show that both the ratio of the x-coordinate of P to that of Q and the ratio of the y-coordinate of P to that of R are independent of the position of P.
- **(ii)** P' is a point on H so that PP' is parallel to the conjugate axis. If O is the centre of H, then OP' cuts the normal at P in N. Show that N lies on the hyperbola $x^2/a^2 - y^2/b^2 - \lambda^2$ where $\lambda = (a^2 + b^2)/(a^2 - b^2)$. If H is rectangular, show that the normal at P and OP' are parallel to one another.

17. The point $P(a\sec t, b\tan t)$ on the hyperbola $H: \; x^2/a^2 - y^2/b^2 = 1$ is joined to the vertices $A(a, 0)$ and $A'(-a, 0)$. The lines $AP, A'P$ meet the asymptote $ay = bx$ at Q, R respectively. Prove that the x-coordinate of Q is $a\cos\frac{1}{2}t/(\cos\frac{1}{2}t - \sin\frac{1}{2}t)$, and that the length of QR is independent of the value of t.

18. Prove that the point P whose coordinates are $\{\frac{1}{2}a(t + 1/t), \frac{1}{2}b(t - 1/t)\}$ lies on the hyperbola $H: \; x^2/a^2 - y^2/b^2 = 1$. Show that the tangent at (x_1, y_1) on H has equation $xx_1/a^2 - yy_1/b^2 = 1$, and hence deduce that the tangent at P is $(t^2 + 1)x/a - (t^2 - 1)y/b = 2t$.

- **(i)** If O is the centre of H and S is either focus, and if the tangent at P meets the asymptote $x/a = y/b$ at X and meets the asymptote $x/a + y/b = 0$ at Y, prove that $t = OZ/OS = OS/OY$.
- **(ii)** If the point P on H is such that the tangent at P, the latus rectum through the focus S and one asymptote are concurrent, prove that SP is parallel to the other asymptote.