NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 46/67

by James Coroneos*

1. Write the following in simplest form

(a) $(\cos\theta+i\sin\theta)^9$; $(\cos\theta-i\sin\theta)^{-4}$; $(\cos 2\theta-i\sin 2\theta)^3$; $(\cos 4\theta+i\sin 4\theta)^{-5}$

(b) $(\sin\theta+i\cos\theta)^4$; $(\sin\theta-i\cos\theta)^6$; $(\sin\theta+i\cos\theta)^{-9}$; $(\sin\theta-i\cos\theta)^{-8}$

2. If $z=\cos\theta+i\sin\theta$, express in simplest form

(i) $z+\frac{1}{z}$ **(b)** $z^8+\frac{1}{z^8}$ **(c)** $z^3-\frac{1}{z^3}$ **(d)** $z^6-\frac{1}{z^6}$

3. Express the following in the form $a+ib$.

(a) $(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})^6$ **(b)** $(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4})^3$ **(c)** $(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8})^4$

(d) $(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})^{-3}$ **(e)** $(\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5})^{-10}$ **(f)** $(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12})^{-4}$

4. If $z=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}$, express as complex numbers in the form $a+ib$

(a) z^4 **(b)** z^{-6} **(c)** $z^3+\frac{1}{z^3}$ **(d)** $z^{10}-z^{-10}$

5. Express each of the following in simplest form

(a) $\frac{(\cos\alpha+i\sin\alpha)^3}{(\cos\beta+i\sin\beta)^4}$ **(b)** $\frac{(\cos\theta-i\sin\theta)^5}{(\cos 2\phi-i\sin 2\phi)^3}$ **(c)** $\frac{(\cos 3\theta+i\sin 3\theta)^7}{(\cos 4\theta-i\sin 4\theta)^5}$

(d) $\frac{(\cos 7\theta+i\sin 7\theta)(\cos\theta+i\sin\theta)^{-5}}{(\cos 4\theta-i\sin 4\theta)}$ **(e)** $\frac{(\cos 3\theta-i\sin 3\theta)^8\cdot(\cos 2\theta+i\sin 2\theta)^7}{(\cos 5\theta+i\sin 5\theta)^4}$

6. **(i)** Express $1+i$ in the form $r(\cos\theta+i\sin\theta)$ and hence show that $(1+i)^{10}$ is purely imaginary.

(ii) Express $z=-\sqrt{3}+i$ in mod-arg form and hence obtain the value of $1/z^{12}$.

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7. **(i)** Simplify the following, expressing each in $a + ib$ form.
(a) $(2 - 2\sqrt{3}i)^3$ **(b)** $(-\frac{\sqrt{3}}{2} - \frac{1}{2}i)^{-9}$ **(c)** $(i-1)^7$
(ii) Express $-1 - i\sqrt{3}$ in mod-arg form, and hence show it is a sixth root of 64.

8. If $w = 1 + i\sqrt{3}$, show on the same Argand diagram, the points representing the complex numbers $w, w^2, w^3, 1/w, 4w, -w, \overline{w}, iw, w^2 + w, w^3 - w, w^{\frac{1}{2}}$

9. If $z = i - 1$, show clearly on an Argand diagram, the points representing the complex numbers $z, z^2, z^3, 1/z, \sqrt{2}z, -z, \overline{z}, iz, z^2 - z, z^3 + z, z^{\frac{1}{2}}$

10. **(i)** Express $z = i\sqrt{3} + 1$ in the form $r(\cos\theta + i\sin\theta)$ and hence express z^{11} and z^{-9} in the form $a + ib$.
(ii) Show that $(-1 + i)^n$ is real if n is a multiple of 4.

11. **(i)** By expressing $\sqrt{3}+i, 1+i$ in mod-arg forms, evaluate $(\sqrt{3}+i)^{50}/(1+i)^{30}$ in the form $a + ib$, where a, b are real.
(ii) Show that $\tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}(\frac{a-b}{1+ab})$. If $z = \frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$, find $|z|$ and $\arg z$ in exact form. Hence evaluate z^6.

12. **(i)** Express $i - \sqrt{3}$ and $-i - \sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$, and hence show that $(i - \sqrt{3})^n - (-i - \sqrt{3})^n = i.2^{n+1}\sin\frac{5n\pi}{6}$.
(ii) Show that $(1 + i)^n + (1 - i)^n$ is real for all positive integer values of n. For what positive integer values of n is $(1 + i)^n + (1 - i)^n = 0$?

13. **(i)** Simplify $z = \frac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta}$ and prove $z^n = \cos 2n\theta = i\sin 2n\theta$
(ii) Prove that $(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n = 2^{n+1}\cos\frac{\theta}{2}\cos\frac{n\theta}{2}$

14. If $z = \cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$ and n is a positive integer, prove $1+z+z^2+\cdots+z^{2n-1} = 0$ and $1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\frac{\pi}{2n}$

15. If n is any integer, prove that
(i) $(1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})^n = -2^n\cos^n\frac{\pi}{n}$
(ii) $(1+i\sqrt{3})^{2n}+(1-i\sqrt{3})^{2n}$ has either the value 2^{2n+1} or -2^{2n}, according as to whether n is or is not a multiple of 3.