NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 7/67

by James Coroneos*

1. Express each of the following in the form $a + ib$, (where a, b are real), and mark on an Argand diagram, the points representing each number.
(i) $(2+3i)^2$ **(ii)** $\frac{1}{2+3i}$ **(iii)** $\frac{1+i}{1-i}$ **(iv)** $\{\frac{1+i}{1-i}\}^2$ **(v)** $(2-3i)(3+2i)$ **(vi)** $\overline{(1+i)^{-1}}$

2. Find the modulus and argument of each of the following complex numbers, and express each in the form $r(\cos\theta + i\sin\theta)$. Illustrate each example geometrically.
(i) $1+i$ **(ii)** $1-i$ **(iii)** $-1+i$ **(iv)** $-1-i$ **(v)** 1 **(vi)** -1 **(vii)** i **(viii)** $-i$
(ix) $1+\sqrt{3}i$ **(x)** $-1-\sqrt{3}i$ **(xi)** $1-\sqrt{3}i$ **(xii)** $-1+\sqrt{3}i$ **(xiii)** $\sqrt{3}i$ **(xiv)** -4
(xv) $\frac{1+i}{\sqrt{2}}$ **(xvi)** $\frac{\sqrt{3}-i}{2}$ **(xvii)** $4+3i$ **(xviii)** $2-3i$ **(xix)** $2[-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}]$
(xx) $-\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}$ **(xxi)** $-3(\cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2})$ **(xxii)** $\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$

3. **(i)** If $z = (1-2i)(i-3)$, find the **(a)** multiplicative inverse **(b)** additive inverse of z.
(ii) If $z = 3-4i$, then $\frac{\overline{z}}{|z|^2}$ is equal to
(A) $\frac{3+4i}{5}$
(B) $\frac{3-4i}{25}$
(C) $\frac{3+4i}{25}$
(D) $\frac{4+3i}{\sqrt{5}}$
(E) None of the above.

4. **(i)** If $z = \frac{1-i}{2+i}$, prove that $\Re(z+z^{-1}) = \frac{7}{10}$ and $\Im(z+z^{-1}) = \frac{9}{10}$.
(ii) If $z = x+iy$, determine $\Re(z^2-2z)$ and $\Im(z^2-2z)$.

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5. If $z = x + iy$ and $\overline{z} = x - iy$

(i) Show the geometric representation on the Argand diagram of **(a)** z **(b)** $\overline{z}$ **(c)** $-\overline{z}$ **(d)** $-z$ **(e)** $\overline{(-z)}$

(ii) Prove that **(a)** $z\overline{z} = |z|^2 = \{\Re(z)\}^2 + \{\Im(z)\}^2$ **(b)** $z + \overline{z} = 2\Re(z)$ **(c)** $z - \overline{z} = 2i\Im(z)$ **(d)** $|z| = |\overline{z}|$ **(e)** $\Im(z) = \Re(-iz)$ **(f)** $|\Re(z)| < |z|$ **(g)** $|\Im(z)| < |z|$

(iii) If $z = \overline{z}$, what can you say about the position of z on the Argand diagram.

(iv) If $z^2 = \overline{z}^2$, show that z is either real or pure imaginary.

6. If z_1, z_2 are complex numbers, prove that

(i) $\Re(z_1 \pm z_2) = \Re(z_1) \pm \Re(z_2)$ **(ii)** $\Im(z_1 \pm z_2) = \Im(z_1) \pm \Im(z_2)$

(iii) In general, $\Re(z_1 z_2) \neq \Re(z_1) \cdot \Re(z_2)$

and $\Im(z_1 z_2) \neq \Im(z_1) \cdot \Im(z_2)$

7. Solve $z^2 + (1+i)z + 2i = 0$, expressing the roots in mod-arg form. If z_1, z_2 are these roots, prove that $|z_1| = |z_2| = \sqrt{2}$, and that $\arg z_1 + \arg z_2 = \frac{\pi}{2}$. [*Hint*, show $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ and $\tan\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$]

8. If a, b, c are real and

(i) z_1, z_2 are complex numbers, show that $az_1\overline{z}_1 + b(z_1\overline{z}_2 + \overline{z}_1 z_2) + cz_2\overline{z}_2$ is real.

(ii) α is an unreal root of the equation $az^2 + bz + c = 0$, show that $a(\alpha + \overline{\alpha}) + b = 0$ and $a\alpha\overline{\alpha} = c$.

9. If $z = x + iy$, $\alpha = a + ib$ are complex numbers and $\overline{z}$, $\overline{\alpha}$ their conjugates, show that if c is real, then

(i) $z\overline{z} - \alpha\overline{z} - \overline{\alpha}z + c = 0$ represents a circle (provided $\alpha\overline{\alpha} - c \geq 0$)

(ii) $\overline{\alpha}z + \alpha\overline{z} + c = 0$, $\alpha \neq 0$, represents a straight line.

(iii) $\overline{\alpha}(\overline{z} - z) + \alpha(\overline{z}) + c = 0$ represents two straight lines.

10. Solve for $z = x + iy$: **(i)** $z^2 + iz = 2$ **(ii)** $(1+i)z + 3i\overline{z} = 2 + i$ **(iii)** $z\overline{z} + 2z = \frac{1}{4} + i$ **(iv)** $z\overline{z} + 3(z - \overline{z}) = 4 - 3i$

11. Find complex numbers $\alpha = a + ib$, $\beta = c + id$ which satisfy

(i) $\begin{cases} \alpha + i\beta & = 1 \\ i\alpha + \beta & = 1 + i \end{cases}$ **(ii)** $\begin{cases} (1+i)\alpha - i\beta & = 3 + i \\ (2+i)\alpha + (2-i)\beta & = 2i \end{cases}$

12. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ prove that

(i) $(\overline{z_1 + z_2}) = \overline{z}_1 + \overline{z}_2$ **(ii)** $(\overline{z_1 - z_2}) = \overline{z}_1 - \overline{z}_2$ **(iii)** $(\overline{z_1 z_2}) = \overline{z}_1 \cdot \overline{z}_2$

(iv) $\overline{z_1 \div z_2} = \overline{z}_1 \div \overline{z}_2$ **(v)** $|z_1 + z_2| = |\overline{z}_1 + \overline{z}_2|$ **(vi)** $\overline{iz_1} = -i \cdot \overline{z}_1$

(vii) $|z_1 z_2|^2 = (z_1\overline{z}_1)(z_2\overline{z}_2)$ **(viii)** $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(ix) $|z_1 + z_2|^2 = z_1\overline{z}_1 + z_2\overline{z}_2 + (z_1\overline{z}_2 + z_2\overline{z}_1) = |z_1|^2 + |z_2|^2 + 2\Re(z_1\overline{z}_2)$

(x) $|z_1 - z_2|^2 = z_1\overline{z}_1 + z_2\overline{z}_2 - 2\Re(z_1\overline{z}_2)$